Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to price corporate securities.
  - The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a] Black & Scholes (1973); Merton (1974).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call[a] on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V.

[a] Recall p. 201.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned} nS &= C \,(\text{market capitalization of XYZ.com}), \\ B &= V - C. \end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 10 (p. 296) and the put-call parity,[a]

$$nS = VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \tag{51}$$

$$B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \tag{52}$$

  – Above,

$$x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

[a]This is sometimes called Merton's (1974) structural model.

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$
\begin{aligned}
& \frac{\ln(X/B)}{\tau} - r \\
= \quad & -\frac{1}{\tau} \ln \left(N(-z) + \frac{1}{\omega} N(z - \sigma\sqrt{\tau}) \right).
\end{aligned}
$$

  - $\omega \stackrel{\Delta}{=} Xe^{-r\tau}/V$.
  - $z \stackrel{\Delta}{=} (\ln \omega)/(\sigma\sqrt{\tau}) + (1/2)\, \sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of the remaining firm value.
- Then Eqs. (51)–(52) on p. 371 become, respectively,

$$\begin{aligned} nS &= Ve^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ B &= (1-\alpha) Ve^{-q\tau} N(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  – Above,

$$x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

  dollars.

  – Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.[a]
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

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[a]See p. 225.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility,[a] dividend, and strike price are under partial control of the stockholders or their boards.

[a] This is called the asset substitution problem (Myers, 1977).

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 378 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

  dollars.
- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain
$$14,187.5 + 1,937.5 - 15,250 = 875$$
dollars.
- The *original* bondholders lose an equal amount,
$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$
  - This is called claim dilution.[a]

[a] Fama & M. H. Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a $14,833.3 cash dividend.
- The stockholders now have $14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning 2/3 of a call on n Merck shares with a strike price of $45,000.
- n such calls are worth $1,937.5 (p. 378).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

- Other Examples:[a]
  - Stock as compound call when company issues coupon bonds.
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
  - Bonds with safety covenants as barrier options.

[a] Cox & Rubinstein (1985); Geske (1977).

Further Topics (concluded)

- Securities issued by firms with a complex capital structure must be solved by trees.[a]

[a] Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).

Distance to Default (DTD)[a]

- Let μ be the total value V's rate of expected return.
- From Eq. (51), on p. 371, the probability of default τ years from now equals

$$N(-\text{DTD}),$$

where

$$\text{DTD} \triangleq \frac{\ln(V/X) + (\mu - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

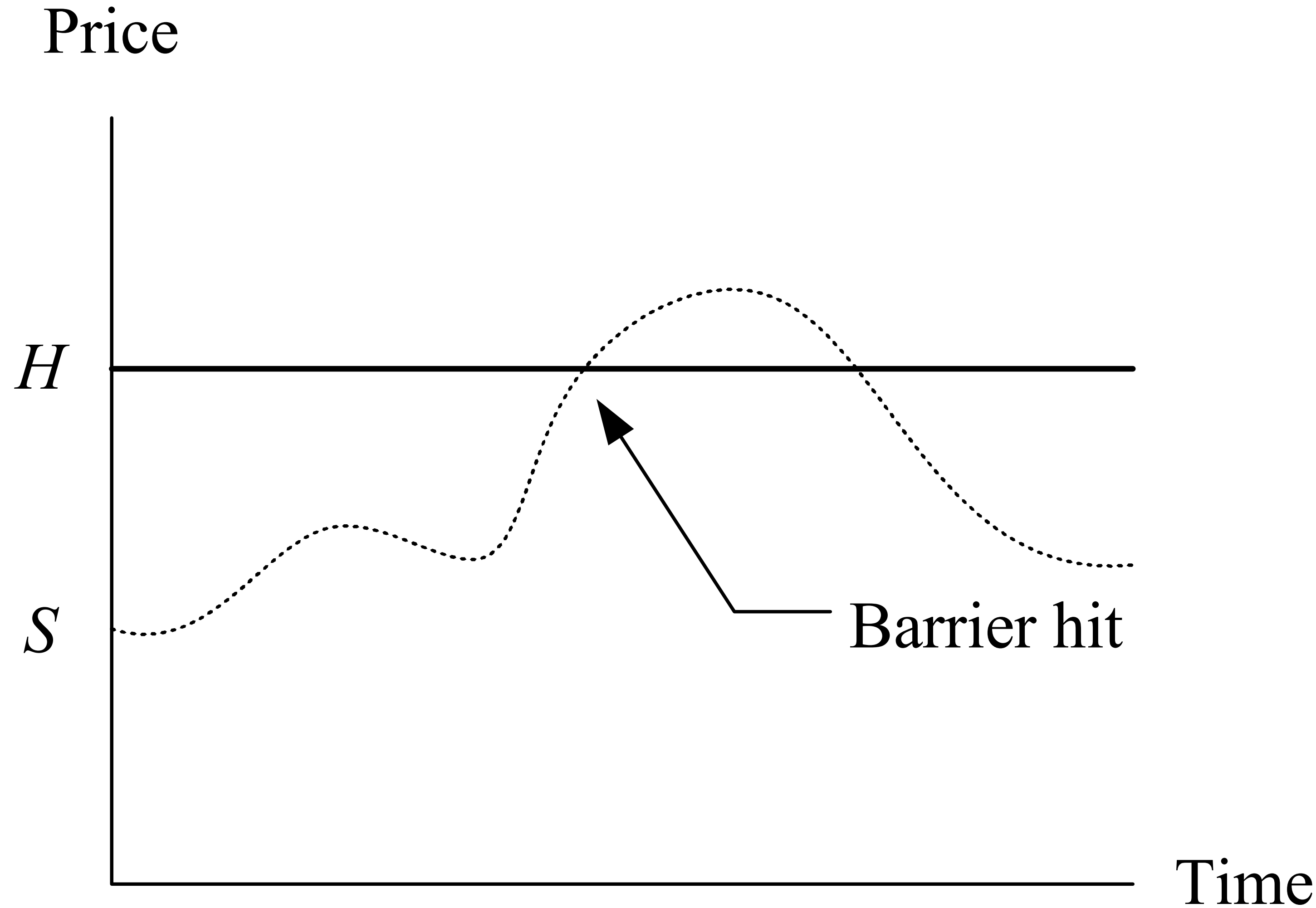
- V/X is called the leverage ratio.

[a] Merton (1974).

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H *throughout its life*.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

[a]A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.


Price
H
S
Barrier hit
Time

Barrier Options (continued)

- A knock-in (KI) option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all the possible barrier options mentioned above.[a]

[a]Haug (2006).

Barrier Options (concluded)

- Knock-out options were issued in the U.S. in 1967.[a]
- Knock-in puts are the most popular barrier options.[b]
- Knock-out puts are the second most popular barrier options.[c]
- Knock-out calls are the most popular among barrier call options.[d]

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[a] Cox & Rubinstein (1985).
[b] Bennett (2014).
[c] Bennett (2014).
[d] Bennett (2014).

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{53}$$

  - $x \stackrel{\Delta}{=} \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \stackrel{\Delta}{=} (r - q + \sigma^2/2)/\sigma^2$.

- A European down-and-out call can be priced via the in-out parity (see text).

[a]Merton (1973). See Exercise 17.1.6 of the textbook for a proof.

A Formula for Up-and-In Puts[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- Again, a European up-and-out put can be priced via the in-out parity.

[a]Merton (1973).

Are American Options Barrier Options?[a]

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be found by backward induction.
- It cannot be specified in an arbitrary way.
- In conrast, the barrier in a barrier option is given by a contract.[b]

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[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on March 25, 2009.

[b]Cox & Rubinstein (1985).

Interesting Observations

- Assume $H < X$.
- Replace S in the Merton pricing formula Eq. (43) on p. 326 for the call with H^2/S.
  - Equation (53) on p. 392 for the down-and-in call becomes Eq. (43) when $r - q = \sigma^2/2$.
  - Equation (53) becomes S/H times Eq. (43) when $r - q = 0$.

Interesting Observations (concluded)

- Replace S in the pricing formula for the down-and-in call, Eq. (53), with H^2/S.
  - Equation (53) becomes Eq. (43) when $r - q = \sigma^2/2$.
  - Equation (53) becomes H/S times Eq. (43) when $r - q = 0$.[a]
- Why?[b]

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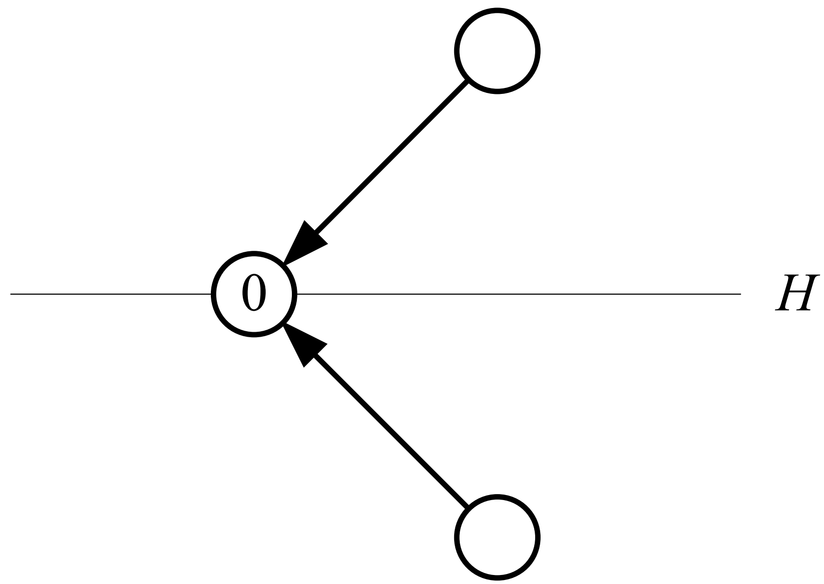
[a] Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.

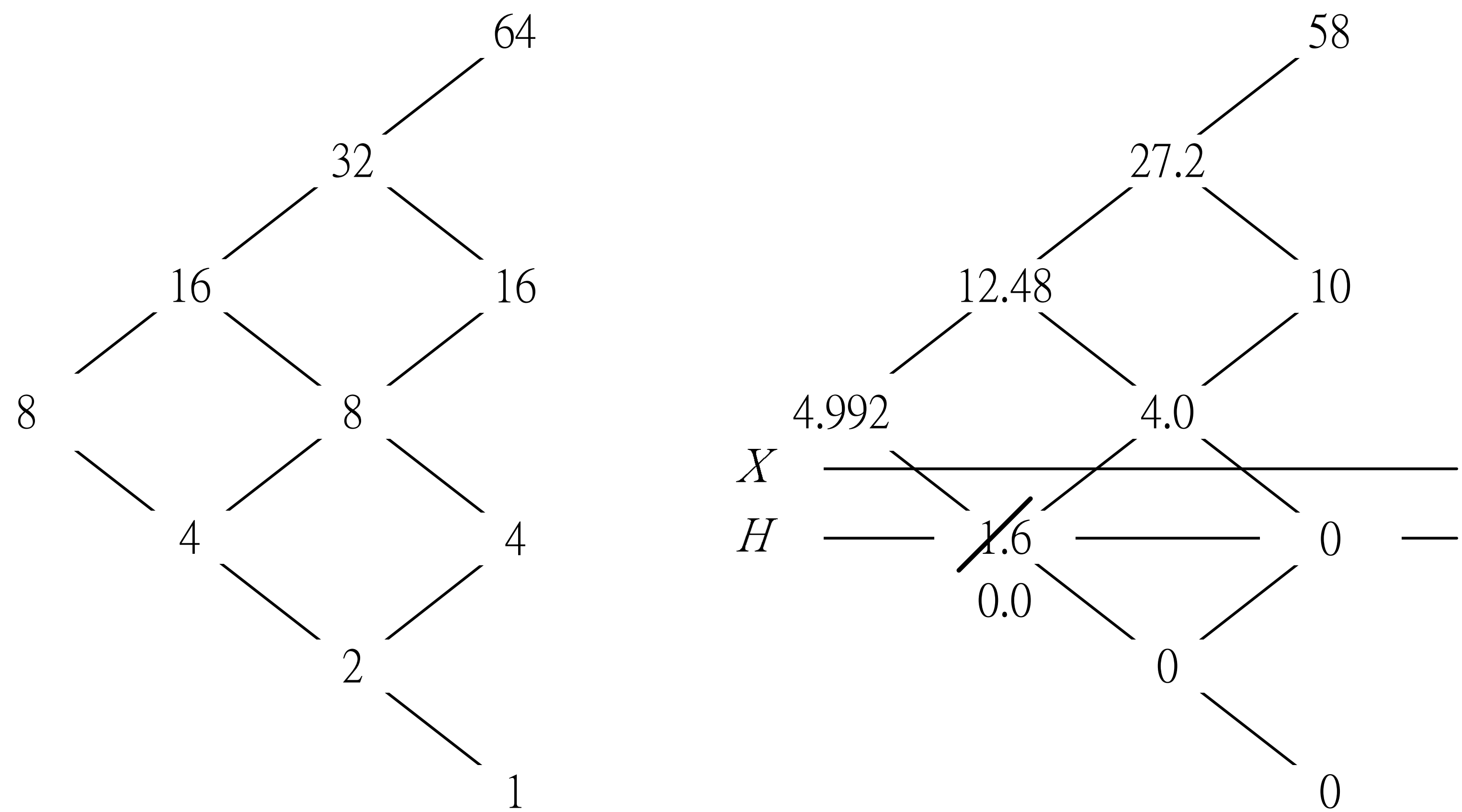
[b] Apply the reflection principle (p. 700), Eq. (42) on p. 289, and Lemma 9 (p. 294).

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.



- Pricing down-and-in options is subtler.



$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (continued)

- But convergence is erratic because H is not at a price level on the tree (see plot on next page).[a]
  - The barrier H is moved lower (or higher) to a close-by node price.
  - This "effective barrier" thus changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]
- Solutions will be presented later.

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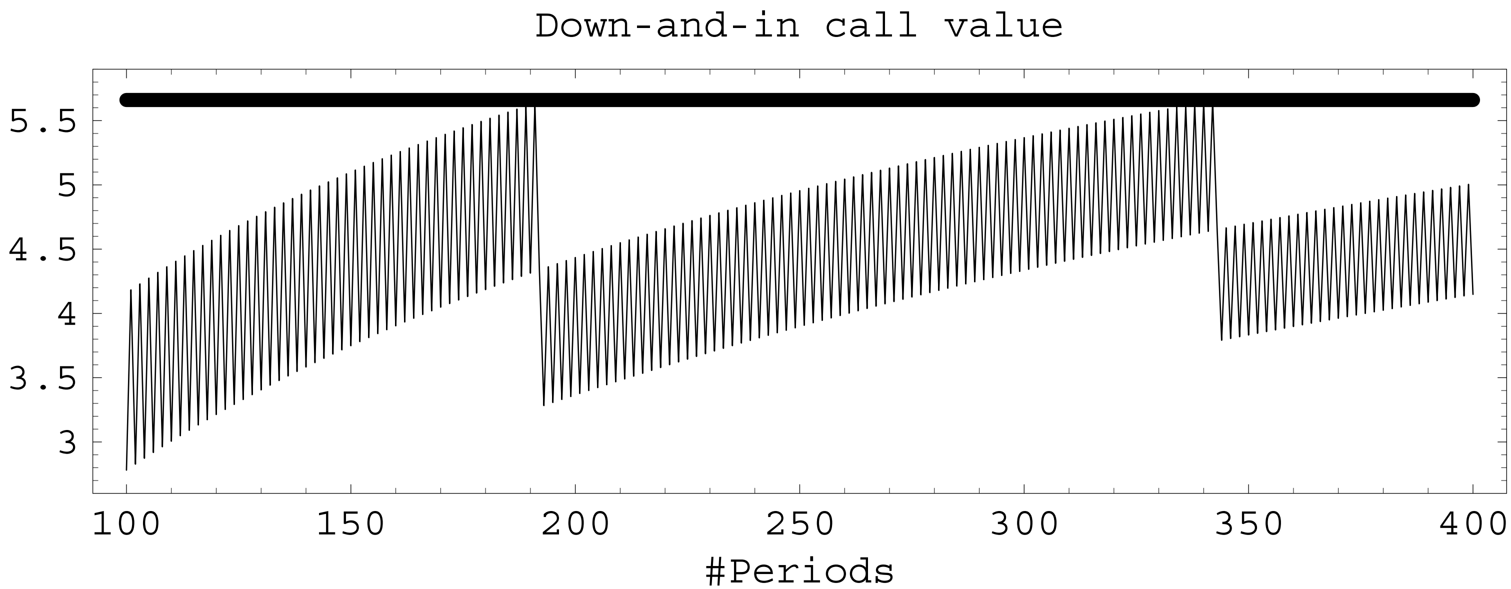
[a]Boyle & Lau (1994).

[b]Tavella & Randall (2000); J. Lin (`R95221010`) (2008).

Binomial Tree Algorithms (concluded)[a]



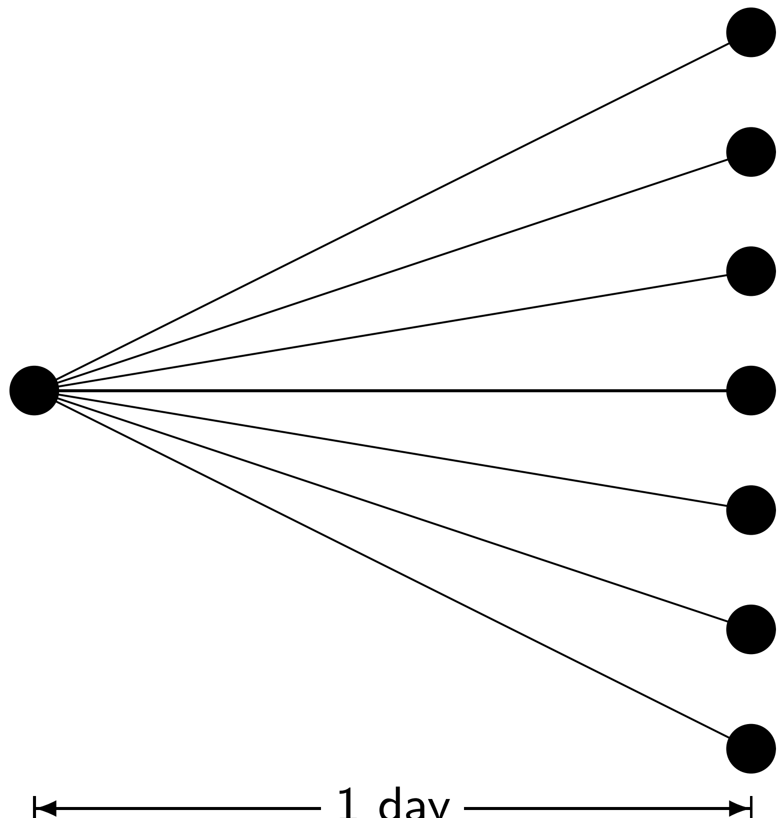
[a]Lyuu (1998).

Daily Monitoring

- Many barrier options monitor the barrier only for daily *closing prices.*
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d+1$ nodes if each day is partitioned into d periods.
- Does this save time or space?[a]

[a]Contributed by Ms. Chen, Tzu-Chun (`R94922003`) and others on April 12, 2006.

A Heptanomial Tree (6 Periods Per Day)



Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.[a]
- They are equally popular for indices.[b]

[a] Bennett (2014).
[b] Bennett (2014).

Data! data! data!

— Arthur Conan Doyle (1892),

The Adventures of Sherlock Holmes

Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
  - By that we mean the number of domestic currencies per unit of foreign currency.[a]
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.

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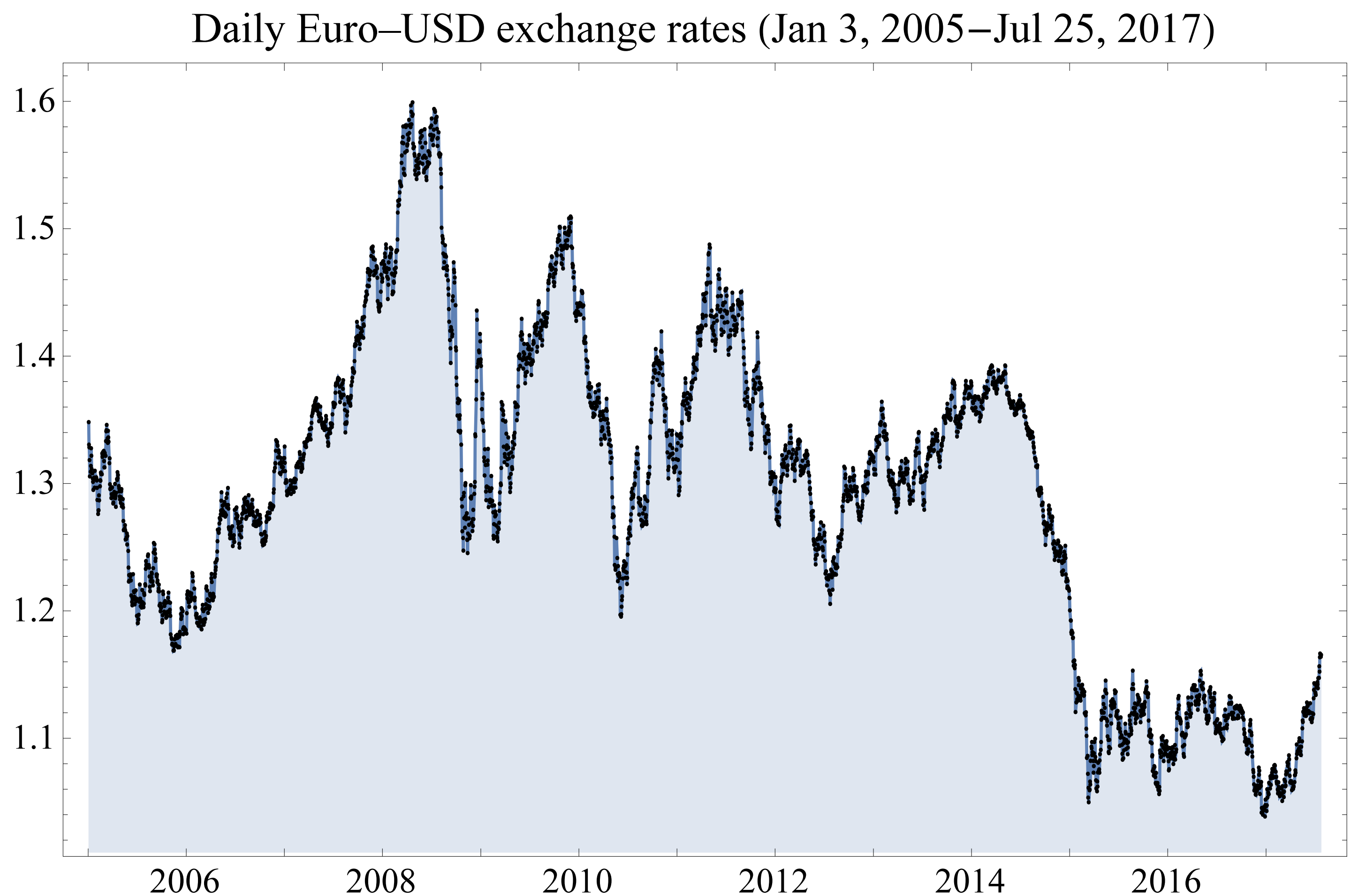
[a] The market convention is the opposite: $\mathrm{A}/\mathrm{B} = x$ means one unit of currency A (the reference currency or base currency) is equal to x units of currency B (the counter-value currency).

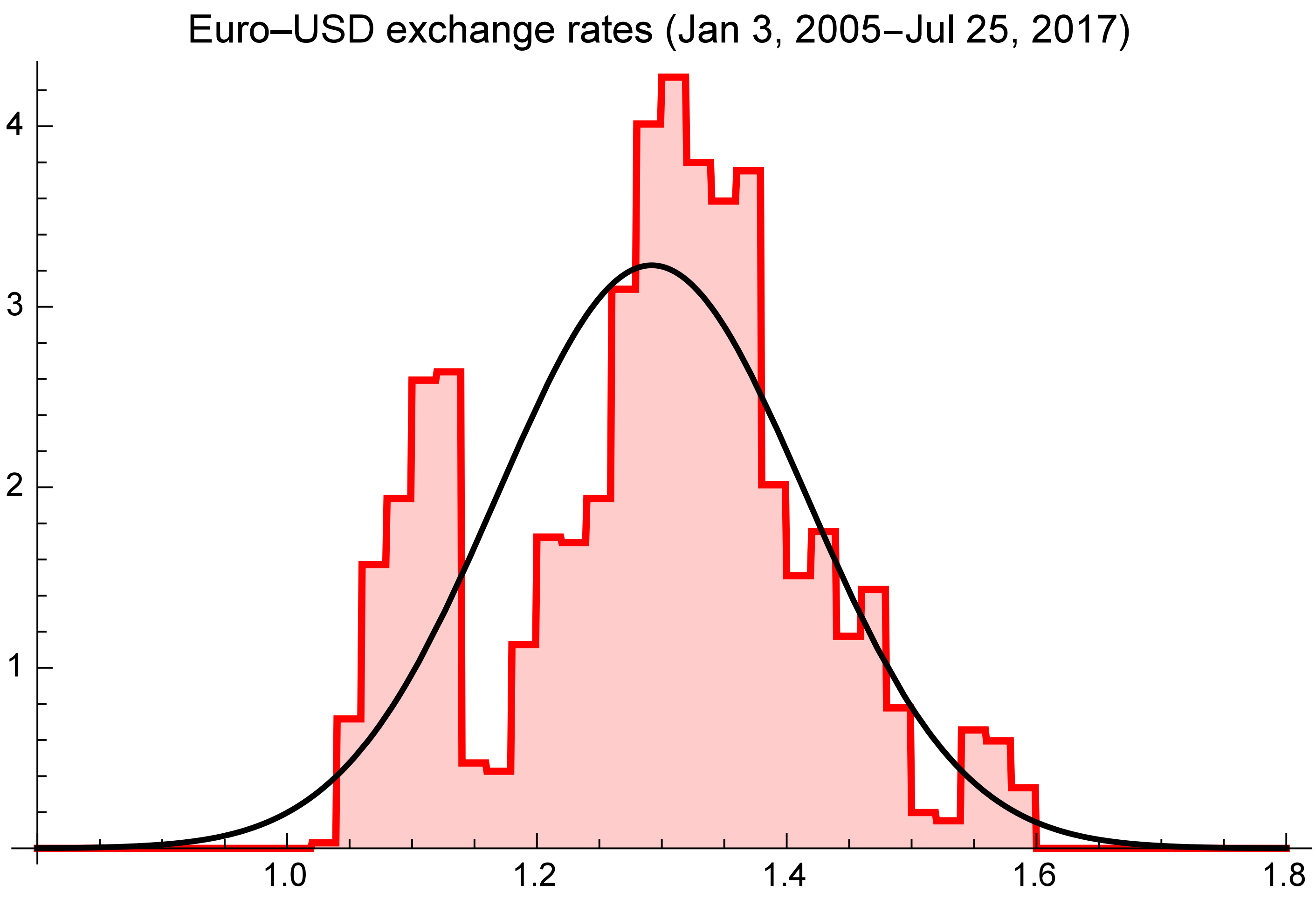
Foreign Currencies (concluded)

- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

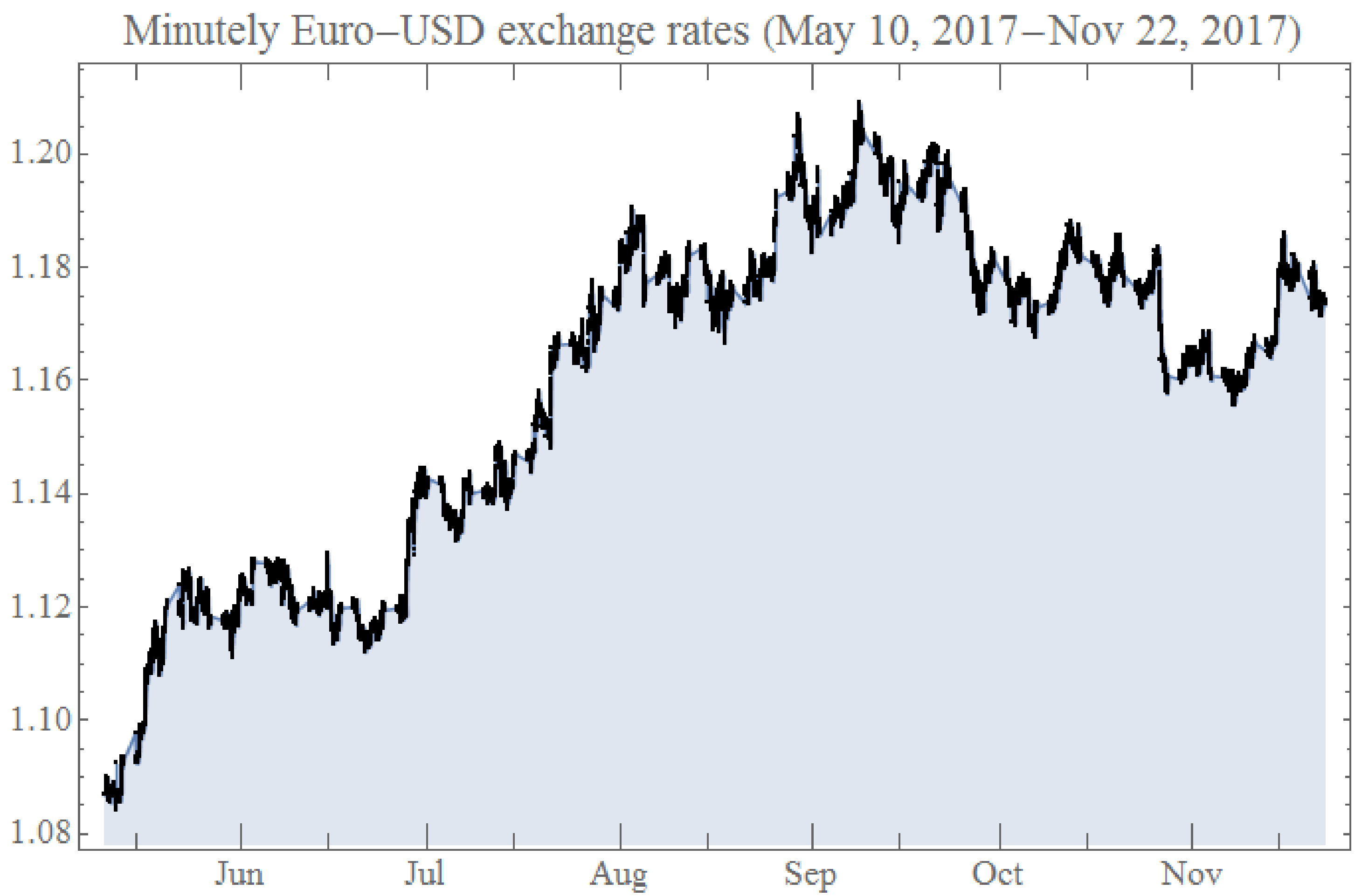
Time Series of the Daily Euro–USD Exchange Rate



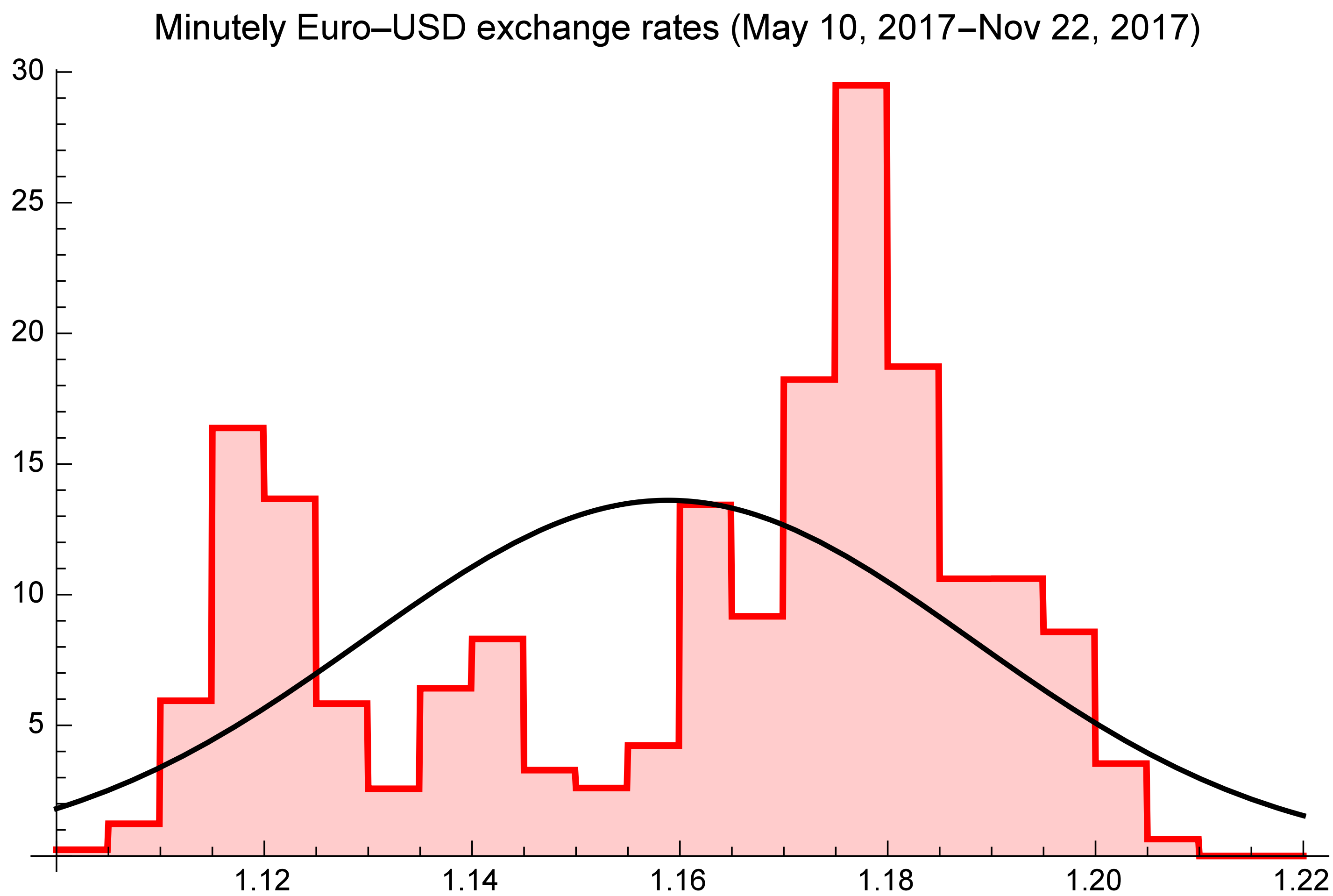
Distribution of the Daily Euro–USD Exchange Rate



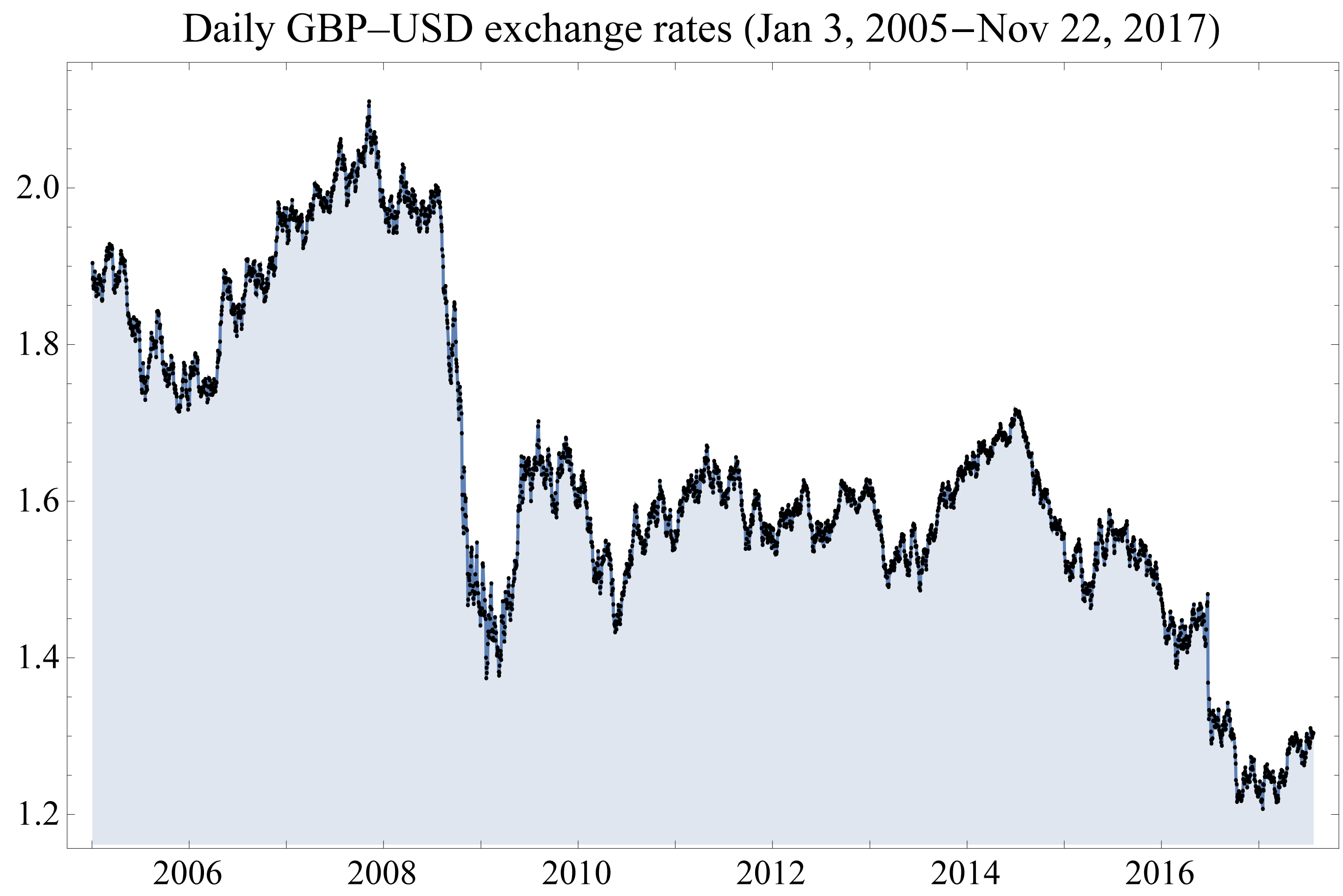
Time Series of the Minutely Euro–USD Exchange Rate



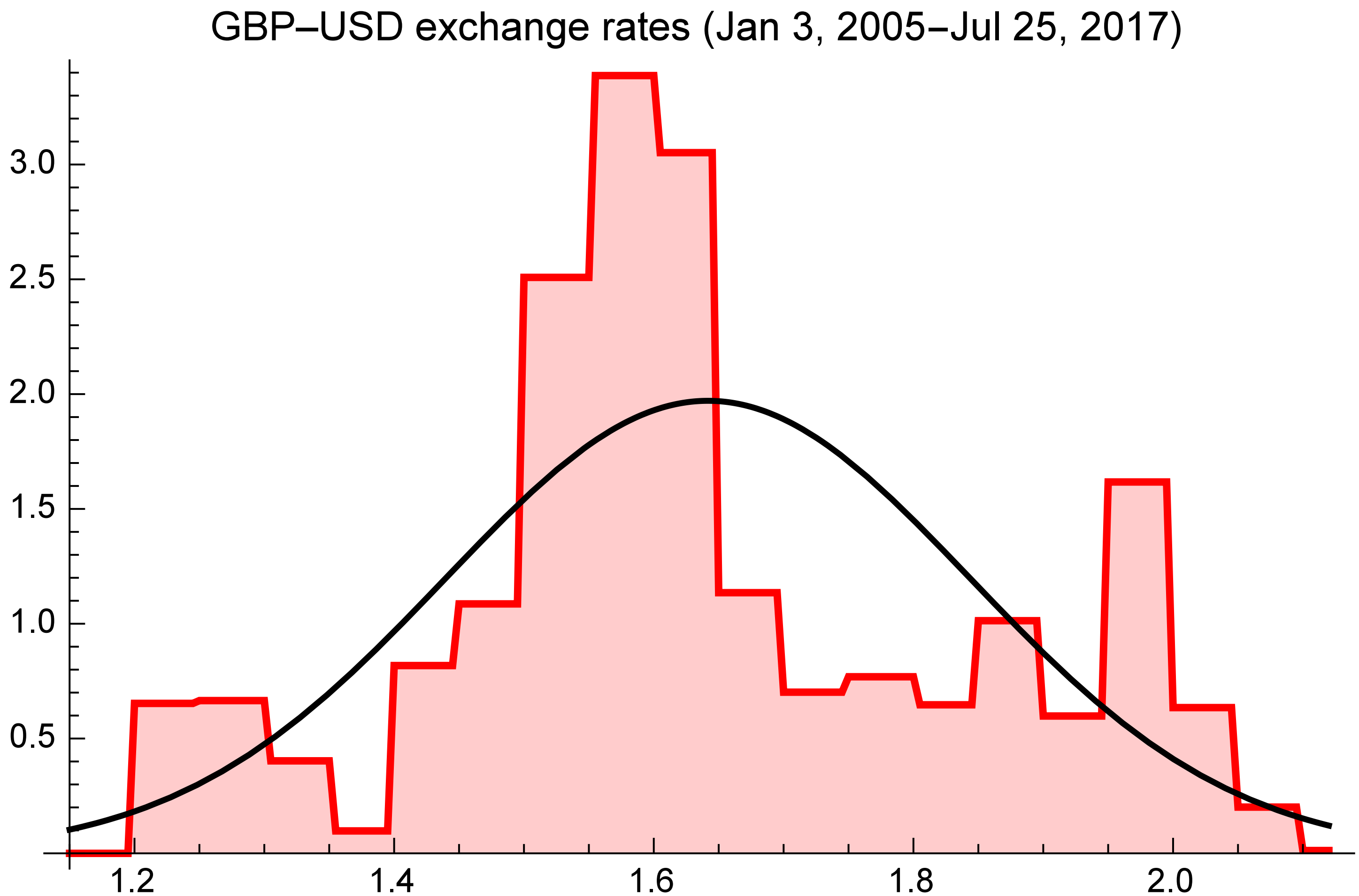
Distribution of the Minutely Euro–USD Exchange Rate



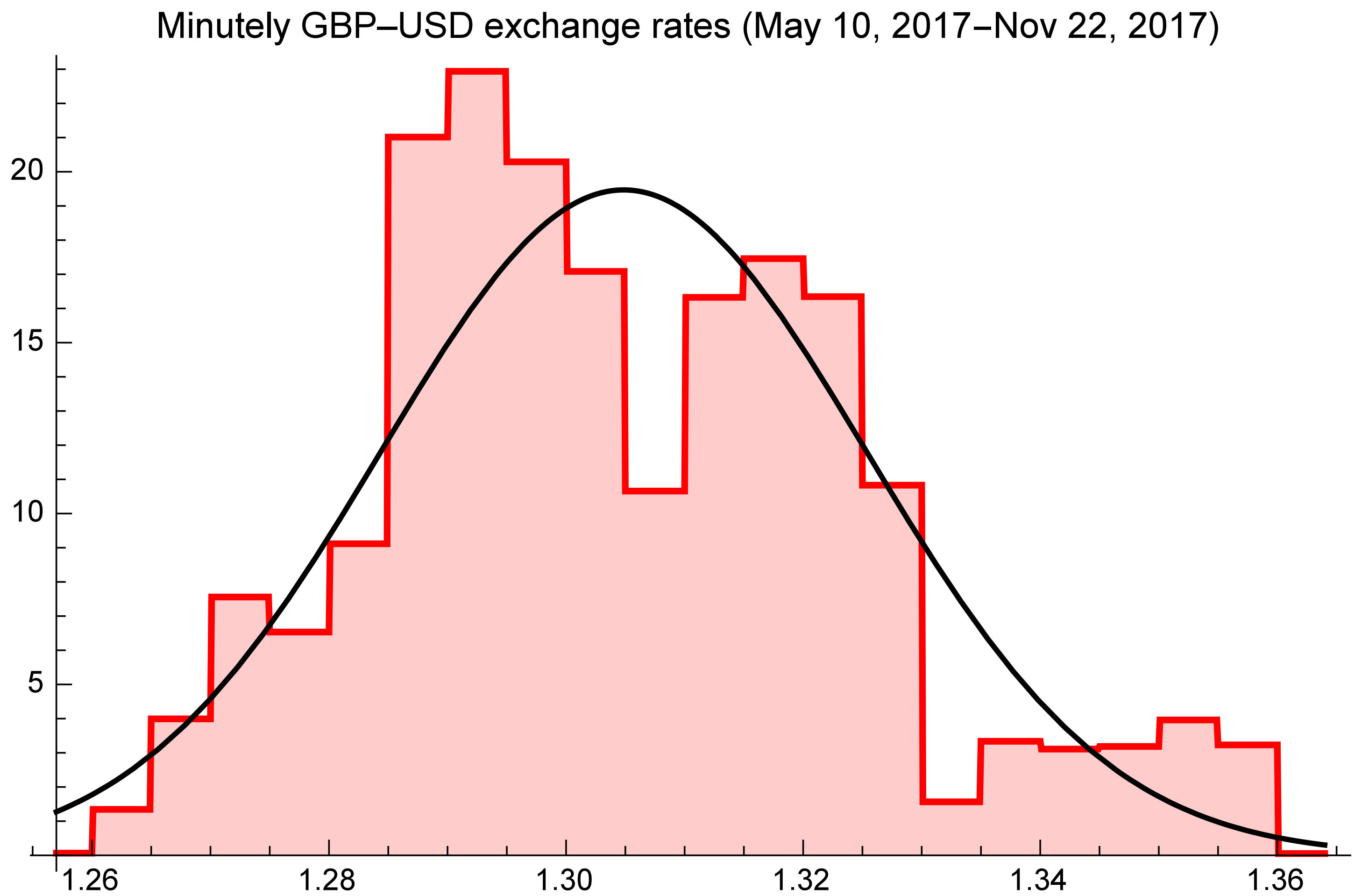
Time Series of the Daily GBP–USD Exchange Rate



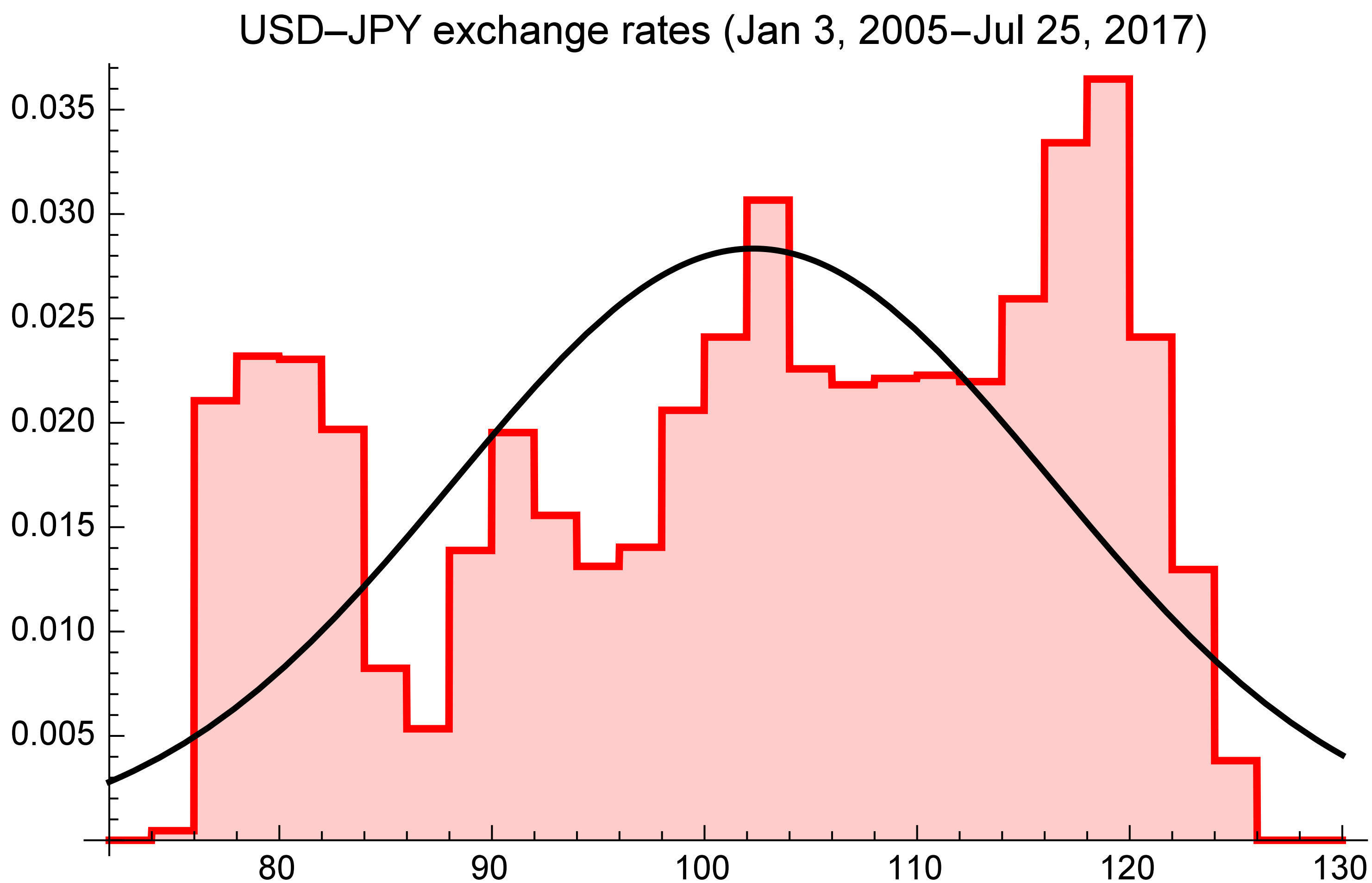
Distribution of the Daily GBP–USD Exchange Rate



Distribution of the Minutely GBP–USD Exchange Rate



Distribution of the Daily JPY–USD Exchange Rate



Foreign Exchange Options

- In 2000 the total notional volume of foreign exchange options was US$13 trillion.[a]
  - 38.5% were vanilla calls and puts with a maturity less than one month.
  - 52.5% were vanilla calls and puts with a maturity between one and 18 months.
  - 4% were barrier options.
  - 1.5% were vanilla calls and puts with a maturity more than 18 months.
  - 1% were digital options (see p. 849).
  - 0.7% were Asian options (see p. 426).

[a] Lipton (2002).

Foreign Exchange Options (continued)

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases

$$\frac{100{,}000{,}000}{6{,}250{,}000} = 16$$

  puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for

$$100{,}000{,}000 \times .0088 = 880{,}000$$

  U.S. dollars.

Foreign Exchange Options (concluded)

- Assume the exchange rate S is lognormally distributed.
- The formulas derived for stock index options in Eqs. (43) on p. 326 apply with the dividend yield equal to $\hat{r}$:

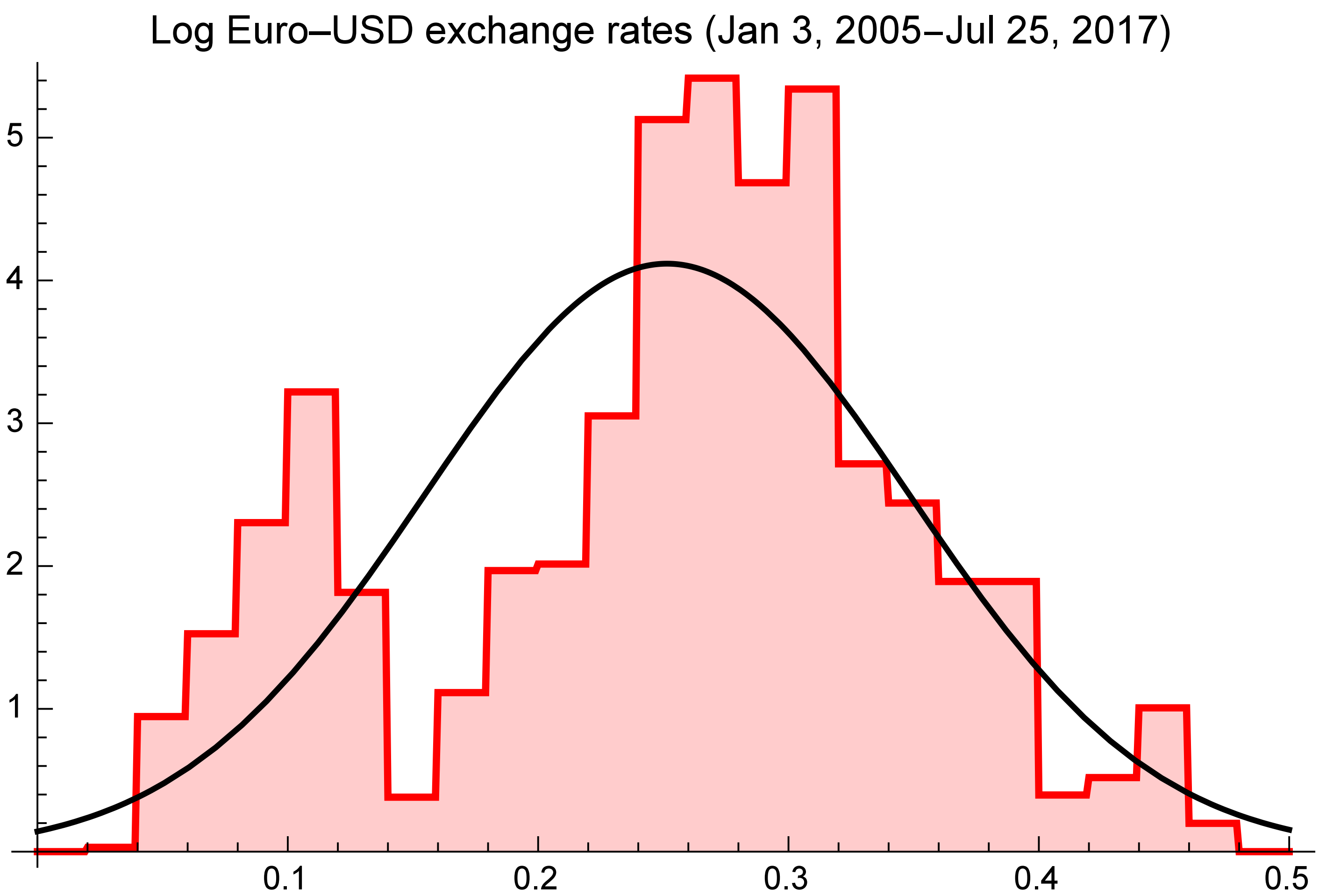
$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{54}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x). \tag{54'}$$

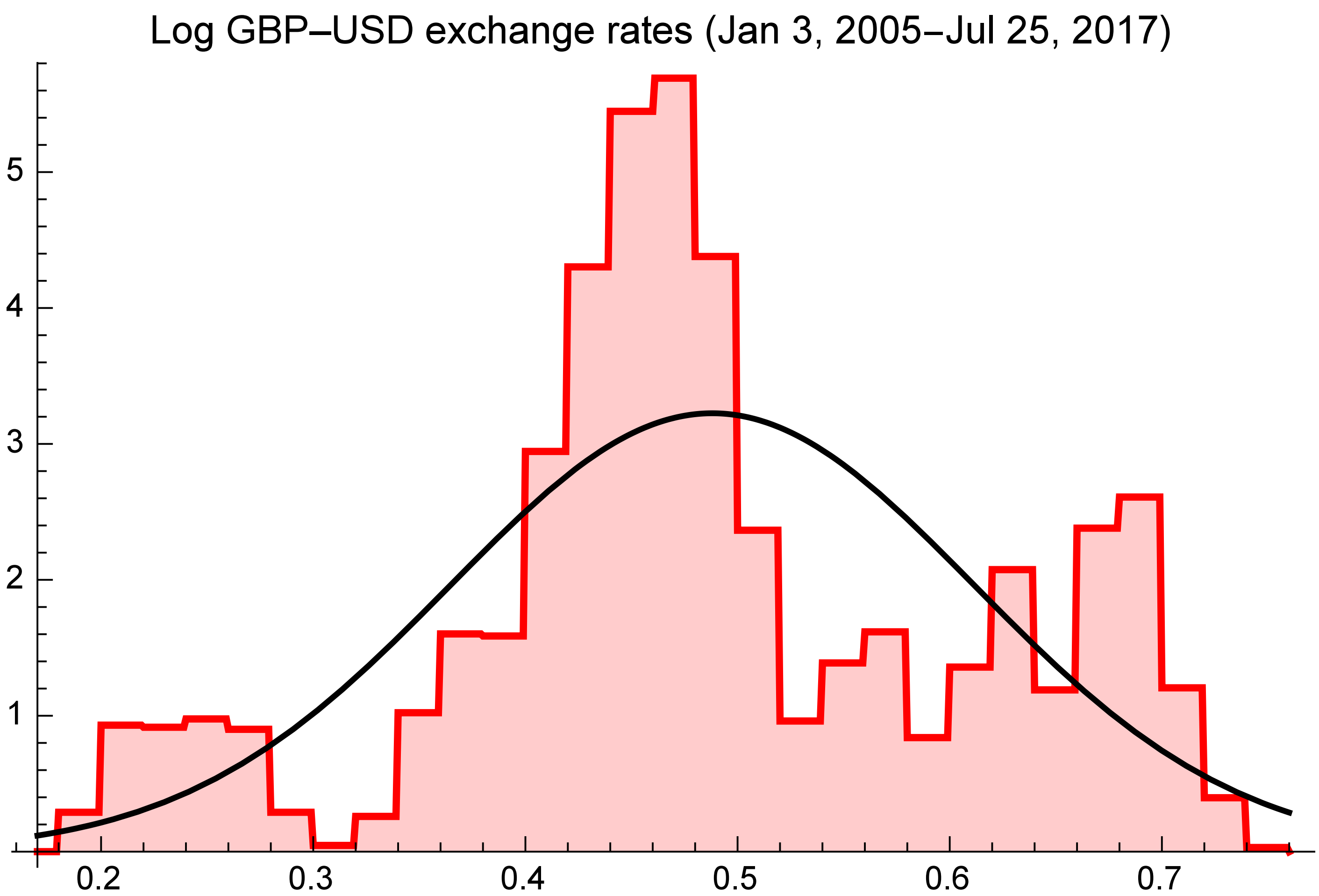
  – Above,

$$x \stackrel{\Delta}{=} \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

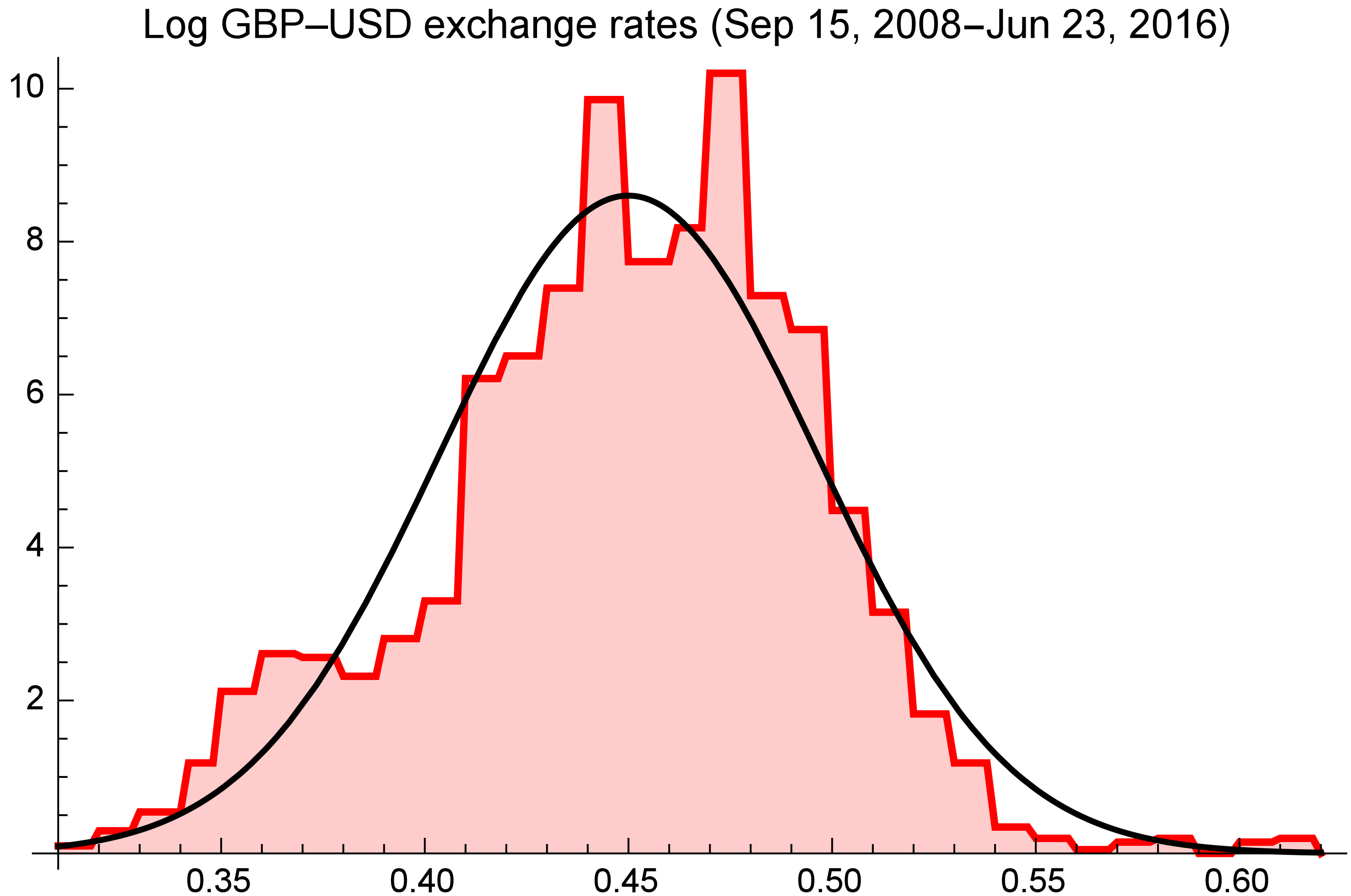
Distribution of the Logarithmic Euro–USD Exchange Rate



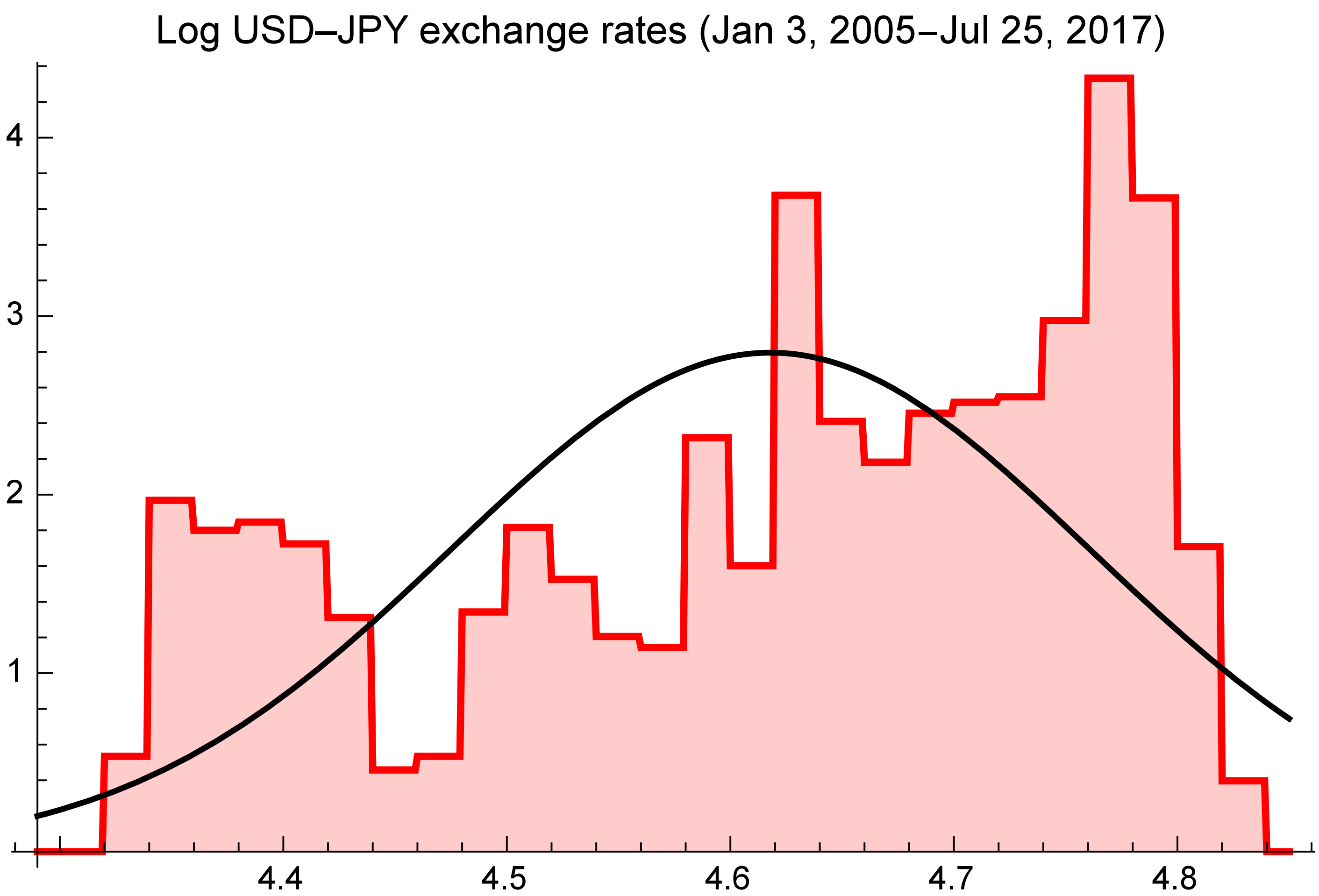
Distribution of the Logarithmic GBP–USD Exchange Rate



Distribution of the Logarithmic GBP–USD Exchange Rate (after the Collapse of Lehman Brothers and before Brexit)



Distribution of the Logarithmic JPY–USD Exchange Rate



Bar the roads!
Bar the paths!
Wert thou to flee from here, wert thou
to find all the roads of the world,
the way thou seekst
the path to that thou'dst find not[.]
— Richard Wagner (1813–1883), *Parsifal*

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends *critically* on the path.
- The (arithmetic) average-rate call has this terminal value:

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

- The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.[a]
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- The averaging clause is also common in convertible bonds and structured notes.

[a] As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen & Sandmann, 2003).

Path-Dependent Derivatives (continued)

- A lookback call option on the minimum has a terminal payoff of

$$S_n - \min_{0 \leq i \leq n} S_i.$$

- A lookback put on the maximum has a terminal payoff of

$$\max_{0 \leq i \leq n} S_i - S_n.$$

Path-Dependent Derivatives (concluded)

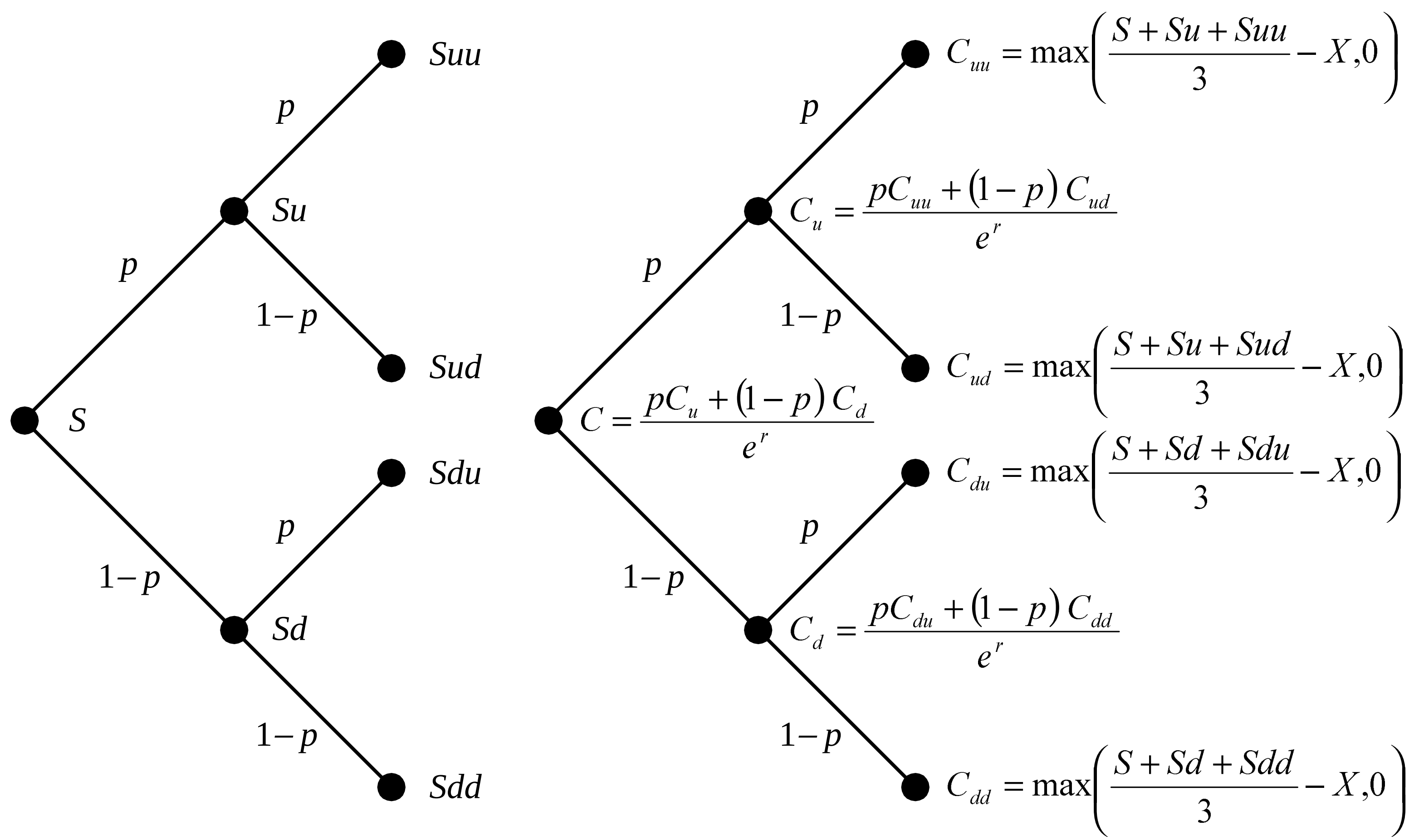
- The fixed-strike lookback option provides a payoff of
  - $\max(\max_{0 \leq i \leq n} S_i - X, 0)$ for the call.
  - $\max(X - \min_{0 \leq i \leq n} S_i, 0)$ for the put.
- Lookback calls and puts on the average (instead of a constant X) are called average-strike options.

Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine (see next page).
- A naive algorithm enumerates the 2^n paths for an n-period binomial tree and then averages the payoffs.
- But the complexity is exponential.[a]
- The Monte Carlo method[b] and approximation algorithms are some of the alternatives left.

[a] Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2007) reduce it to $2^{O(\sqrt{n})}$.

[b] See pp. 836ff.



Suu
p
Su
p
$1-p$
Sud
S
Sdu
p
$1-p$
Sd
$1-p$
Sdd
$C_{uu} = \max\left(\frac{S+Su+Suu}{3} - X, 0\right)$
$C_u = \frac{pC_{uu} + (1-p)\,C_{ud}}{e^r}$
$C_{ud} = \max\left(\frac{S+Su+Sud}{3} - X, 0\right)$
$C = \frac{pC_u + (1-p)\,C_d}{e^r}$
$C_{du} = \max\left(\frac{S+Sd+Sdu}{3} - X, 0\right)$
$C_d = \frac{pC_{du} + (1-p)\,C_{dd}}{e^r}$
$C_{dd} = \max\left(\frac{S+Sd+Sdd}{3} - X, 0\right)$

States and Their Transitions

- The tuple

$$(i, S, P)$$

captures the state[a] for the Asian option.

  - i: the time.
  - S: the prevailing stock price.
  - P: the running sum.[b]

---

[a] A "sufficient statistic," if you will.

[b] When the average is a moving average, a different technique is needed (C. Kao (`R89723057`) & Lyuu, 2003).

States and Their Transitions (concluded)

- For the binomial model, the state transition is:

$$
\begin{array}{ccl}
 & & (i+1, Su, P+Su), \quad \text{for the up move} \\
 & \nearrow & \\
(i, S, P) & & \\
 & \searrow & \\
 & & (i+1, Sd, P+Sd), \quad \text{for the down move}
\end{array}
$$

- This leads to an exponential-time algorithm.

Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
  - Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0 \right),$$

$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0 \right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas:[a]

$$C = Se^{-q_{\mathrm{a}}\tau} N(x) - Xe^{-r\tau} N(x - \sigma_{\mathrm{a}}\sqrt{\tau}), \tag{55}$$

$$P = Xe^{-r\tau} N(-x + \sigma_{\mathrm{a}}\sqrt{\tau}) - Se^{-q_{\mathrm{a}}\tau} N(-x), \tag{55'}$$

  - With the volatility set to $\sigma_{\mathrm{a}} \stackrel{\Delta}{=} \sigma/\sqrt{3}$.
  - With the dividend yield set to $q_{\mathrm{a}} \stackrel{\Delta}{=} (r + q + \sigma^2/6)/2$.
  - $x \stackrel{\Delta}{=} \frac{\ln(S/X) + \left(r - q_{\mathrm{a}} + \sigma_{\mathrm{a}}^2/2\right)\tau}{\sigma_{\mathrm{a}}\sqrt{\tau}}$.

[a] See Angus (1999), for example.

An Approximate Formula for Asian Calls[a]

$$C = e^{-r\tau} \left[\frac{S}{\tau} \int_0^\tau e^{\mu t + \sigma^2 t/2} N\left(\frac{-\gamma + (\sigma t/\tau)(\tau - t/2)}{\sqrt{\tau/3}} \right) dt \right.$$
$$\left. -XN\left(\frac{-\gamma}{\sqrt{\tau/3}} \right) \right],$$

where

- $\mu \stackrel{\Delta}{=} r - \sigma^2/2$.
- γ is the unique value that satisfies

$$\frac{S}{\tau} \int_0^\tau e^{3\gamma\sigma t(\tau - t/2)/\tau^2 + \mu t + \sigma^2 [t - (3t^2/\tau^3)(\tau - t/2)^2]/2}\, dt = X.$$

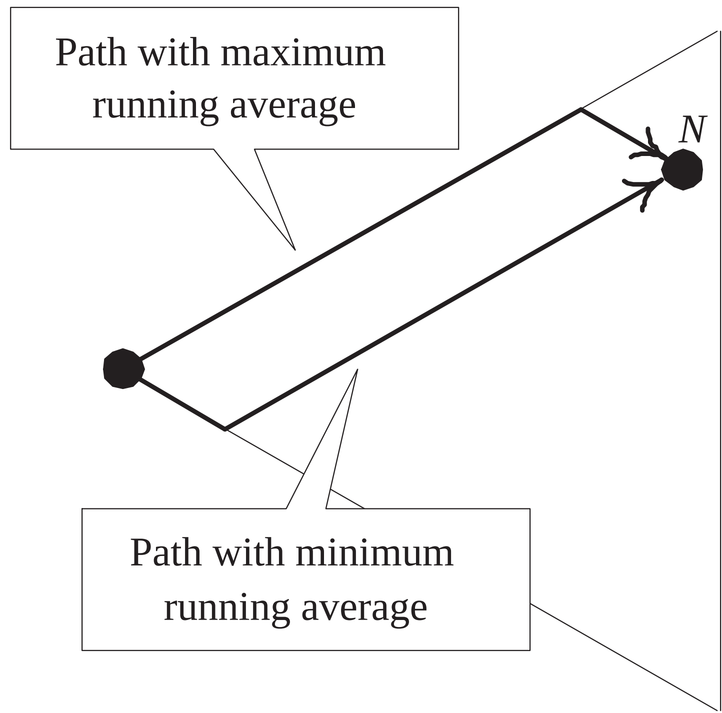
[a]Rogers & Shi (1995); Thompson (1999); K. Chen (`R92723061`) (2005); K. Chen (`R92723061`) & Lyuu (2006).

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j,i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i}d + \cdots + u^{j-i}d^i}^{j})$$

$$= S_0 \frac{1 - u^{j-i+1}}{1-u} + S_0 u^{j-i} d \, \frac{1-d^i}{1-d}.$$


Path with maximum
running average
N
Path with minimum
running average

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j + 1$ and call it $A_{\max}(j, i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1 + \overbrace{d + d^2 + \cdots + d^i + d^i u + \cdots + d^i u^{j-i}}^{j})$$
$$= S_0 \, \frac{1 - d^{i+1}}{1 - d} + S_0 d^i u \, \frac{1 - u^{j-i}}{1 - u}.$$

- Divide this value by $j + 1$ and call it $A_{\min}(j, i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.

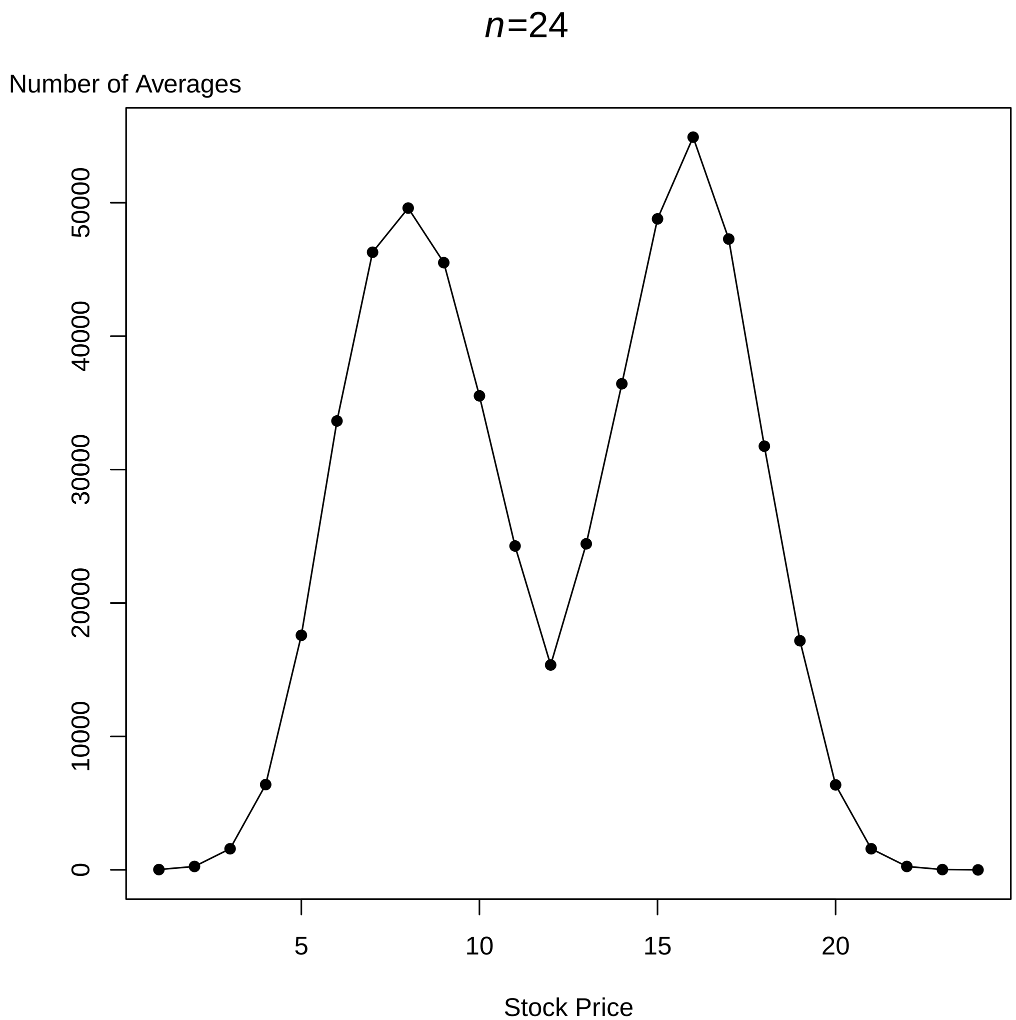
Approximation Algorithm for Asian Options (continued)

- The number of paths to $N(j, i)$ are far too many: $\binom{j}{i}$.
  - For example,

$$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)}\,.$$

- The number of distinct running averages for the nodes at any given time step n seems to be bimodal for n big enough.[a]
  - In the plot on the next page, $u = 5/4$ and $d = 4/5$.

[a]Contributed by Mr. Liu, Jun (R99944027) on April 15, 2014.



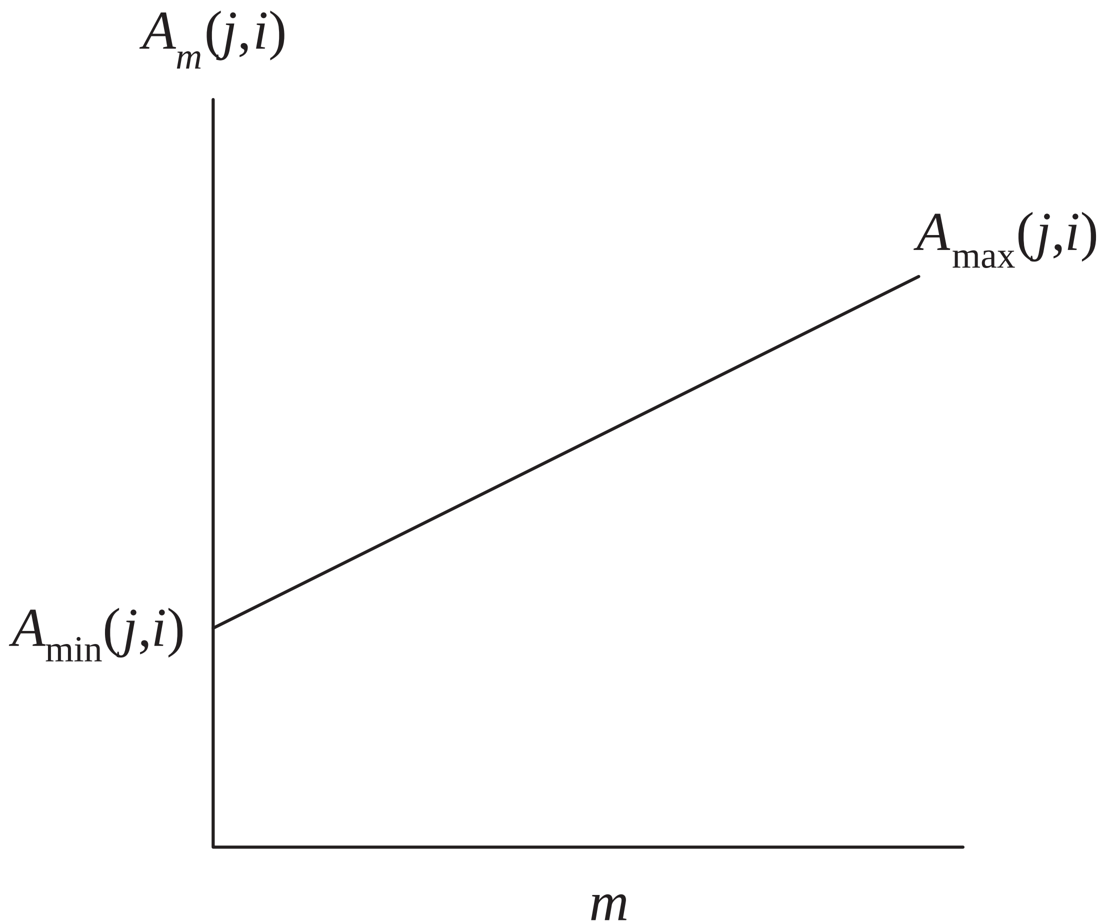
n=24
Number of Averages
0
10000
20000
30000
40000
50000
5
10
15
20
Stock Price

Approximation Algorithm for Asian Options (continued)

- But all averages must lie between $A_{\min}(j,i)$ and $A_{\max}(j,i)$.
- Pick $k+1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j,i) \triangleq \left(\frac{k-m}{k}\right) A_{\min}(j,i) + \left(\frac{m}{k}\right) A_{\max}(j,i)$$

for $m = 0, 1, \ldots, k$.



$A_m(j,i)$
$A_{\max}(j,i)$
$A_{\min}(j,i)$
m

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" introduces errors, but it works reasonably well in practice.[a]
- A better alternative picks values whose logarithms are equally spaced.[b]
- Still other alternatives are possible (considering the distribution of averages on p. 440).

---

[a] Hull & White (1993); Ritchken, Sankarasubramanian, & Vijh (1993).
[b] Called log-linear interpolation.

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.
- Suppose the current node is $N(j, i)$ and the running average is a.
- Assume the next node is $N(j+1, i)$, after an up move.
- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the new running average

$$A_{\mathrm{u}} \triangleq \frac{(j+1)\, a + S_0 u^{j+1-i} d^i}{j+2}.$$

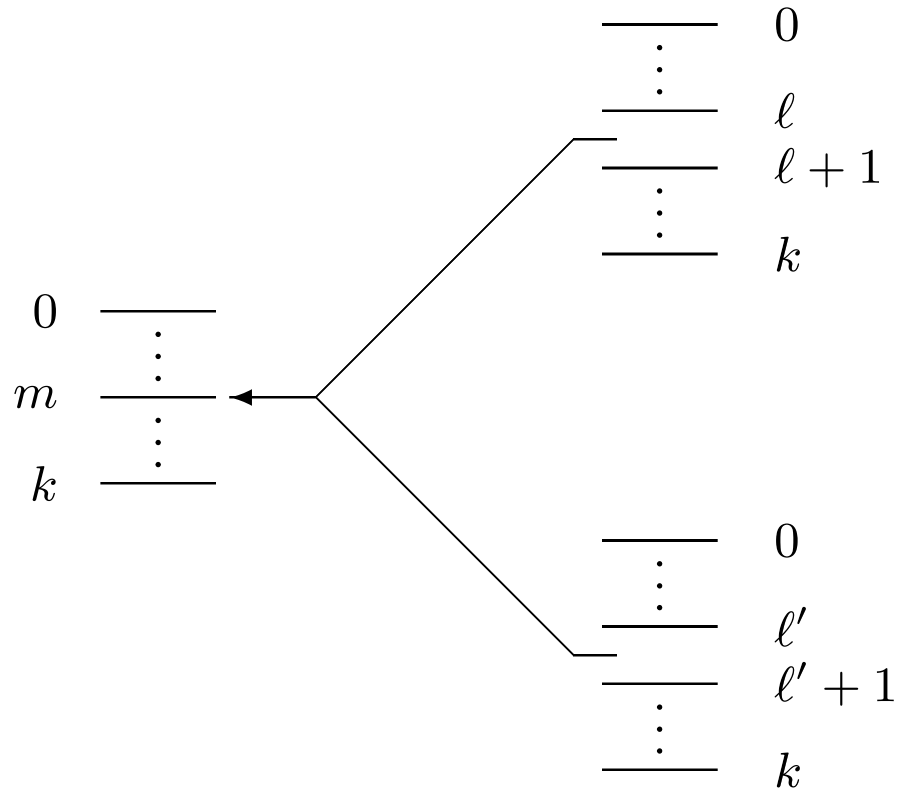
Approximation Algorithm for Asian Options (continued)

- But A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1,i)$!
- Find the 2 running averages that bracket it:

$$A_{\ell}(j+1,i) \leq A_{\mathrm{u}} < A_{\ell+1}(j+1,i).$$

- In "most" cases, the fastest way to nail ℓ is via

$$\ell = \left\lfloor \frac{A_{\mathrm{u}} - A_{\min}(j+1,i)}{[\,A_{\max}(j+1,i) - A_{\min}(j+1,i)\,]/k} \right\rfloor.$$


0
ℓ
$\ell + 1$
k
0
m
k
0
ℓ'
$\ell' + 1$
k

Approximation Algorithm for Asian Options (continued)

- But watch out for the rare case where

$$A_{\mathrm{u}} = A_\ell(j+1, i)$$

for some ℓ.

- Also watch out for the case where

$$A_{\mathrm{u}} = A_{\max}(j, i).$$

- Finally, watch out for the degenerate case where $A_0(j+1, i) = \cdots = A_k(j+1, i)$.
  - It will happen along extreme paths!

Approximation Algorithm for Asian Options (continued)

- Express A_{u} as a linearly interpolated value of the two running averages,

$$A_{\mathrm{u}} = x A_{\ell}(j+1, i) + (1-x)\, A_{\ell+1}(j+1, i), \quad 0 < x \leq 1.$$

- Obtain the approximate option value given the running average A_{u} via

$$C_{\mathrm{u}} \stackrel{\Delta}{=} x C_{\ell}(j+1, i) + (1-x)\, C_{\ell+1}(j+1, i).$$

  - $C_{\ell}(t, s)$ denotes the option value at node $N(t, s)$ with running average $A_{\ell}(t, s)$.

- This interpolation introduces the second source of error.

Approximation Algorithm for Asian Options (continued)

- The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value C_{d}.
- Finally obtain the option value as

$$[\, pC_{\mathrm{u}} + (1-p)\, C_{\mathrm{d}}\,]\, e^{-r\Delta t}.$$

- The running time is $O(kn^2)$.
  - There are $O(n^2)$ nodes.
  - Each node has $O(k)$ buckets.

Approximation Algorithm for Asian Options (continued)

- For the calculations at time step $n-1$, no interpolation is needed.[a]
  - The option values are simply (for calls):

$$\begin{aligned} C_{\mathrm{u}} &= \max(A_{\mathrm{u}} - X, 0), \\ C_{\mathrm{d}} &= \max(A_{\mathrm{d}} - X, 0). \end{aligned}$$

  - That saves $O(nk)$ calculations.

[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on April 9, 2014.

Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: no historical average to deal with.
- This problem can be easily addressed.[a]
- How about the Greeks?[b]

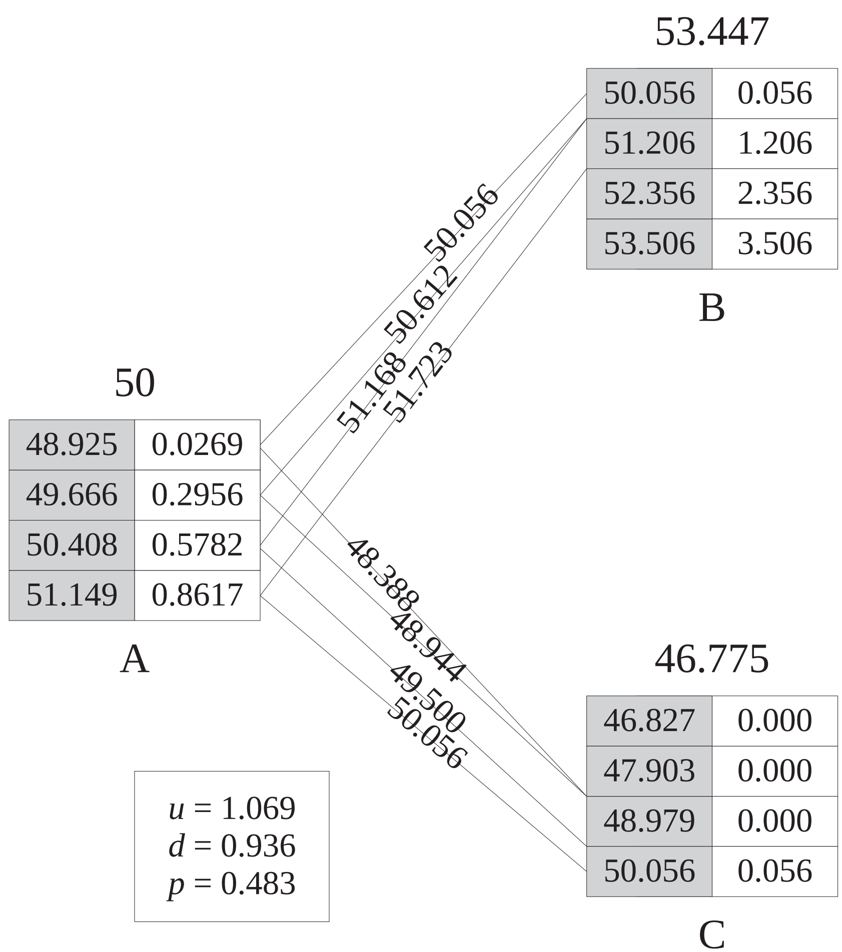
---

[a]See Exercise 11.7.4 of the textbook.

[b]Thanks to lively class discussions on March 31, 2004, and April 9, 2014.

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 453 is 48.925.
- The maximum running average at node A in the same figure is 51.149.


53.447
50.056 0.056
51.206 1.206
52.356 2.356
53.506 3.506
B
50.056
50.612
51.168
51.723
50
48.925 0.0269
49.666 0.2956
50.408 0.5782
51.149 0.8617
A
48.388
48.944
49.500
50.056
46.775
46.827 0.000
47.903 0.000
48.979 0.000
50.056 0.056
C
u = 1.069
d = 0.936
p = 0.483

A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option value corresponding to running average 50.056 at node B is 0.056.
- The option values corresponding to running average 51.206 at node B is 1.206.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be
$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$
- With 48.944 lying between 47.903 and 48.979 at node C, we solve
$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$
to obtain $x \approx 0.033$.

A Numerical Example (concluded)

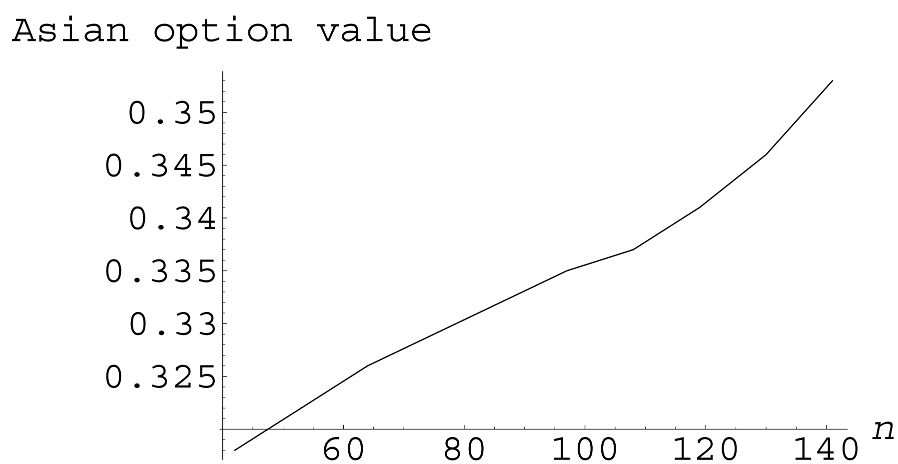
- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1 - p) \times 0.0 \approx 0.2956,$$

where $p = 0.483$.

- The remaining three option values at node A can be computed similarly.

Convergence Behavior of the Approximation Algorithm with $k = 50000$[a]



[a]Dai (B82506025, R86526008, D8852600) & Lyuu (2002).