Example

• A hedger is short 10,000 European calls.

• $S = 50$, $\sigma = 30\%$, and $r = 6\%$.

• This call’s expiration is four weeks away, its strike price is $50$, and each call has a current value of $f = 1.76791$.

• As an option covers 100 shares of stock, $N = 1,000,000$.

• The trader adjusts the portfolio weekly.

• The calls are replicated well if the cumulative cost of trading stock is close to the call premium’s FV.$^a$

---

$a$This takes the replication viewpoint: One starts with zero dollar.
Example (continued)

• As $\Delta = 0.538560$

\[
N \times \Delta = 538,560
\]

shares are purchased for a total cost of

\[
538,560 \times 50 = 26,928,000
\]
dollars to make the portfolio delta-neutral.

• The trader finances the purchase by borrowing

\[
B = N \times \Delta \times S - N \times f = 25,160,090
\]
dollars net.\(^a\)

\(^a\)This takes the hedging viewpoint: One starts with the option premium. See Exercise 16.3.2 of the text.
Example (continued)

- At 3 weeks to expiration, the stock price rises to $51.
- The new call value is $f' = 2.10580$.
- So before rebalancing, the portfolio is worth

$$- N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622.$$ (91)
Example (continued)

• A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.

• The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

• In fact, the tracking error over one rebalancing act is positive about 68% of the time, but its expected value is essentially zero.\(^a\)

• The tracking error at maturity is proportional to vega.\(^b\)

\(^a\)Boyle & Emanuel (1980).
\(^b\)Kamal & Derman (1999).
Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.

- With a higher delta $\Delta' = 0.640355$, the trader buys

$$N \times (\Delta' - \Delta) = 101,795$$

shares for $5,191,545$.

- The number of shares is increased to $N \times \Delta' = 640,355$. 
Example (continued)

- The cumulative cost is\(^a\)

\[ 26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634. \]

- The portfolio is again delta-neutral.

\(^a\)We take the replication viewpoint here.
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</table>

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).
Example (continued)

• At expiration, the trader has 1,000,000 shares.

• They are exercised against by the in-the-money calls for $50,000,000.

• The trader is left with an obligation of

  \[ 51,524,853 - 50,000,000 = 1,524,853, \]

  which represents the replication cost.

• That means if we started with the PV of $1,524,853, we could replicate 10,000 such calls in this scenario.
Example (concluded)

- The FV of the call premium equals
  \[ 1,767,910 \times e^{0.06 \times 4/52} = 1,776,088. \]

- That means the net gain will be
  \[ 1,776,088 - 1,524,853 = 251,235 \]
  if we are hedging 10,000 European calls.
Tracking Error Revisited

• Define the dollar gamma as \( S^2 \Gamma \).

• The change in value of a delta-hedged long option position after a duration of \( \Delta t \) is proportional to the dollar gamma.

• It is about

\[
(1/2) S^2 \Gamma \left[ (\Delta S/S)^2 - \sigma^2 \Delta t \right].
\]

\(- (\Delta S/S)^2 \) is called the daily realized variance.
Tracking Error Revisited (continued)

- In our particular case,

\[ S = 50, \Gamma = 0.0957074, \Delta S = 1, \sigma = 0.3, \Delta t = 1/52. \]

- The estimated tracking error is

\[ -(1/2) \times 50^2 \times 0.0957074 \times \left[ (1/50)^2 - (0.09/52) \right] = 159,205. \]

- It is very close to our earlier number of 171,622.\(^a\)

- Delta hedge is also called gamma scalping.\(^b\)

\(^a\)Recall Eq. (91) on p. 684.
\(^b\)Bennett (2014).
Tracking Error Revisited (continued)

• Let the rebalancing times be $t_1, t_2, \ldots, t_n$.
• Let $\Delta S_i = S_{i+1} - S_i$.
• The total tracking error at expiration is about
  \[
  \sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[ \left( \frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].
  \]
• The tracking error is path dependent.
• It is also known that\(^a\)
  \[
  \sum_{i=0}^{n-1} \left( \frac{\Delta S_i}{S_i} \right)^2 \to \sigma^2 T.
  \]

\(^a\)Protter (2005).
Tracking Error Revisited (concluded)\textsuperscript{a}

- The tracking error\textsuperscript{b} $\epsilon_n$ over $n$ rebalancing acts has about the same probability of being positive as being negative.

- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\epsilon_n^2]}$ is $O(1/\sqrt{n})$.\textsuperscript{c}

- The root-mean-square tracking error increases with $\sigma$ at first and then decreases.

\textsuperscript{a}Bertsimas, Kogan, & Lo (2000).
\textsuperscript{b}Such as 251,235 on p. 690.
\textsuperscript{c}Grannan & Swindle (1996).
Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, $\Delta f$, due to changes in the stock price, $\Delta S$.

- When $\Delta S$ is not small, the second-order term, gamma $\Gamma \equiv \frac{\partial^2 f}{\partial S^2}$, helps (theoretically).

- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.

- To meet this extra condition, one more security needs to be brought in.
Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call $f_2$ is brought in.
- To set up a delta-gamma hedge, we solve

\[
-N \times f + n_1 \times S + n_2 \times f_2 - B = 0 \quad \text{(self-financing)},
\]
\[
-N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 = 0 \quad \text{(delta neutrality)},
\]
\[
-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 = 0 \quad \text{(gamma neutrality)},
\]

for $n_1$, $n_2$, and $B$.

- The gammas of the stock and bond are 0.

- See the numerical example on pp. 231–232 of the text.
Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

All those holes and pebbles.
Who could count them?
— James Joyce, *Ulysses* (1922)

And though the holes were rather small,
they had to count them all.
The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.

- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.

- We first used this method in the linear-time algorithm for standard European option pricing on p. 279.

- We will now apply it to price barrier options.
The Reflection Principle\textsuperscript{a}

- Imagine a particle at position $(0, -a)$ on the integral lattice that is to reach $(n, -b)$.
- Without loss of generality, assume $a > 0$ and $b \geq 0$.
- This particle’s movement:
  
  $$(i, j) \rightarrow (i + 1, j - 1) \quad \text{down move} \quad S \rightarrow Sd$$
  $$(i, j) \rightarrow (i + 1, j + 1) \quad \text{up move} \quad S \rightarrow Su$$

- How many paths touch the $x$ axis?

\textsuperscript{a}André (1887).
The Reflection Principle (continued)

- For a path from $(0, -a)$ to $(n, -b)$ that touches the $x$ axis, let $J$ denote the first point this happens.
- Reflect the portion of the path from $(0, -a)$ to $J$.
- A path from $(0, a)$ to $(n, -b)$ is constructed.
- It also hits the $x$ axis at $J$ for the first time.
- The one-to-one mapping shows the number of paths from $(0, -a)$ to $(n, -b)$ that touch the $x$ axis equals the number of paths from $(0, a)$ to $(n, -b)$.
The Reflection Principle (concluded)

• A path of this kind has \((n + b + a)/2\) down moves and \((n - b - a)/2\) up moves.\(^{a}\)

• Hence there are

\[
\binom{n}{\frac{n+a+b}{2}} = \binom{n}{\frac{n-a-b}{2}} \tag{92}
\]

such paths for even \(n + a + b\).

– Convention: \(\binom{n}{k} = 0\) for \(k < 0\) or \(k > n\).

\(^{a}\)Verify it!
Pricing Barrier Options (Lyuu, 1998)

• Focus on the down-and-in call with barrier \( H < X \).
• Assume \( H < S \) without loss of generality.
• Define

\[
a \triangleq \left\lceil \frac{\ln \left( \frac{X}{S d^n} \right) }{\ln(u/d) } \right\rceil = \left\lceil \frac{\ln(X/S) }{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\
h \triangleq \left\lfloor \frac{\ln \left( \frac{H}{S d^n} \right) }{\ln(u/d) } \right\rfloor = \left\lfloor \frac{\ln(H/S) }{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.
\]

- \( a \) is such that \( \tilde{X} \triangleq S u^a d^{n-a} \) is the terminal price that is closest to \( X \) from above.
- \( h \) is such that \( \tilde{H} \triangleq S u^h d^{n-h} \) is the terminal price that is closest to \( H \) from below.\(^a\)

\(^a\)So we underestimate the price.
Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.

- A process with $n$ moves hence ends up in the money if and only if the number of up moves is at least $a$.

- The price $S u^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price $S d^n$ on the binomial tree.

\[ S u^k d^{n-k} = S d^{-k} d^{n-k} = S d^{n-2k}. \]  

(93)
$S u^j d^{n-j}$

$\tilde{X} = S u^a d^{n-a}$

$\tilde{H} = S u^h d^{n-h}$
Pricing Barrier Options (continued)

- A path from $S$ to the terminal price $Su^j d^{m-j}$ has probability $p^j(1-p)^{n-j}$ of being taken.

- With reference to p. 707, the reflection principle (p. 702) can be applied with
  
  \[
  a = n - 2h, \\
  b = 2j - 2h, 
  \]

  in Eq. (92) on p. 704 by treating the \( \tilde{H} \) line as the \( x \) axis.
Pricing Barrier Options (continued)

• Therefore,

\[
\binom{n}{\frac{n + (n-2h) + (2j-2h)}{2}} = \binom{n}{n-2h+j}
\]

paths hit \( \tilde{H} \) in the process for \( h \leq n/2 \).

• The terminal price \( Su^j d^{n-j} \) is reached by a path that hits the effective barrier with probability

\[
\binom{n}{n-2h+j} p^j (1-p)^{n-j}, \quad j \leq 2h.
\]
Pricing Barrier Options (concluded)

- The option value equals

\[
\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1 - p)^{n-j} \frac{(Su^j d^{n-j} - X)}{R^n}.
\]

- \( R \triangleq e^{r\tau/n} \) is the riskless return per period.

- It yields a linear-time algorithm.\(^a\)

\(^a\)Lyuu (1998).
Convergence of BOPM

- Equation (94) results in the same sawtooth-like convergence shown on p. 400 (repeated on next page).
- The reasons are not hard to see.
- The true barrier $H$ most likely does not equal the effective barrier $\tilde{H}$. 
Convergence of BOPM (continued)

Down-and-in call value

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Convergence of BOPM (continued)

- Convergence is actually good if we limit \( n \) to certain values—191, for example.

- These values make the true barrier coincide with or just above one of the stock price levels, that is,

\[
H \approx S d^j = S e^{-j\sigma \sqrt{\tau/n}}
\]

for some integer \( j \).

- The preferred \( n \)'s are thus

\[
n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots
\]
Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.
- However, the effective barrier above, $Sd^j$, corresponds to a terminal stock price only when $n - j$ is even.\(^a\)
- To close this gap, we decrement $n$ by one, if necessary, to make $n - j$ an even number.

\(^a\)This is because $j = n - 2k$ for some $k$ by Eq. (93) on p. 706. Of course we could have adopted the form $Sd^j (-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.
Convergence of BOPM (concluded)

- The preferred $n$’s are now

$$n = \begin{cases} 
\ell, & \text{if } \ell - j \text{ is even}, \\
\ell - 1, & \text{otherwise}, 
\end{cases}$$

$j = 1, 2, 3, \ldots$, where

$$\ell \overset{\Delta}{=} \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (94) on p. 710 only with the $n$’s above.
Practical Implications

• This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.\(^a\)

• Now that barrier options can be efficiently priced, we can afford to pick very large $n$’s (p. 718).

• This has profound consequences.\(^b\)

---

\(^a\) J. Lin (R95221010) (2008); J. Lin (R95221010) & Palmer (2010).

\(^b\) See pp. 732ff.
| \( n \) | Combinatorial method  |
|---|---|---|
| 21 | Value 5.507548 | Time (milliseconds) 0.30 |
| 84 | 5.597597 | 0.90 |
| 191 | 5.635415 | 2.00 |
| 342 | 5.655812 | 3.60 |
| 533 | 5.652253 | 5.60 |
| 768 | 5.654609 | 8.00 |
| 1047 | 5.658622 | 11.10 |
| 1368 | 5.659711 | 15.00 |
| 1731 | 5.659416 | 19.40 |
| 2138 | 5.660511 | 24.70 |
| 2587 | 5.660592 | 30.20 |
| 3078 | 5.660099 | 36.70 |
| 3613 | 5.660498 | 43.70 |
| 4190 | 5.660388 | 44.10 |
| 4809 | 5.659955 | 51.60 |
| 5472 | 5.660122 | 68.70 |
| 6177 | 5.659981 | 76.70 |
| 6926 | 5.660263 | 86.90 |
| 7717 | 5.660272 | 97.20 |
Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because
  \[ n \sim \frac{1}{\ln^2(S/H)}. \]
  - This is called the barrier-too-close problem.

- This observation is indeed true of standard quadratic-time binomial tree algorithms.

- But it no longer applies to linear-time algorithms (see p. 720).
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</table>

(All times in milliseconds.)
Trinomial Tree

• Set up a trinomial approximation to the geometric Brownian motion\(^a\)

\[
\frac{dS}{S} = r \, dt + \sigma \, dW.
\]

• The three stock prices at time \(\Delta t\) are \(S\), \(Su\), and \(Sd\), where \(ud = 1\).

• Let the mean and variance of the stock price be \(SM\) and \(S^2V\), respectively.

\(^a\)Boyle (1988).
Trinomial Tree (continued)

• By Eqs. (30) on p. 179,

\[ M \triangleq e^{r\Delta t}, \]
\[ V \triangleq M^2(e^{\sigma^2\Delta t} - 1). \]

• Impose the matching of mean and that of variance:

\[ 1 = p_u + p_m + p_d, \]
\[ SM = (p_u u + p_m + (p_d/u)) S, \]
\[ S^2V = p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2. \]
Trinomial Tree (continued)

- Use linear algebra to verify that

\[ p_u = \frac{u(V + M^2 - M) - (M - 1)}{(u - 1)(u^2 - 1)}, \]

\[ p_d = \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}. \]

- We must also make sure the probabilities lie between 0 and 1.
Trinomial Tree (concluded)

- There are countless variations.

- But all converge to the Black-Scholes option pricing model.\(^a\)

- The trinomial model has a linear-time algorithm for European options.\(^b\)

---

\(^a\) Madan, Milne, & Shefrin (1989).
\(^b\) T. Chen (R94922003) (2007).
A Trinomial Tree

- Use \( u = e^{\lambda \sigma \sqrt{\Delta t}} \), where \( \lambda \geq 1 \) is a tunable parameter.
- Then

\[
\begin{align*}
p_u & \rightarrow \frac{1}{2\lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2\lambda \sigma}, \\
p_d & \rightarrow \frac{1}{2\lambda^2} - \frac{(r - 2\sigma^2) \sqrt{\Delta t}}{2\lambda \sigma}.
\end{align*}
\]

- A nice choice for \( \lambda \) is \( \sqrt{\pi/2} \).\(^a\)

\(^a\)Omberg (1988).
Barrier Options Revisited

• BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.

• The trinomial model solves the problem by adjusting $\lambda$ so that the barrier is hit exactly.$^a$

• When

$$S e^{-h \lambda \sigma \sqrt{\Delta t}} = H,$$

it takes $h$ down moves to go from $S$ to $H$, if $h$ is an integer.

• Then

$$h = \frac{\ln(S/H)}{\lambda \sigma \sqrt{\Delta t}}.$$

---

Barrier Options Revisited (continued)

• This is easy to achieve by adjusting $\lambda$.

• Typically, we find the smallest $\lambda \geq 1$ such that $h$ is an integer.$^a$
  
  – Such a $\lambda$ may not exist for very small $n$’s.
  
  – This is not hard to check.

• Toward that end, we find the largest integer $j \geq 1$ that satisfies \( \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1 \) to be our $h$.

• Then let
  \[
  \lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.
  \]

$^a$Why must $\lambda \geq 1$?
Barrier Options Revisited (continued)

• Alternatively, we can pick

\[ h = \left\lfloor \frac{\ln(S/H)}{\sigma \sqrt{\Delta t}} \right\rfloor. \]

• Make sure \( h \geq 1 \).

• Then let

\[ \lambda = \frac{\ln(S/H)}{h\sigma \sqrt{\Delta t}}. \]
Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.

- The following probabilities may be used,

\[
pu = \frac{1}{2\lambda^2} + \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}, \\
\rho_m = 1 - \frac{1}{\lambda^2}, \\
\rho_d = \frac{1}{2\lambda^2} - \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}. \\
\]

\[\mu' \triangleq r - (\sigma^2/2).\]
Algorithms Comparison\textsuperscript{a}

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the $n$ value at which they converge.
  - The one with the smallest $n$ wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not $n$.\textsuperscript{b}

\textsuperscript{a}Lyuu (1998).
\textsuperscript{b}Patterson & Hennessy (1994).
Algorithms Comparison (continued)

• Pages 712 and 731 seem to show the trinomial model converges at a smaller $n$ than BOPM.

• It is in this sense when people say trinomial models converge faster than binomial ones.

• But does it make the trinomial model better then?
Algorithms Comparison (concluded)

• The linear-time binomial tree algorithm actually performs better than the trinomial one.

• See the next page, expanded from p. 718.

• The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.\(^a\)
  
  – See pp. 745ff for an alternative solution.

\(^a\)Lyuu (1998).
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(All times in milliseconds.)
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
  - They make up “less than 5% of the light exotic market.”

- Assume $L < S < H$.

- The binomial model produces oscillating option values (see plot on next page).

---

\( ^a \text{Bennett (2014).} \)

\( ^b \text{Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2005).} \)
Double-Barrier Options (concluded)

- The combinatorial method yields a linear-time algorithm.\textsuperscript{a}

- This binomial model is $O(1/\sqrt{n})$ convergent in general.\textsuperscript{b}

- If the barriers $L$ and $H$ depend on time, we have moving-barrier options.\textsuperscript{c}

\textsuperscript{a}See p. 241 of the textbook.
\textsuperscript{b}Gobet (1999).
\textsuperscript{c}Rogers & Zane (1998).
Double-Barrier Knock-Out Options

• We knew how to pick the $\lambda$ so that one of the layers of the trinomial tree coincides with one barrier, say $H$.

• This choice, however, does not guarantee that the other barrier, $L$, is also hit.

• One way to handle this problem is to lower the layer of the tree just above $L$ to coincide with $L$.

  – More general ways to make the trinomial model hit both barriers are available.

---

\[ a \] Ritchken (1995); Hull (1999).

\[ b \] Hsu (R7526001, D89922012) & Lyuu (2006). Dai (B82506025, R86526008, D8852600) & Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 745ff).
Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above $L$ must be adjusted.

- Let $\ell$ be the positive integer such that
  \[ Sd^{\ell+1} < L < Sd^\ell. \]

- Hence the layer of the tree just above $L$ has price $Sd^\ell$.\(^a\)

\(^a\)You probably cannot do the same thing for binomial models (why?). Thanks to a lively discussion on April 25, 2012.
Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying
  \[ L = Sd^{\ell-1}e^{-\gamma \lambda \sigma \sqrt{\Delta t}}. \]

- The prices between the barriers are (from low to high)
  \[ L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H. \]

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are
  \[ p'_u = \frac{b + a\gamma}{1 + \gamma}, \quad p'_d = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d, \]

  where $a \equiv \mu' \sqrt{\Delta t}/(\lambda \sigma)$ and $b \equiv 1/\lambda^2$. 
Convergence: Binomial vs. Trinomial

![Graph showing convergence between Binomial and Trinomial options values.](image)

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Ideas for Binomial Trees To Handle Two Barriers

\[
\begin{align*}
\ln(H/L) \\
2\sigma \sqrt{\Delta t} \\
\ln(H) \\
\ln(S) \\
\ln(L)
\end{align*}
\]
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.\(^a\)
- The resulting tree is called the binomial-trinomial tree.\(^b\)
- Suppose a binomial tree will be built with \(\Delta t\) as the duration of one period.
- Node \(X\) at time \(t\) needs to pick three nodes on the binomial tree at time \(t + \Delta t'\) as its successor nodes.
  \[- \Delta t \leq \Delta t' < 2\Delta t.\]

\(^a\) Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).
\(^b\) The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.
The Binomial-Trinomial Tree (continued)

\[ \Delta t' \]

\[ \hat{\mu} + 2\sigma \sqrt{\Delta t} \]

\[ \hat{\mu} - 2\sigma \sqrt{\Delta t} \]

\[ p_u \]

\[ p_m \]

\[ p_d \]

\[ |\alpha| \]

\[ |\beta| \]

\[ |\gamma| \]

\[ 2\sigma \sqrt{\Delta t} \]

\[ 2\sigma \sqrt{\Delta t} \]

\[ 2\sigma \sqrt{\Delta t} \]

\[ \Delta t \]

\[ \Delta t \]
The Binomial-Trinomial Tree (continued)

• These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.

• Let $S$ be the stock price at node $X$.

• Use $s(z)$ to denote the stock price at node $z$. 
The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return
  \( \ln(S_{t+\Delta t'}/S) \) at time \( t + \Delta t' \) equals\(^{a}\)
  \[
  \mu \doteq \left( r - \frac{\sigma^2}{2} \right) \Delta t'.
  \] (95)

- Its variance equals
  \[
  \text{Var} \doteq \sigma^2 \Delta t'.
  \] (96)

- Let node B be the node whose logarithmic return
  \( \hat{\mu} \doteq \ln(s(B)/S) \) is closest to \( \mu \) among all the nodes at
  time \( t + \Delta t' \).

\(^{a}\)See p. 294.
The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at \( 2\sigma \sqrt{\Delta t} \) apart.
- Review the illustration on p. 746.
The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.

- Recall that
  \[
  \hat{\mu} \triangleq \ln(s(B)/S)
  \]
  is the logarithmic return of the middle node B.

- Let $\alpha$, $\beta$, and $\gamma$ be the differences between $\mu$ and the logarithmic returns
  \[
  \ln(s(Z)/S), \quad Z = A, B, C,
  \]
  in that order.
The Binomial-Trinomial Tree (continued)

• In other words,

\[ \alpha \triangleq \hat{\mu} + 2\sigma \sqrt{\Delta t} - \mu = \beta + 2\sigma \sqrt{\Delta t}, \quad (97) \]
\[ \beta \triangleq \hat{\mu} - \mu, \quad (98) \]
\[ \gamma \triangleq \hat{\mu} - 2\sigma \sqrt{\Delta t} - \mu = \beta - 2\sigma \sqrt{\Delta t}. \quad (99) \]

• The three branching probabilities \( p_u, p_m, p_d \) then satisfy

\[ p_u \alpha + p_m \beta + p_d \gamma = 0, \quad (100) \]
\[ p_u \alpha^2 + p_m \beta^2 + p_d \gamma^2 = \text{Var}, \quad (101) \]
\[ p_u + p_m + p_d = 1. \quad (102) \]
The Binomial-Trinomial Tree (concluded)

- Equation (100) matches the mean (95) of the logarithmic return $\ln(S_{t+\Delta t}/S)$ on p. 748.
- Equation (101) matches its variance (96) on p. 748.
- The three probabilities can be proved to lie between 0 and 1 by Cramer’s rule.
Pricing Double-Barrier Options

- Consider a double-barrier option with two barriers $L$ and $H$, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- The idea is to choose a $\Delta t$ such that
  \[
  \frac{\ln(H/L)}{2\sigma \sqrt{\Delta t}}
  \] (103)
  is a positive integer.
  - The distance between two adjacent nodes such as nodes $Y$ and $Z$ in the figure on p. 754 is $2\sigma \sqrt{\Delta t}$. 
Pricing Double-Barrier Options (continued)

\[
\begin{align*}
\ln(H/L) & \quad \ln(H/S) \\
2\sigma\sqrt{\Delta t} & \quad \ln(L/S) + 4\sigma\sqrt{\Delta t} \\
& \quad \ln(L/S) + 2\sigma\sqrt{\Delta t} \\
& \quad 0 \\
& \quad \ln(L/S) \\
\Delta t' & \quad \Delta t \\
T & \quad \Delta t \\
\end{align*}
\]
Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with \( \sim m \) periods.
- Suppose we pick \( \Delta \tau \overset{\Delta}{=} T/m \) for the length of each period.
- There is no guarantee that \( \frac{\ln(H/L)}{2\sigma \sqrt{\Delta \tau}} \) is an integer.
- So we pick a \( \Delta t \) that is close to, but does not exceed, \( \Delta \tau \) and makes \( \frac{\ln(H/L)}{2\sigma \sqrt{\Delta t}} \) some integer \( \kappa \).
- Specifically, we select

\[
\Delta t = \left( \frac{\ln(H/L)}{2\kappa \sigma} \right)^2,
\]

where \( \kappa = \left\lceil \frac{\ln(H/L)}{2\sigma \sqrt{\Delta \tau}} \right\rceil \).
Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier \( L \) upward.
- Automatically, a layer coincides with the high barrier \( H \).
- It is unlikely that \( \Delta t \) divides \( T \), however.
- So the position at time 0 and with logarithmic return \( \ln(S/S) = 0 \) is not occupied by a binomial node to serve as the root node (recall p. 754).
Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.

- Between time 0 and time $T$, the binomial tree spans $\lfloor T/\Delta t \rfloor$ periods.

- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1\right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span $T$ years.

- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$. 
Pricing Double-Barrier Options (continued)

• Start with the root node at time 0 and at a price with logarithmic return \( \ln(S/S') = 0 \).

• Find the three nodes on the binomial tree at time \( \Delta t' \) as described earlier.

• Calculate the three branching probabilities to them.

• Grow the binomial tree from these three nodes until time \( T \) to obtain a binomial-trinomial tree with \( \lfloor T/\Delta t \rfloor \) periods.

• Review the illustration on p. 754.
Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But a linear-time algorithm exists for double-barrier options on the *binomial* tree.\(^a\)
- Apply that algorithm to price the double-barrier option’s prices at the three nodes at time \(\Delta t'\).
  - That is, nodes A, B, and C on p. 754.
- Then calculate their expected discounted value for the root node.

\(^a\)See p. 241 of the textbook; Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2008).
Pricing Double-Barrier Options (continued)

• The overall running time is only linear!

• Binomial trees have troubles pricing barrier options.\(^a\)

• Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 761 and p. 762).

• In fact, the binomial-trinomial tree has an error of \(O(1/n)\) for single-barrier options.\(^b\)

• It has an error of \(O(1/n^{1-a})\) for any \(0 < a < 1\) for double-barrier options.\(^c\)

\(^a\)See p. 400, p. 737, and p. 743.
\(^b\)Lyuu & Palmer (2010).
\(^c\)Appolloni, Gaudenziy, & Zanette (2014).
Pricing Double-Barrier Options\textsuperscript{a} (continued)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig}
\caption{Comparison of option prices using different tree models.}
\end{figure}

\textsuperscript{a}Generated by Mr. Lin, Ying-Hung (R01723029) on June 6, 2014.
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).