Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

\[
\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}.
\]  

(12)
Full Price (continued)

\[ C(1 - \omega) \]

coupon payment date \( (1 - \omega) \) \quad \omega \quad \text{coupon payment date}
Full Price (concluded)

- The price is now calculated by

\[
PV = \frac{C}{(1 + \frac{r}{m})^\omega} + \frac{C}{(1 + \frac{r}{m})^{\omega+1}} \cdots \\
= \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}. \tag{13}
\]
Accrued Interest

• The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

• The buyer pays the invoice price: the quoted price plus the accrued interest (AI).

• The accrued interest equals

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega). \]
Accrued Interest (concluded)

• The yield to maturity is the $r$ satisfying Eq. (13) on p. 85 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.$$
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.
Example ("30/360") (concluded)

• The yield to maturity is 3%.

• This can be verified by Eq. (13) on p. 85 with
  - \( \omega = 60/180, \)
  - \( n = 4, \)
  - \( m = 2, \)
  - \( F = 100, \)
  - \( C = 5, \)
  - \( PV = 111.2891 + 3.3333, \)
  - \( r = 0.03. \)
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.\(^a\)
  – The short reason: Exponential growth to \(C\) is replaced by linear growth, hence “overpaying.”

\(^a\)See Exercise 3.5.6 of the textbook for proof.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[ -\frac{\partial P}{\partial y} \cdot \frac{1}{P} \]  \hspace{1cm} (14)
Price Volatility of Bonds

- The price volatility of a level-coupon bond is

\[ -\frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)} \].

- \( F \) is the par value.
- \( C \) is the coupon payment per period.
- Formula can be simplified a bit with \( C = Fc/m \).

For the above bond,

\[ -\frac{\partial P}{\partial y} > 0. \]
Macaulay Duration\(^{a}\)

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

- The weights are the cash flows’ PVs divided by the asset’s price.

- Formally,

\[
\text{MD} \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} i.
\]

- The Macaulay duration, in periods, is equal to

\[
\text{MD} = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \tag{15}
\]

\(^{a}\)Macaulay (1938).
MD of Bonds

- The MD of a level-coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \tag{16}
\]

- It can be simplified to

\[
MD = \frac{c(1+y)[(1+y)^n - 1] + n y (y - c)}{cy[(1+y)^n - 1] + y^2},
\]

where \( c \) is the period coupon rate.

- The MD of a zero-coupon bond equals \( n \), its term to maturity.

- The MD of a level-coupon bond is less than \( n \).
Remarks

• Equations (15) on p. 96 and (16) on p. 97 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility\(^a\) may decrease.

\(^a\)As originally defined in Eq. (14) on p. 94.
How *Not* To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- Many, if not most, duration-related terminology can only be comprehended as measuring volatility.
Conversion

• For the MD to be year-based, modify Eq. (16) on p. 97 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \left( \frac{C}{(1 + \frac{y}{k})^i} \right) + \frac{n}{k} \left( \frac{F}{(1 + \frac{y}{k})^n} \right) \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Equation (15) on p. 96 also becomes

\[
\text{MD} = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \( \frac{\text{MD (in periods)}}{k} \).
Modified Duration

• Modified duration is defined as

\[
\text{modified duration} \triangleq - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \quad (17)
\]

• By the Taylor expansion,

percent price change \approx - \text{modified duration} \times \text{yield change}. 
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[ -11.54 \times 0.001 = -0.01154 = -1.154\%. \]
Modified Duration of a Portfolio

• By calculus, the modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

– \( D_i \) is the modified duration of the \( i \)th asset.
– \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[
\frac{P_- - P_+}{P_0(y_+ - y_-)}.
\]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
Effective Duration (concluded)

• One can compute the effective duration of just about any financial instrument.

• An alternative is to use

\[ \frac{P_0 - P_+}{P_0 \Delta y}. \]

  – More economical but theoretically less accurate.
The Practices

- Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^a\)

- The percentage price change expressed in percentage terms is then approximated by

\[-D\% \times \Delta r\]

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

- $D\%$ in fact equals modified duration (prove it!).

\(^a\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[ \text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}. \]

- The approximate dollar price change is

\[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

- One can hedge a bond portfolio with a dollar duration \( D \) by bonds with a dollar duration \( -D \).
Convexity

- Convexity is defined as

  \[
  \text{convexity (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.
  \]

- The convexity of a level-coupon bond is positive (prove it!).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

• The approximation $\Delta P/P \approx - \text{duration} \times \text{yield change}$ works for small yield changes.

• For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$ 

• Recall the figure on p. 110.
The Practices

• Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

• $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as
  \[ \frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2}, \]
  - \( P_- \) is the price if the yield is decreased by \( \Delta y \).
  - \( P_+ \) is the price if the yield is increased by \( \Delta y \).
  - \( P_0 \) is the initial price, \( y \) is the initial yield.
  - \( \Delta y \) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- Numerically, choosing the right \( \Delta y \) is a delicate matter.
Approximate $d^2 f(x)^2 / dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $[(1 + \Delta x)^2 + (1 - \Delta x)^2 - 2]/(\Delta x)^2$ and 2:

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 852ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

\begin{itemize}
  \item If you have one, you have the other.
  \item So they are just two names given to the same thing: cost of fund.
  \item Traders most likely work with prices.
  \item Banks most likely work with interest rates.
\end{itemize}

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.

- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.

- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

• The term “term structure” often refers exclusively to the yields of zero-coupon bonds.

• A yield curve plots the yields to maturity of coupon bonds against maturity.

• A par yield curve is constructed from bonds trading near par.
Yield Curve of U.S. Treasuries as of July 24, 2015

Yield (%)
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is by definition $[1 + S(i)]^{-i}$.
  - It is the price of an $i$-period zero-coupon bond.\(^{a}\)
- The one-period spot rate is called the short rate.
- Spot rate curve:\(^{b}\) Plot of spot rates against maturity:
  \[S(1), S(2), \ldots, S(n)\].

\(^{a}\)Recall Eq. (9) on p. 69.
\(^{b}\)That is, term structure.
Problems with the PV Formula

• In the bond price formula (4) on p. 41,

\[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]

every cash flow is discounted at the same yield \( y \).

• Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

\[
PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \\
PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}.
\]
Problems with the PV Formula (concluded)

• The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.

• But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

\[ C_1 \quad \uparrow \quad C_2 \quad \uparrow \quad C_3 \quad \uparrow \quad \ldots \quad \uparrow \quad C_n \]

\[ 1 \quad 2 \quad 3 \quad \ldots \quad n \]
Spot Rate Discount Methodology (concluded)

• So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \] (18)

• This pricing method incorporates information from the term structure.

• It discounts each cash flow at the corresponding spot rate.
Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2,\ldots, C_n$ should have a market price of

\[ P = \sum_{i=1}^{n} C_i d(i). \]

- Above, $d(i) \triangleq [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.

- $d(i)$ is the PV of one dollar $i$ periods from now.

- This formula—now just a definition—will be justified on p. 220.

- The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.

- Compute $S(2)$ from the two-period coupon bond price $P$ by solving
  \[
P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.
\]
Extracting Spot Rates from Yield Curve (concluded)

• Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n - 1)$.

• Then $S(n)$ can be computed from Eq. (18) on p. 127, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$  

• The running time can be made to be $O(n)$ (see text).

• The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.$^a$

$^a$Often without economic justifications.
Which One (from P. 121)?
Yield Spread

- Consider a risky bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.
Static Spread

- Were the risky bond riskless, it would fetch
  \[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift *in parallel* to price the risky bond:
  \[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

- Unlike the yield spread, the static spread explicitly incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

• $y_k$: yield to maturity for the $k$-period coupon bond.
• $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
• $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
• $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
• $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

• If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

• The yield curve contains information regarding future interest rates currently “expected” by the market.

• Invest $1 for $j$ periods to end up with $[1 + S(j)]^j$ dollars at time $j$.
  – The maturity strategy.

• Invest $1$ in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.

• Will have $[1 + S(i)]^i[1 + S(i, j)]^{j-i}$ dollars at time $j$.
  – $S(i, j)$: $(j - i)$-period spot rate $i$ periods from now.
  – The rollover strategy.
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$

we will end up with $[1 + S(j)]^j$ dollars again.

- By definition, $f(0, j) = S(j)$.

- $f(i, j)$ are called the (implied) forward rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

- $f(0, 1)$
- $f(1, 2)$
- $f(2, 3)$
- $f(3, 4)$

Time 0

- $S(1)$
- $S(2)$
- $S(3)$
- $S(4)$
Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.  
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

\[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

\[ f(i, j) < S(j) < \cdots < S(i). \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)]. \]
Forward Rates ≡ Spot Rates ≡ Yield Curves (concluded)

• Since they are identical,

\[ S(n) = \left\{ [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)] \right\}^{1/n} - 1. \quad (20) \]

• Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

• Other equivalencies can be derived similarly, such as

\[ f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \quad (21) \]
Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ $m$-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time $n$ there will be a cash inflow of $1$.
- At time $m$ there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
Locking in the Forward Rate \( f(n, m) \) (concluded)

- This implies the interest rate between times \( n \) and \( m \) equals \( f(n, m) \) by Eq. (19) on p. 138.

\[
(1 + S(m))^m / (1 + S(n))^n
\]
Forward Loans

- We had generated the cash flow of a type of forward contract called the forward loan.

- Agreed upon today, it enables one to
  - Borrow money at time $n$ in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.

- Can the spot rate curve be an arbitrary curve?\(^a\)

\(^a\)Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.
Synthetic Bonds

• We had seen that

\[
\text{forward loan} = n\text{-period zero} - [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• Thus

\[
\text{n-period zero} = \text{forward loan} + [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• We have created a \textit{synthetic} zero-coupon bond with forward loans and other zero-coupon bonds.

• Very useful if the \textit{n-period} zero is unavailable or illiquid.
Spot and Forward Rates under Continuous Compounding

- The pricing formula:
  \[ P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}. \]

- The market discount function:
  \[ d(n) = e^{-nS(n)}. \]

- The spot rate is an arithmetic average of forward rates,\(^a\)
  \[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}. \]

\(^a\)Compare it with Eq. (20) on p. 144.
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

\[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]  \hspace{1cm} (22)

- Compare the above formula with Eq. (19) on p. 138.

- The one-period forward rate:\(^{a}\)

\[ f(j, j + 1) = - \ln \frac{d(j + 1)}{d(j)}. \]

\(^{a}\)Compare it with Eq. (21) on p. 144.
Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

\[
 f(T) \overset{\Delta}{=} \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}.
\]  

(23)

- So \( f(T) > S(T) \) if and only if \( \frac{\partial S}{\partial T} > 0 \) (i.e., a normal spot rate curve).

- If \( S(T) < -T(\partial S/\partial T) \), then \( f(T) < 0 \).\(^a\)

\(^a\)Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
An Example

• Let the interest rates be continuously compounded.

• Suppose the spot rate curve is\(^a\)

\[
S(T) \equiv 0.08 - 0.05 e^{-0.18T}.
\]

• Then by Eq. (23) on p. 151, the forward rate curve is

\[
f(T) = S(T) + TS'(T)
\]

\[
= 0.08 - 0.05 e^{-0.18T} + 0.009T e^{-0.18T}.
\]

\(^a\)Hull & White (1994).
Unbiased Expectations Theory

- Forward rate equals the average future spot rate,
  
  \[ f(a, b) = E[S(a, b)]. \]  
  \( (24) \)

- It does not imply that the forward rate is an accurate predictor for the future spot rate.

- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on average.
Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - \( f(j, j + 1) > S(j + 1) \) if and only if \( S(j + 1) > S(j) \) from Eq. (19) on p. 138.
  - So \( E[S(j, j + 1)] > S(j + 1) > \cdots > S(1) \) if and only if \( S(j + 1) > \cdots > S(1) \).

- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
A “Bad” Expectations Theory

• The expected returns\(^a\) on all possible riskless bond strategies are equal for \textit{all} holding periods.

• So

\[
(1 + S(2))^2 = (1 + S(1)) E[1 + S(1, 2)] \tag{25}
\]

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

• After rearrangement,

\[
\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]

\(^a\)More precisely, the one-plus returns.
A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  - The expected return is
    \[
    E[(1 + S(1, 2))^{-1}] / (1 + S(2))^{-2}.
    \]
  - Strategy two buys a one-period bond with a return of \(1 + S(1)\).
A “Bad” Expectations Theory (continued)

- The theory says the returns are equal:

\[
\frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
\]

- Combine this with Eq. (25) on p. 155 to obtain

\[
E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.
\]
A “Bad” Expectations Theory (concluded)

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that $E[g(X)] > g(E[X])$
    for any nondegenerate random variable $X$ and
    strictly convex function $g$ (i.e., $g''(x) > 0$).
  – Use
    $$g(x) \triangleq (1 + x)^{-1}$$
    to prove our point.
Local Expectations Theory

• The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1, n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \quad \text{for all } n > 1.
\]

• This theory is the basis of many interest rate models.