Set Things in Motion (concluded)

• The term structure of (yield) volatilities\(^a\) can be estimated from:
  – Historical data (historical volatility).
  – Or interest rate option prices such as cap prices (implied volatility).

• The binomial tree should be found that is consistent with both term structures.

• Here we focus on the term structure of interest rates.

\(^a\)Or simply the volatility (term) structure.
Model Term Structures

- The model price is computed by backward induction.
- Refer back to the figure on p. 1004.
- Given that the values at nodes B and C are $P_B$ and $P_C$, respectively, the value at node A is then
  \[
  \frac{P_B + P_C}{2(1 + r)} + \text{cash flow at node A}.
  \]
- We compute the values column by column (see next page).
- This takes $O(n^2)$ time and $O(n)$ space.
Cash flows:
Term Structure Dynamics

- An \( n \)-period zero-coupon bond’s price can be computed by assigning $1 to every node at period \( n \) and then applying backward induction.
- Repeating this step for \( n = 1, 2, \ldots \), one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics.
  - Taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.
Sample Term Structure

• We shall construct interest rate trees consistent with the sample term structure in the following table.
  – This is calibration (the reverse of pricing).

• Assume the short rate volatility is such that

\[ v \equiv \frac{r_h}{r_\ell} = 1.5, \]

independent of time.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate (%)</td>
<td>4</td>
<td>4.2</td>
<td>4.3</td>
</tr>
<tr>
<td>One-period forward rate (%)</td>
<td>4</td>
<td>4.4</td>
<td>4.5</td>
</tr>
<tr>
<td>Discount factor</td>
<td>0.96154</td>
<td>0.92101</td>
<td>0.88135</td>
</tr>
</tbody>
</table>
An Approximate Calibration Scheme

- Start with the implied one-period forward rates.
- Equate the expected short rate with the forward rate.\textsuperscript{a}
- For the first period, the forward rate is today’s one-period spot rate.
- In general, let $f_j$ denote the forward rate in period $j$.
- This forward rate can be derived from the market discount function via\textsuperscript{b}

\[ f_j = \frac{d(j)}{d(j + 1)} - 1. \]

\textsuperscript{a}See Exercise 5.6.6 in text.
\textsuperscript{b}See Exercise 5.6.3 in text.
An Approximate Calibration Scheme (continued)

• Since the \( i \)th short rate \( r_j v_j^{i-1} \), \( 1 \leq i \leq j \), occurs with probability \( 2^{-(j-1)} \binom{j-1}{i-1} \), this means

\[
\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.
\]

• Thus

\[
r_j = \left( \frac{2}{1 + v_j} \right)^{j-1} f_j. \tag{133}
\]

• This binomial interest rate tree is trivial to set up (implicitly), in \( O(n) \) time.
An Approximate Calibration Scheme (continued)

• The ensuing tree for the sample term structure appears in figure next page.

• For example, the price of the zero-coupon bond paying $1 at the end of the third period is

\[
\frac{1}{4} \times \frac{1}{1.04} \times \left( \frac{1}{1.0352} \times \left( \frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left( \frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)
\]

or 0.88155, which exceeds discount factor 0.88135.

• The tree is thus not calibrated.
Baseline rates:

A 4.0%
B 3.52%
C 2.88%
D

Implied forward rates:

4.0% 4.4% 4.5%

period 1 period 2 period 3
An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree *overprices* the bonds.\(^a\)
- Suppose we replace the baseline rates \(r_j\) by \(r_j v_j\).
- Then the resulting tree *underprices* the bonds.\(^b\)
- The true baseline rates are thus bounded between \(r_j\) and \(r_j v_j\).

---

\(^a\)See Exercise 23.2.4 in text.
Issues in Calibration

• The model prices generated by the binomial interest rate tree should match the observed market prices.

• Perhaps the most crucial aspect of model building.

• Treat the backward induction for the model price of the $m$-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate $r_m$ for period $m$.

• A root-finding method is applied to solve $f(r_m) = P$ for $r_m$ given the zero’s price $P$ and $r_1, r_2, \ldots, r_{m-1}$.

• This procedure is carried out for $m = 1, 2, \ldots, n$.

• It runs in $O(n^3)$ time.
Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.\(^a\)

- The scheme records how much $1 at a node contributes to the model price.

- This number is called the state price (p. 209), the Arrow-Debreu price, or Green’s function.
  - It is the price of a state contingent claim that pays $1 at that particular node (state) and 0 elsewhere.

- The column of state prices will be established by moving forward from time 0 to time $n$.

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time $j$ and there are $j + 1$ nodes.
  - The unknown baseline rate for period $j$ is $r \triangleq r_j$.
  - The multiplicative ratio is $v \triangleq v_j$.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at earlier time $j - 1$.
  - They have rates $r, rv, \ldots, rv^{j-1}$ for period $j$.\(^a\)

- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j - 1)$-period zero-coupon bond.

- We want to find $r$ based on $P_1, P_2, \ldots, P_j$ and the price of the $j$-period zero-coupon bond.

\(^a\)Recall p. 1009, repeated on next page.
Binomial Interest Rate Tree Calibration (continued)

Baseline rates

$A \rightarrow B \rightarrow C \rightarrow D$

$\begin{align*}
A & : r_1 \\
B & : r_2 v_2 \\
C & : r_3 v_3 \\
D & : r_3 v_3^2
\end{align*}$
Binomial Interest Rate Tree Calibration (continued)

- One dollar at time $j$ has a known market value of $1/[1 + S(j)]^j$, where $S(j)$ is the $j$-period spot rate.

- Alternatively, this dollar has a present value of

$$g(r) \triangleq \frac{P_1}{1 + r} + \frac{P_2}{(1 + rv)} + \frac{P_3}{(1 + rv^2)} + \cdots + \frac{P_j}{(1 + rv^{j-1})}$$

(see next plot).

- So we solve

$$g(r) = \frac{1}{[1 + S(j)]^j} \quad (134)$$

for $r$. 

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Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for $r$ is guaranteed.
- The state prices at time $j$ can now be calculated (see panel (a) next page).
- We call a tree with these state prices a binomial state price tree (see panel (b) next page).
- The calibrated tree is depicted on p. 1031.
Implied forward rates: 4.0%  4.4%  4.5%

period 1  period 2  period 3
Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the $r$ in Eq. (134) on p. 1027 as $g'(r)$ is easy to evaluate.

- The monotonicity and the convexity of $g(r)$ also facilitate root finding.

- The total running time is $O(n^2)$, as each root-finding routine consumes $O(j)$ time.

- With a good initial guess,\(^a\) the Newton-Raphson method converges in only a few steps.\(^b\)

---

\(^a\)Such as $r_j = \left(\frac{2}{1+v_j}\right)^{j-1} f_j$ on p. 1019.

\(^b\)Lyuu (1999).
A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, $r_2$, satisfies

$$\frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.$$ 

- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in panel (b) on p. 1030 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.
A Numerical Example (concluded)

• The baseline rate for the third period, $r_3$, satisfies

\[
\frac{0.232197}{1 + r_3} + \frac{0.460505}{1 + 1.5 \times r_3} + \frac{0.228308}{1 + (1.5)^2 \times r_3} = 0.88135.
\]

• The result is $r_3 = 2.895\%$.

• Now, redo the calculation on p. 1020 using the new rates:

\[
\frac{1}{4} \times \frac{1}{1.04} \times \left[ \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) \right] + \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right),
\]

which equals 0.88135, an exact match.

• The tree on p. 1031 prices without bias the benchmark securities.
Spread of Nonbenchmark Bonds

- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.
Spread of Nonbenchmark Bonds (continued)

• We illustrate the idea with an example.

• Start with the tree on p. 1037.

• Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$.

• Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895 + s} + \frac{C_3}{1.04343 + s} \right) \right) \right] + \frac{1}{2} \times \frac{1}{1.05289 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343 + s} + \frac{C_3}{1.06514 + s} \right) \right).$$

• Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$. 
Implied forward rates: 4.0%  4.4%  4.5%

<table>
<thead>
<tr>
<th>Period</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.00%+s</td>
</tr>
<tr>
<td>2</td>
<td>5.289%+s</td>
</tr>
<tr>
<td>3</td>
<td>6.514%+s</td>
</tr>
<tr>
<td>4</td>
<td>2.895%+s</td>
</tr>
<tr>
<td>5</td>
<td>3.526%+s</td>
</tr>
<tr>
<td>6</td>
<td>4.343%+s</td>
</tr>
</tbody>
</table>
Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of $s$.

- We will employ the Newton-Raphson root-finding method to solve

$$p(s) - P = 0$$

for $s$.

- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.

- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.
Spread of Nonbenchmark Bonds (continued)

• Consider an arbitrary node A in the tree associated with the short rate $r$.

• In the process of computing the model price $p(s)$, a price $p_A(s)$ is computed at A.

• Prices computed at A’s two successor nodes B and C are discounted by $r + s$ to obtain $p_A(s)$ as follows,

$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)},$$

where $c$ denotes the cash flow at A.
Spread of Nonbenchmark Bonds (continued)

• To compute $p'_A(s)$ as well, node A calculates

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}.$$  \hfill (135)

• This is easy if $p'_B(s)$ and $p'_C(s)$ are also computed at nodes B and C.

• When A is a terminal node, simply use the payoff function for $p_A(s)$.

\footnotesize\textsuperscript{a} Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.
\[
p_A(s) = c + \frac{p_b(s) + p_c(s)}{2(1+r+s)}
\]

\[
p_A'(s) = \frac{p_b'(s) + p_c'(s)}{2(1+r+s)} - \frac{p_b(s) + p_c(s)}{2(1+r+s)^2}
\]
Spread of Nonbenchmark Bonds (continued)

• Apply the above procedure inductively to yield \( p(s) \) and \( p'(s) \) at the root (p. 1041).

• This is called the differential tree method.\(^a\)
  
  – Similar ideas can be found in automatic differentiation (AD)\(^b\) and backpropagation\(^c\) in artificial neural networks.

• The total running time is \( O(n^2) \).

• The memory requirement is \( O(n) \).

\(^a\) Lyuu (1999).
\(^b\) Rall (1981).
\(^c\) Werbos (1974); Rumelhart, Hinton, & Williams (1986).
Spread of Nonbenchmark Bonds (continued)

<table>
<thead>
<tr>
<th>Number of partitions $n$</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>7.850</td>
<td>5</td>
<td>10500</td>
<td>3503.410</td>
<td>5</td>
</tr>
<tr>
<td>1500</td>
<td>71.650</td>
<td>5</td>
<td>11500</td>
<td>4169.570</td>
<td>5</td>
</tr>
<tr>
<td>2500</td>
<td>198.770</td>
<td>5</td>
<td>12500</td>
<td>4912.680</td>
<td>5</td>
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<tr>
<td>3500</td>
<td>387.460</td>
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<td>13500</td>
<td>5714.440</td>
<td>5</td>
</tr>
<tr>
<td>4500</td>
<td>641.400</td>
<td>5</td>
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<td>6589.360</td>
<td>5</td>
</tr>
<tr>
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<td>951.800</td>
<td>5</td>
<td>15500</td>
<td>7548.760</td>
<td>5</td>
</tr>
<tr>
<td>6500</td>
<td>1327.900</td>
<td>5</td>
<td>16500</td>
<td>8502.950</td>
<td>5</td>
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<tr>
<td>7500</td>
<td>1761.110</td>
<td>5</td>
<td>17500</td>
<td>9523.900</td>
<td>5</td>
</tr>
<tr>
<td>8500</td>
<td>2269.750</td>
<td>5</td>
<td>18500</td>
<td>10617.370</td>
<td>5</td>
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<tr>
<td>9500</td>
<td>2834.170</td>
<td>5</td>
<td>9500</td>
<td>3503.410</td>
<td>5</td>
</tr>
</tbody>
</table>

75MHz Sun SPARCstation 20.
Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 1045).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 133) and static spread (p. 134) of the nonbenchmark bond over an otherwise identical benchmark bond.
Cash flows:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>105</td>
<td></td>
</tr>
</tbody>
</table>

A: 4.50% 100.569
B: 4.026% 106.754
C: 5.789% 103.436
D: 3.395% 106.552
C: 4.843% 105.150
B: 7.014% 103.118
D: 105
More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)\textsuperscript{a}

<table>
<thead>
<tr>
<th>Number of partitions</th>
<th>Running time</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.008210</td>
<td>2</td>
<td>100</td>
<td>0.013845</td>
<td>3</td>
</tr>
<tr>
<td>200</td>
<td>0.033310</td>
<td>2</td>
<td>200</td>
<td>0.036335</td>
<td>3</td>
</tr>
<tr>
<td>300</td>
<td>0.072940</td>
<td>2</td>
<td>300</td>
<td>0.120455</td>
<td>3</td>
</tr>
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<td>400</td>
<td>0.129180</td>
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<td>0.214100</td>
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<td>500</td>
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<td>600</td>
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<td>600</td>
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<td>800</td>
<td>0.522040</td>
<td>2</td>
<td>800</td>
<td>0.569605</td>
<td>2</td>
</tr>
</tbody>
</table>

Intel 166MHz Pentium, running on Microsoft Windows 95.

\textsuperscript{a}Lyuu (1999).
Fixed-Income Options

- Consider a 2-year 99 European call on the 3-year, 5% Treasury.

- Assume the Treasury pays annual interest.

- From p. 1048 the 3-year Treasury’s price minus the $5 interest at year 2 could be $102.046, $100.630, or $98.579 two years from now.
  - The accrued interest is not included as it belongs to the original bondholder.

- Now compare the strike price against the bond prices.

- The call is in the money in the first two scenarios out of the money in the third.
Fixed-Income Options (continued)

- The option value is calculated to be $1.458 on p. 1048(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth $98.579 without the accrued interest.
- The option value is computed to be $0.096 on p. 1048(b).
Fixed-Income Options (concluded)

• The present value of the strike price is
  
  \[ PV(X) = 99 \times 0.92101 = 91.18. \]

• The Treasury is worth \( B = 101.955 \). 

• The present value of the interest payments during the life of the options is\(^a\)
  
  \[ PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275. \]

• The call and the put are worth \( C = 1.458 \) and \( P = 0.096 \), respectively.

• Hence the put-call parity is preserved:
  
  \[ C = P + B - PV(I) - PV(X). \]

\(^a\)There is no coupon today.
Delta or Hedge Ratio

• How much does the option price change in response to changes in the price of the underlying bond?

• This relation is called delta (or hedge ratio) defined as

$$\frac{O_h - O_\ell}{P_h - P_\ell}.$$ 

• In the above $P_h$ and $P_\ell$ denote the bond prices if the short rate moves up and down, respectively.

• Similarly, $O_h$ and $O_\ell$ denote the option values if the short rate moves up and down, respectively.
Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 1048 as examples.
- Their deltas are

\[
\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441,
\]
\[
\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,
\]

respectively.
Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an \( n \)-period zero-coupon bond.
- First find its yield to maturity \( y_h \) (\( y_\ell \), respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

\[
\frac{1}{2} \ln \left( \frac{y_h}{y_\ell} \right). \quad (136)
\]
Volatility Term Structures (continued)

- For example, take the tree on p. 1031 (repeated on next page).

- The two-year zero’s yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.

- Its yield volatility is therefore

\[
\frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.
\]

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Volatility Term Structures (continued)

Implied forward rates: 4.0% 4.4% 4.5%

<table>
<thead>
<tr>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0%</td>
<td>4.4%</td>
<td>4.5%</td>
</tr>
</tbody>
</table>
Volatility Term Structures (continued)

• Consider the three-year zero-coupon bond.

• If the short rate rises, the price of the zero one year from now will be

\[
\frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.
\]

• Thus its yield is \( \sqrt{\frac{1}{0.90096}} - 1 = 0.053531 \).

• If the short rate declines, the price of the zero one year from now will be

\[
\frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.
\]
Volatility Term Structures (continued)

• Thus its yield is \( \sqrt{\frac{1}{0.93225}} - 1 = 0.0357 \).

• The yield volatility is hence

\[
\frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\% ,
\]

slightly less than the one-year yield volatility.

• This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.\(^a\)

• The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

\(^a\)The relation is reversed for *price* volatilities (duration).
Spot rate volatility

Short rate volatility given a flat %10 volatility structure.
Volatility Term Structures (concluded)

- We started with $v_i$ and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The $v_i$—hence the short rate volatilities via Eq. (131) on p. 1008—and the $r_i$ are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.\textsuperscript{a}

\textsuperscript{a}Black, Derman, & Toy (1990).
Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

— Roger Lowenstein,

*When Genius Failed* (2000)
[The] fixed-income traders I knew seemed smarter than the equity trader […] there’s no competitive edge to being smart in the equities business.[.] — Emanuel Derman, *My Life as a Quant* (2004)

Bond market terminology was designed less to convey meaning than to bewilder outsiders. — Michael Lewis, *The Big Short* (2011)
Terminology

• A period denotes a unit of elapsed time.
  – Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one — unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

$t$: a point in time.

$r(t)$: the one-period riskless rate prevailing at time $t$ for repayment one period later.$^a$

$P(t,T)$: the present value at time $t$ of one dollar at time $T$.

$^a$Alternatively, the instantaneous spot rate, or short rate, at time $t$. 
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period.$^a$

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

$^a$In other words, the $(T - t)$-period spot rate at time $t$. 
Standard Notations (concluded)

$f(t,T,L)$: the $L$-period forward rate at time $T$ implied at time $t$ stated on a per-period basis and compounded once per period.

$f(t,T)$: the one-period or instantaneous forward rate at time $T$ as seen at time $t$ stated on a per period basis and compounded once per period.

- It is $f(t,T,1)$ in the discrete-time model and $f(t,T,dt)$ in the continuous-time model.
- Note that $f(t,t)$ equals the short rate $r(t)$. 
Fundamental Relations

• The price of a zero-coupon bond equals

\[ P(t, T) = \begin{cases} 
(1 + r(t, T))^{-(T-t)} , & \text{in discrete time}, \\
e^{-r(t,T)(T-t)} , & \text{in continuous time}.
\end{cases} \] (137)

• \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

• By definition,

\[ f(t, t) = \begin{cases} 
r(t, t + 1) , & \text{in discrete time}, \\
r(t, t) , & \text{in continuous time}.
\end{cases} \]
Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \]  (138)

- The forward price equals the future value at time \( T \) of the underlying asset.\(^a\)

- Equation (138) holds whether the model is discrete-time or continuous-time.

\(^a\)See Exercise 24.2.1 of the textbook for proof.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by
  \[
  f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1
  \]
  (139)
  in discrete time.

- The analog to Eq. (139) under simple compounding is
  \[
  f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right).
  \]
Fundamental Relations (continued)

• In continuous time,

\[
 f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L}
\]

(140)

by Eq. (138) on p. 1068.

• Furthermore,

\[
 f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t, T)}{\partial T}
\]

\[
 = -\frac{\partial P(t, T)/\partial T}{P(t, T)}. 
\]
Fundamental Relations (continued)

• So

\[ f(t, T) \triangleq -\frac{\partial \ln P(t, T)}{\partial T} = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \]

(141)

• Because Eq. (141) is equivalent to

\[ P(t, T) = e^{-\int_t^T f(t, s) \, ds}, \]

(142)

the spot rate curve is

\[ r(t, T) = \frac{\int_t^T f(t, s) \, ds}{T - t}. \]
Fundamental Relations (concluded)

- The discrete analog to Eq. (142) is

\[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \]

- The short rate and the market discount function are related by

\[ r(t) = -\frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
Risk-Neutral Pricing

• Assume the local expectations theory.

• The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  
  – For all $t + 1 < T$,

  \[
  \frac{E_t[P(t + 1, T)]}{P(t, T)} = 1 + r(t). \tag{143}
  \]

  – Relation (143) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^a\)Theorem 16 on p. 554.
Risk-Neutral Pricing (continued)

• The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.

• Equation (143) on p. 1073 can also be expressed as

$$E_t[P(t + 1, T)] = F(t, t + 1, T).$$

- Verify that with, e.g., Eq. (138) on p. 1068.

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.\(^a\)

\(^a\)Under the local expectations theory. But the forward rate is not an unbiased estimator of the expected future short rate (p. 1022).
Risk-Neutral Pricing (continued)

- Rewrite Eq. (143) on p. 1073 as

\[
E_t^\pi \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = P(t, T).
\]  \hspace{1cm} (144)

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (concluded)

• Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = E_t^\pi \left[ \frac{E_{t+1}^\pi \left[ P(t + 2, T) \right]}{(1 + r(t))(1 + r(t + 1))} \right] = \cdots \\
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right].
\]
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \]  

(145)

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

• Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.

• For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.

• The fixed rate is $c$ per annum.

• The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.

• The payoff at time $t_{i+1}$ for the fixed-rate payer is $(f_i - c) \Delta t$. 
Interest Rate Swaps (continued)

\[(f_0 - c) \Delta t \]
\[(f_{n-1} - c) \Delta t \]
\[(f_1 - c) \Delta t \]

\(t \quad t_0 \quad t_1 \quad t_2 \quad t_n\)
Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence \( f_i \) satisfies
  \[
P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.
  \]
- If \( t < t_0 \), we have a forward interest rate swap.
- The ordinary swap corresponds to \( t = t_0 \).
Interest Rate Swaps (continued)

• The value of the swap at time $t$ is thus

$$\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t_i} r(s) \, ds} \left( f_{i-1} - c \right) \Delta t \right]$$

$$= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t_i} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c \Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t_i} r(s) \, ds} \left( e^{\int_{t_{i-1}}^{t_i} r(s) \, ds} - (1 + c \Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c \Delta t) \times P(t, t_i) \right]$$

$$= P(t, t_0) - P(t, t_n) - c \Delta t \sum_{i=1}^{n} P(t, t_i).$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present-value calculations.
Swap Rate

• The swap rate, which gives the swap zero value, equals

\[ S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \] (146)

• The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

• For an ordinary swap, \( P(t, t_0) = 1 \).

• The swap rate is called a forward swap rate if \( t_0 > t \).