Implementing Delta Hedge

• We want to hedge $N$ short derivatives.
• Assume the stock pays no dividends.
• The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus $B$ borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$  

• At next rebalancing point when the delta is $\Delta'$, buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares.
• Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.
Example

- A hedger is short 10,000 European calls.
- $S = 50$, $\sigma = 30\%$, and $r = 6\%$.
- This call’s expiration is four weeks away, its strike price is $50$, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1,000,000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading stock is close to the call premium’s FV.\(^a\)

\(^a\)This takes the replication viewpoint: One starts with zero dollar.
Example (continued)

- As $\Delta = 0.538560$
  
  $$N \times \Delta = 538,560$$

  shares are purchased for a total cost of
  
  $$538,560 \times 50 = 26,928,000$$

dollars to make the portfolio delta-neutral.

- The trader finances the purchase by borrowing
  
  $$B = N \times \Delta \times S - N \times f = 25,160,090$$

dollars net.a

  aThis takes the hedging viewpoint: One starts with the option premium. See Exercise 16.3.2 of the text.
Example (continued)

- At 3 weeks to expiration, the stock price rises to $51.
- The new call value is \( f' = 2.10580 \).
- So before rebalancing, the portfolio is worth

\[
-N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622.
\]  
(91)
Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.

- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

- In fact, the tracking error over one rebalancing act is positive about 68% of the time, but its expected value is essentially zero.\(^a\)

- The tracking error at maturity is proportional to vega.\(^b\)

---

\(^a\) Boyle & Emanuel (1980).
\(^b\) Kamal & Derman (1999).
Example (continued)

• In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.

• With a higher delta $\Delta' = 0.640355$, the trader buys

$$N \times (\Delta' - \Delta) = 101,795$$

shares for $5,191,545$.

• The number of shares is increased to $N \times \Delta' = 640,355$. 
Example (continued)

- The cumulative cost is\(^{a}\)
  \[
  26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634. 
  \]

- The portfolio is again delta-neutral.

\(^{a}\)We take the replication viewpoint here.
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<th>$\Delta$</th>
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</tbody>
</table>

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).
Example (continued)

- At expiration, the trader has 1,000,000 shares.

- They are exercised against by the in-the-money calls for $50,000,000.

- The trader is left with an obligation of

  \[ 51,524,853 - 50,000,000 = 1,524,853, \]

  which represents the replication cost.

- That means if we started with $1,524,853, we could replicate 10,000 such calls in this scenario.
Example (concluded)

• The FV of the call premium equals

\[ 1,767,910 \times e^{0.06 \times \frac{4}{52}} = 1,776,088. \]

• That means the net gain will be

\[ 1,776,088 - 1,524,853 = 251,235 \]

if we are hedging 10,000 European calls.
Tracking Error Revisited

- Define the dollar gamma as $S^2 \Gamma$.
- The change in value of a delta-hedged *long* option position after a duration of $\Delta t$ is proportional to the dollar gamma.
- It is about

$$\frac{1}{2} S^2 \Gamma [ (\Delta S/S)^2 - \sigma^2 \Delta t ] .$$

- $(\Delta S/S)^2$ is called the daily realized variance.
Tracking Error Revisited (continued)

- In our particular case,

\[ S = 50, \Gamma = 0.0957074, \Delta S = 1, \sigma = 0.3, \Delta t = 1/52. \]

- The estimated tracking error is

\[ -\frac{1}{2} \times 50^2 \times 0.0957074 \times \left[ \frac{1}{50}^2 - \frac{0.09}{52} \right] = 159,205. \]

- It is very close to our earlier number of 171,622.a

- Delta hedge is also called gamma scalping.b

---

a Recall Eq. (91) on p. 684.
b Bennett (2014).
Tracking Error Revisited (continued)

• Let the rebalancing times be $t_1, t_2, \ldots, t_n$.

• Let $\Delta S_i = S_{i+1} - S_i$.

• The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[ \left( \frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].$$

• The tracking error is path dependent.

• It is also known that\(^a\)

$$\sum_{i=0}^{n-1} \left( \frac{\Delta S_i}{S_i} \right)^2 \to \sigma^2 T.$$

\(^a\)Protter (2005).
Tracking Error Revisited (concluded)\textsuperscript{a}

- The tracking error\textsuperscript{b} $\epsilon_n$ over $n$ rebalancing acts has about the same probability of being positive as being negative.

- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\epsilon_n^2]}$ is $O(1/\sqrt{n})$.\textsuperscript{c}

- The root-mean-square tracking error increases with $\sigma$ at first and then decreases.

\textsuperscript{a}Bertsimas, Kogan, & Lo (2000).
\textsuperscript{b}Such as 251,235 on p. 690.
\textsuperscript{c}Grannan & Swindle (1996).
Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, $\Delta f$, due to changes in the stock price, $\Delta S$.

- When $\Delta S$ is not small, the second-order term, gamma $\Gamma \overset{\Delta}{=} \frac{\partial^2 f}{\partial S^2}$, helps (theoretically).

- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.

- To meet this extra condition, one more security needs to be brought in.
Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call $f_2$ is brought in.
- To set up a delta-gamma hedge, we solve

\[
-N \times f + n_1 \times S + n_2 \times f_2 - B = 0 \quad \text{(self-financing)},
\]

\[
-N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 = 0 \quad \text{(delta neutrality)},
\]

\[
-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 = 0 \quad \text{(gamma neutrality)},
\]

for $n_1$, $n_2$, and $B$.

- The gammas of the stock and bond are 0.
- See the numerical example on pp. 231–232 of the text.
Other Hedges

- If volatility changes, delta-gamma hedge may not work well.

- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.

- To accomplish this, one more security has to be brought into the process.

- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

All those holes and pebbles.
Who could count them?
— James Joyce, Ulysses (1922)

And though the holes were rather small,
they had to count them all.
— The Beatles, A Day in the Life (1967)
The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 279.
- We will now apply it to price barrier options.
The Reflection Principle\textsuperscript{a}

- Imagine a particle at position \((0, -a)\) on the integral lattice that is to reach \((n, -b)\).
- Without loss of generality, assume \(a > 0\) and \(b \geq 0\).
- This particle’s movement:
  \[
  \begin{array}{c}
  (i, j) \\
  (i + 1, j + 1) \text{ up move } S \rightarrow Su \\
  (i + 1, j - 1) \text{ down move } S \rightarrow Sd
  \end{array}
  \]
- How many paths touch the \(x\) axis?

\textsuperscript{a}André (1887).
The Reflection Principle (continued)

- For a path from $(0, -a)$ to $(n, -b)$ that touches the $x$ axis, let $J$ denote the first point this happens.
- Reflect the portion of the path from $(0, -a)$ to $J$.
- A path from $(0, a)$ to $(n, -b)$ is constructed.
- It also hits the $x$ axis at $J$ for the first time.
- The one-to-one mapping shows the number of paths from $(0, -a)$ to $(n, -b)$ that touch the $x$ axis equals the number of paths from $(0, a)$ to $(n, -b)$.
The Reflection Principle (concluded)

- A path of this kind has \((n + b + a)/2\) down moves and \((n - b - a)/2\) up moves.\(^a\)

- Hence there are

\[
\binom{n}{\frac{n+a+b}{2}} = \binom{n}{\frac{n-a-b}{2}}
\]  

such paths for even \(n + a + b\).

- Convention: \(\binom{n}{k} = 0\) for \(k < 0\) or \(k > n\).

\(^a\)Verify it!
Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define
  \[
  a \triangleq \left\lfloor \frac{\ln(X/(Sd^n))}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor,
  \]
  \[
  h \triangleq \left\lceil \frac{\ln(H/(Sd^n))}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil.
  \]

- $a$ is such that $\tilde{X} \triangleq Su^a d^{n-a}$ is the terminal price that is closest to $X$ from above.
- $h$ is such that $\tilde{H} \triangleq Su^h d^{n-h}$ is the terminal price that is closest to $H$ from below.\textsuperscript{a}

\textsuperscript{a}So we underestimate the price.
Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with $n$ moves hence ends up in the money if and only if the number of up moves is at least $a$.
- The price $S u^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price $S d^n$ on the binomial tree.

\[
S u^k d^{n-k} = S d^{-k} d^{n-k} = S d^{n-2k}. \tag{93}
\]
\[ S u^j d^{m-j} \]
\[ \tilde{X} = S u^a d^{m-a} \]
\[ \tilde{H} = S u^h d^{m-h} \]
Pricing Barrier Options (continued)

• A path from $S$ to the terminal price $Su^j d^{n-j}$ has probability $p^j(1-p)^{n-j}$ of being taken.

• With reference to p. 707, the reflection principle (p. 702) can be applied with

$$a = n - 2h,$$
$$b = 2j - 2h,$$

in Eq. (92) on p. 704 by treating the $\tilde{H}$ line as the $x$ axis.
Pricing Barrier Options (continued)

- Therefore,

\[
\binom{n}{n + (n - 2h) + (2j - 2h) \over 2} = \binom{n}{n - 2h + j}
\]

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

- The terminal price $S u^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

\[
\binom{n}{n - 2h + j} p^j (1 - p)^{n-j}, \quad j \leq 2h.
\]
Pricing Barrier Options (concluded)

- The option value equals

\[
\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1 - p)^{n-j} \frac{(Su^j d^{n-j} - X)}{R^n}.
\]  

\[R \triangleq e^{r\tau/n}\] is the riskless return per period.

- It yields a linear-time algorithm.\(^{a}\)

\(^{a}\)Lyuu (1998).
Convergence of BOPM

- Equation (94) results in the same sawtooth-like convergence shown on p. 400 (repeated on next page).
- The reasons are not hard to see.
- The true barrier $H$ most likely does not equal the effective barrier $\tilde{H}$. 
Convergence of BOPM (continued)

Down-and-in call value

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Convergence of BOPM (continued)

- Convergence is actually good if we limit $n$ to certain values—191, for example.

- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$$

for some integer $j$.

- The preferred $n$’s are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$
Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.
- However, the effective barrier above, $Sd^j$, corresponds to a terminal stock price only when $n - j$ is even.\(^a\)
- To close this gap, we decrement $n$ by one, if necessary, to make $n - j$ an even number.

\(^a\)This is because $j = n - 2k$ for some $k$ by Eq. (93) on p. 706. Of course we could have adopted the form $Sd^j$ ($-n \leq j \leq n$) for the effective barrier. It makes a good exercise.
Convergence of BOPM (concluded)

• The preferred $n$’s are now

$$n = \begin{cases} 
\ell, & \text{if } \ell - j \text{ is even}, \\
\ell - 1, & \text{otherwise,}
\end{cases}$$

$j = 1, 2, 3, \ldots$, where

$$\ell \triangleq \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

• Evaluate pricing formula (94) on p. 710 only with the $n$’s above.
Down-and-in call value

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Practical Implications

• This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.\(^a\)

• Now that barrier options can be efficiently priced, we can afford to pick very large $n$’s (p. 718).

• This has profound consequences.\(^b\)

\(^a\) J. Lin (R95221010) (2008); J. Lin (R95221010) & Palmer (2010).
\(^b\) See pp. 732ff.
\begin{tabular}{|c|c|c|}
\hline
\textit{n} & Combinatorial method & \\
& Value & Time (milliseconds) \\
\hline
21 & 5.507548 & 0.30 \\
84 & 5.597597 & 0.90 \\
191 & 5.635415 & 2.00 \\
342 & 5.655812 & 3.60 \\
533 & 5.652253 & 5.60 \\
768 & 5.654609 & 8.00 \\
1047 & 5.658622 & 11.10 \\
1368 & 5.659711 & 15.00 \\
1731 & 5.659416 & 19.40 \\
2138 & 5.660511 & 24.70 \\
2587 & 5.660592 & 30.20 \\
3078 & 5.660099 & 36.70 \\
3613 & 5.660498 & 43.70 \\
4190 & 5.660388 & 44.10 \\
4809 & 5.659955 & 51.60 \\
5472 & 5.660122 & 68.70 \\
6177 & 5.659981 & 76.70 \\
6926 & 5.660263 & 86.90 \\
7717 & 5.660272 & 97.20 \\
\hline
\end{tabular}
Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because
  \[ n \sim 1/\ln^2(S/H). \]
  - This is called the barrier-too-close problem.

- This observation is indeed true of standard quadratic-time binomial tree algorithms.

- But it no longer applies to linear-time algorithms (see p. 720).
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<th>Value</th>
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<th>$n$</th>
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</table>

(All times in milliseconds.)
Trinomial Tree

• Set up a trinomial approximation to the geometric Brownian motion\(^a\)

\[
\frac{dS}{S} = r \, dt + \sigma \, dW.
\]

• The three stock prices at time \(\Delta t\) are \(S\), \(Su\), and \(Sd\), where \(ud = 1\).

• Let the mean and variance of the stock price be \(SM\) and \(S^2V\), respectively.

\(^a\)Boyle (1988).
Trinomial Tree (continued)

• By Eqs. (30) on p. 179,

\[ M \triangleq e^{r \Delta t}, \]
\[ V \triangleq M^2(e^{\sigma^2 \Delta t} - 1). \]

• Impose the matching of mean and that of variance:

\[ 1 = p_u + p_m + p_d, \]
\[ SM = (p_u u + p_m + (p_d/u)) S, \]
\[ S^2V = p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2. \]
Trinomial Tree (continued)

• Use linear algebra to verify that

\[ p_u = \frac{u(V + M^2 - M) - (M - 1)}{(u - 1)(u^2 - 1)}, \]

\[ p_d = \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}. \]

– We must also make sure the probabilities lie between 0 and 1.
Trinomial Tree (concluded)

- There are countless variations.
- But all converge to the Black-Scholes option pricing model.\textsuperscript{a}
- The trinomial model has a linear-time algorithm for European options.\textsuperscript{b}

\textsuperscript{a}Madan, Milne, & Shefrin (1989).
\textsuperscript{b}T. Chen (\texttt{R94922003}) (2007).
A Trinomial Tree

- Use $u = e^{\lambda \sigma \sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

\[
pu \rightarrow \frac{1}{2\lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2\lambda\sigma},
\]

\[
pd \rightarrow \frac{1}{2\lambda^2} - \frac{(r - 2\sigma^2) \sqrt{\Delta t}}{2\lambda\sigma}.
\]

- A nice choice for $\lambda$ is $\sqrt{\pi/2}$.\(^a\)

\(^a\)Omberg (1988).
Barrier Options Revisited

• BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.

• The trinomial model solves the problem by adjusting $\lambda$ so that the barrier is hit exactly.$^a$

• When

$$S e^{-h\lambda \sigma \sqrt{\Delta t}} = H,$$

it takes $h$ down moves to go from $S$ to $H$, if $h$ is an integer.

• Then

$$h = \frac{\ln(S/H)}{\lambda \sigma \sqrt{\Delta t}}.$$

---

Barrier Options Revisited (continued)

• This is easy to achieve by adjusting $\lambda$.

• Typically, we find the smallest $\lambda \geq 1$ such that $h$ is an integer.$^a$
  
  – Such a $\lambda$ may not exist for very small $n$’s.
  
  – This is not hard to check.

• Toward that end, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ to be our $h$.

• Then let

$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

$^a$Why must $\lambda \geq 1$?
Barrier Options Revisited (continued)

- Alternatively, we can pick
  \[ h = \left\lfloor \frac{\ln(S/H)}{\sigma \sqrt{\Delta t}} \right\rfloor. \]

- Make sure \( h \geq 1 \).

- Then let
  \[ \lambda = \frac{\ln(S/H)}{h\sigma \sqrt{\Delta t}}. \]
Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.

- The following probabilities may be used,

\[ p_u = \frac{1}{2\lambda^2} + \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}, \]

\[ p_m = 1 - \frac{1}{\lambda^2}, \]

\[ p_d = \frac{1}{2\lambda^2} - \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}. \]

\[ - \mu' \Delta = r - (\sigma^2 / 2). \]
Down-and-in call value

0 50 100 150 200

#Periods
Algorithms Comparison\textsuperscript{a}

\begin{itemize}
\item So which algorithm is better, binomial or trinomial?
\item Algorithms are often compared based on the $n$ value at which they converge.
\begin{itemize}
\item The one with the smallest $n$ wins.
\end{itemize}
\item So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
\item Performance must be based on actual running times, not $n$.\textsuperscript{b}
\end{itemize}

\textsuperscript{a}Lyuu (1998).
\textsuperscript{b}Patterson & Hennessy (1994).
Algorithms Comparison (continued)

• Pages 712 and 731 seem to show the trinomial model converges at a smaller $n$ than BOPM.

• It is in this sense when people say trinomial models converge faster than binomial ones.

• But does it make the trinomial model better then?
Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 718.
- The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.\(^a\)
  - See pp. 745ff for an alternative solution.

\(^a\)Lyuu (1998).
<table>
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<tr>
<th>n</th>
<th>Combinatorial method</th>
<th>Trinomial tree algorithm</th>
</tr>
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<td>76.70</td>
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</table>

(All times in milliseconds.)
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
  - They make up “less than 5% of the light exotic market.”

- Assume $L < S < H$.

- The binomial model produces oscillating option values (see plot on next page).

---

\(^a\) Bennett (2014).

\(^b\) Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2005).
Double-Barrier Options (concluded)

- The combinatorial method yields a linear-time algorithm.\(^a\)
- This binomial model is \(O(1/\sqrt{n})\) convergent in general.\(^b\)
- If the barriers \(L\) and \(H\) depend on time, we have moving-barrier options.\(^c\)

\(^{a}\text{See p. 241 of the textbook.}\)
\(^{b}\text{Gobet (1999).}\)
\(^{c}\text{Rogers \& Zane (1998).}\)
Double-Barrier Knock-Out Options

• We knew how to pick the $\lambda$ so that one of the layers of the trinomial tree coincides with one barrier, say $H$.

• This choice, however, does not guarantee that the other barrier, $L$, is also hit.

• One way to handle this problem is to lower the layer of the tree just above $L$ to coincide with $L$.\textsuperscript{a}
  
  – More general ways to make the trinomial model hit both barriers are available.\textsuperscript{b}

\textsuperscript{a}Ritchken (1995); Hull (1999).

\textsuperscript{b}Hsu (R7526001, D89922012) & Lyuu (2006). Dai (B82506025, R86526008, D8852600) & Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 745ff).
Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above $L$ must be adjusted.

- Let $\ell$ be the positive integer such that
  
  \[ Sd^{\ell+1} < L < Sd^\ell. \]

- Hence the layer of the tree just above $L$ has price $Sd^\ell$.\(^a\)

\(^a\)You probably cannot do the same thing for binomial models (why?). Thanks to a lively discussion on April 25, 2012.
Double-Barrier Knock-Out Options (concluded)

• Define $\gamma > 1$ as the number satisfying

$$L = S d^{\ell-1} e^{-\gamma \lambda \sigma \sqrt{\Delta t}}.$$ 

– The prices between the barriers are (from low to high)

$L, S d^{\ell-1}, \ldots, S d^2, S d, S, S u, S u^2, \ldots, S u^{h-1}, S u^h = H.$

• The probabilities for the nodes with price equal to $S d^{\ell-1}$ are

$$p'_u = \frac{b + a \gamma}{1 + \gamma}, \quad p'_d = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \overset{\Delta}{=} \mu' \sqrt{\Delta t}/(\lambda \sigma)$ and $b \overset{\Delta}{=} 1/\lambda^2$. 
Convergence: Binomial vs. Trinomial
Ideas for Binomial Trees To Handle Two Barriers

\[
\begin{align*}
\ln(H) &= \ln(H) \\
\ln(H/L) &= \ln(H/L) \\
2\sigma \sqrt{\Delta t} &= 2\sigma \sqrt{\Delta t} \\
\Delta t &= \Delta t \\
T &= T
\end{align*}
\]
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.\textsuperscript{a}
- The resulting tree is called the binomial-trinomial tree.\textsuperscript{b}
- Suppose a binomial tree will be built with $\Delta t$ as the duration of one period.
- Node X at time $t$ needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - $\Delta t \leq \Delta t' < 2\Delta t$.

\textsuperscript{a}Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).
\textsuperscript{b}The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.
The Binomial-Trinomial Tree (continued)

\[ \Delta t' \]

\[ \Delta t \]

\[ \Delta t \]

\[ \hat{\mu} + 2\sigma \sqrt{\Delta t} \]

\[ \hat{\mu} \]

\[ 0 \]

\[ \hat{\mu} - 2\sigma \sqrt{\Delta t} \]

\[ \mu \]

\[ \beta \]

\[ \gamma \]

\[ \alpha \]

\[ p_u \]

\[ p_m \]

\[ p_d \]
The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.

- Let $S$ be the stock price at node $X$.

- Use $s(z)$ to denote the stock price at node $z$. 
The Binomial-Trinomial Tree (continued)

• Recall that the expected value of the logarithmic return
  \( \ln(S_{t+\Delta t'}/S) \) at time \( t + \Delta t' \) equals\(^\text{a}\)

  \[
  \mu \triangleq \left( r - \frac{\sigma^2}{2} \right) \Delta t'.
  \] (95)

• Its variance equals

  \[
  \text{Var} \triangleq \sigma^2 \Delta t'.
  \] (96)

• Let node B be the node whose logarithmic return
  \( \hat{\mu} \triangleq \ln(s(B)/S) \) is closest to \( \mu \) among all the nodes at
  time \( t + \Delta t' \).

\(^\text{a}\)See p. 294.
The Binomial-Trinomial Tree (continued)

• The middle branch from node X will end at node B.

• The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.

• Recall that adjacent nodes on the binomial tree are spaced at $2\sigma \sqrt{\Delta t}$ apart.

• Review the illustration on p. 746.
The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return \( \ln(S_{t+\Delta t'}/S) \).

- Recall that 
  \[ \hat{\mu} \overset{\Delta}{=} \ln(s(B)/S) \]
  is the logarithmic return of the middle node B.

- Let \( \alpha, \beta, \) and \( \gamma \) be the differences between \( \mu \) and the logarithmic returns 
  \[ \ln(s(Z)/S), \quad Z = A, B, C, \]
  in that order.
The Binomial-Trinomial Tree (continued)

- In other words,

\[ \alpha \triangleq \hat{\mu} + 2\sigma \sqrt{\Delta t} - \mu = \beta + 2\sigma \sqrt{\Delta t}, \quad (97) \]

\[ \beta \triangleq \hat{\mu} - \mu, \quad (98) \]

\[ \gamma \triangleq \hat{\mu} - 2\sigma \sqrt{\Delta t} - \mu = \beta - 2\sigma \sqrt{\Delta t}. \quad (99) \]

- The three branching probabilities \( p_u, p_m, p_d \) then satisfy

\[ p_u \alpha + p_m \beta + p_d \gamma = 0, \quad (100) \]

\[ p_u \alpha^2 + p_m \beta^2 + p_d \gamma^2 = \text{Var}, \quad (101) \]

\[ p_u + p_m + p_d = 1. \quad (102) \]
The Binomial-Trinomial Tree (concluded)

- Equation (100) matches the mean (95) of the logarithmic return $\ln(S_{t+\Delta} / S)$ on p. 748.
- Equation (101) matches its variance (96) on p. 748.
- The three probabilities can be proved to lie between 0 and 1 by Cramer’s rule.
Pricing Double-Barrier Options

• Consider a double-barrier option with two barriers \( L \) and \( H \), where \( L < S < H \).

• We need to make each barrier coincide with a layer of the binomial tree for better convergence.

• The idea is to choose a \( \Delta t \) such that

\[
\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}
\]

is a positive integer.

  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 754 is \( 2\sigma\sqrt{\Delta t} \).
Pricing Double-Barrier Options (continued)

\[
\begin{align*}
\ln(H/L) & \quad \ln(L/S) + 4\sigma \sqrt{\Delta t} \\
2\sigma \sqrt{\Delta t} & \quad \ln(L/S) + 2\sigma \sqrt{\Delta t} \\
& \quad 0 \\
& \quad \ln(L/S) \\
\end{align*}
\]
Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with \( \sim m \) periods.
- Suppose we pick \( \Delta \tau \overset{\Delta}{=} T/m \) for the length of each period.
- There is no guarantee that \( \frac{\ln(H/L)}{2\sigma\sqrt{\Delta \tau}} \) is an integer.
- So we pick a \( \Delta t \) that is close to, but does not exceed, \( \Delta \tau \) and makes \( \frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}} \) some integer \( \kappa \).
- Specifically, we select

\[
\Delta t = \left( \frac{\ln(H/L)}{2\kappa\sigma} \right)^2,
\]

where \( \kappa = \left\lfloor \frac{\ln(H/L)}{2\sigma\sqrt{\Delta \tau}} \right\rfloor \).
Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier \( L \) upward.
- Automatically, a layer coincides with the high barrier \( H \).
- It is unlikely that \( \Delta t \) divides \( T \), however.
- So the position at time 0 and with logarithmic return \( \ln(S/S) = 0 \) is not occupied by a binomial node to serve as the root node (recall p. 754).
Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.

- Between time 0 and time $T$, the binomial tree spans $\lfloor T/\Delta t \rfloor$ periods.

- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

  $$\Delta t' = T - \left( \left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span $T$ years.

- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$. 
Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.

- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.

- Calculate the three branching probabilities to them.

- Grow the binomial tree from these three nodes until time $T$ to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.

- Review the illustration on p. 754.
Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But a linear-time algorithm exists for double-barrier options on the binomial tree.\(^a\)
- Apply that algorithm to price the double-barrier option’s prices at the three nodes at time \(\Delta t'\).
  - That is, nodes A, B, and C on p. 754.
- Then calculate their expected discounted value for the root node.

\(^a\)See p. 241 of the textbook; Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2008).
Pricing Double-Barrier Options (continued)

• The overall running time is only linear!

• Binomial trees have troubles pricing barrier options.\textsuperscript{a}

• Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 761 and p. 762).

• In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.\textsuperscript{b}

• It has an error of $O(1/n^{1-a})$ for any $0 < a < 1$ for double-barrier options.\textsuperscript{c}

\textsuperscript{a}See p. 400, p. 737, and p. 743.
\textsuperscript{b}Lyuu & Palmer (2010).
\textsuperscript{c}Appolloni, Gaudenziy, & Zanette (2014).
Pricing Double-Barrier Options\textsuperscript{a} (continued)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Comparison of option prices using Trinomial Tree and Bi-Trinomial Tree with different tree sizes.}
\end{figure}

\textsuperscript{a}Generated by Mr. Lin, Ying-Hung (R01723029) on June 6, 2014.
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).
The Barrier-Too-Close Problem (p. 719) Revisited

- Our idea solves it even if one barrier is very close to $S$.
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 727ff).
  - Unlike an earlier solution using combinatorics (p. 710), now the choice of $n$ is not that restricted.
- So it combines the strengths of binomial and trinomial trees.
- This holds for single-barrier options too.
- Here is how.
The Barrier-Too-Close Problem Revisited (continued)

- We can build the tree treating $S$ as if it were a second barrier.
- So both $H$ and $S$ are matched.
- Alternatively, we can pick $\Delta \tau \equiv T/m$ as our length of a period $\Delta t$ without the subsequent adjustment.\(^a\)
- Then build the tree from the price $H$ down.
- So $H$ is matched.
- As before, the initial price $S$ will be matched by the \textit{trinomial} structure.

\(^a\)There is no second barrier to match!
The Barrier-Too-Close Problem Revisited (concluded)

• The earlier trinomial tree is impractical as it needs a very large $n$ when the barrier $H$ is very close to $S$.\(^a\)
  
  – It needs at least one up move to connect $S$ to $H$ as its middle branch is flat.
  
  – But when $S \approx H$, that up move must take a very small step, necessitating a small $\Delta t$.

• Our trinomial structure’s middle branch is not necessarily flat.

• So $S$ can be connected to $H$ via the middle branch, and the need of a very large $n$ no longer exists!

\(^a\)Recall the table on p. 720.
Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.

- They are less common than the continuously monitored versions for single stocks.\(^a\)

- The main difficulty with pricing discrete barrier options lies in matching the monitored *times*.

- Here is why.

\(^a\)Bennett (2014).
Pricing Discrete Barrier Options (continued)

• Suppose each period has a duration of $\Delta t$ and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$

• It is unlikely that all monitored times coincide with the end of a period on the tree, or $\Delta t$ divides $t_i$ for all $i$.

• The binomial-trinomial tree can handle discrete options with ease, however.

• Simply build a binomial-trinomial tree from time 0 to time $t_1$, followed by one from time $t_1$ to time $t_2$, and so on until time $t_\ell$. 
$2\sigma \sqrt{\Delta t_1}$

$2\sigma \sqrt{\Delta t_2}$

$\Delta t'_1, \Delta t_1, \Delta t_1, \Delta t'_2$
Pricing Discrete Barrier Options (concluded)

- This procedure works even if each $t_i$ is associated with a distinct barrier or if each window $[t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

- For typical discrete barriers, placing barriers midway between two price levels on the tree may increase accuracy.\(^a\)

\(^a\)Steiner & Wallmeier (1999); Tavella & Randall (2000).
Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.\(^a\)
  1. The one we saw earlier\(^b\) models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

\(^a\)Frishling (2002).
\(^b\)On p. 318.
Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., \( \alpha(t)S_0 + \beta(t)S_t \)), for practitioners.\(^a\)

- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.

- But they result in binomial trees that grow exponentially (recall p. 317).

- The binomial-trinomial tree can avoid this problem in most cases.

\(^a\)Henry-Labordère (2009).
Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is $D$ dollars and the ex-dividend date is at time $t$.

- So there are $m = t/\Delta t$ periods between time 0 and the ex-dividend date.\(^a\)

- To avoid negative stock prices, we need to make sure the lowest stock price at time $t$ is at least $D$, i.e.,

  $$S e^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D.$$  

  - Or,

  $$\Delta t \geq \left[ \frac{t\sigma}{\ln(S/D)} \right]^2.$$  

\(^a\)That is, $m$ is an integer input and $\Delta t \triangleq t/m$. 
Build a CRR tree from time 0 to time \( t \) as before.

Subtract \( D \) from all the stock prices on the tree at time \( t \) to represent the price drop on the ex-dividend date.

Assume the top node’s price equals \( S' \).
- As usual, its two successor nodes will have prices \( S'u \) and \( S'u^{-1} \).

The remaining nodes’ successor nodes will choose from prices
\[
S'u, S', S'u^{-1}, S'u^{-2}, S'u^{-3}, \ldots ,
\]
same as the CRR tree.
A Stair Tree
Options on a Stock That Pays Known Dividends (continued)

- For each node at time $t$ below the top node, we build the trinomial connection.

- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 746.
Options on a Stock That Pays Known Dividends (concluded)

- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.\(^a\)

\(^a\)Dai (B82506025, R86526008, D8852600) & Lyuu (2004); Dai (B82506025, R86526008, D8852600) (2009).
Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.\(^a\)
- Option pricing with stochastic volatilities.\(^b\)
- Efficient Parisian option pricing.\(^c\)
- Option pricing with time-varying volatilities and time-varying barriers.\(^d\)
- Defaultable bond pricing.\(^e\)

---

\(^a\) H. Wu (R96723058) (2009).
\(^b\) C. Huang (R97922073) (2010).
\(^c\) Y. Huang (R97922081) (2010).
\(^d\) C. Chou (R97944012) (2010); C. I. Chen (R98922127) (2011).
Mean Tracking\textsuperscript{a}

- The general idea behind the binomial-trinomial tree on pp. 745ff is very powerful.
- One finds the successor middle node as the one closest to the mean.
- The two flanking successor nodes are then spaced at $c\sigma\sqrt{\Delta t}$ from the middle node for a suitably large $c > 0$.
- The resulting trinomial structure are then guaranteed to have valid branching probabilities.

\textsuperscript{a}Lyuu & C. Wu (R90723065) (2003, 2005).
Default Prediction

- Under the structural model, the default boundary is modeled as a barrier.\(^a\)

- The constant barrier can be inferred from the closed-form formula given the firm’s market capitalization, etc.\(^b\)

- More generally, the moving barrier can be inferred from the term structure of default probabilities with the binomial-trinomial tree.\(^d\)

---

\(^a\) Recall p. 366.
\(^b\) Black & Cox (1976).
\(^c\) Brockman & Turtle (2003).
\(^d\) Y. Lu (R06723032) (2019).
Default Prediction (continued)

- We then predict default if the firm’s estimated future asset value is ever below the implied barrier.

- For example, on July 1, 2008, this method predicted Freddie Mac would default.

- Indeed, Freddie Mac was placed under the conservatorship of the Federal Housing Finance Agency in September 2008.
Default Prediction\textsuperscript{a} (concluded)

Federal Home Loan Mortgage as of July 1, 2008

\begin{figure}
  \centering
  \includegraphics[width=\textwidth]{default_prediction_concluded.png}
  \caption{Expected Asset Value and Default Boundary for the Federal Home Loan Mortgage as of July 1, 2008.}
  \label{fig:default_prediction_concluded}
\end{figure}

\textsuperscript{a}Plot supplied by Mr. Lu, Yu-Ming (R06723032) on July 2, 2019.
Time-Varying Double Barriers under Time-Dependent Volatility\textsuperscript{a}

- More general models allow a time-varying $\sigma(t)$ (p. 309).
- Let the two barriers $L(t)$ and $H(t)$ be functions of time.\textsuperscript{b}
- They do not have to be differentiable or even continuous.
- Still, we can price double-barrier options in $O(n^2)$ time or less with trinomial trees.
- Continuously monitored double-barrier knock-out options with time-varying barriers are called hot dog options.\textsuperscript{c}

\textsuperscript{a}Y. Zhang (R05922052) (2019).
\textsuperscript{b}So the barriers are continuously monitored.
\textsuperscript{c}El Babsiri & Noel (1998).