Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.[a]
- Need to calibrate the BOPM's parameters u, d, and R to make it converge to the continuous-time model.
- We now skim through the proof.

[a] Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- First, $\hat{r} = r\tau/n$.
  - Each period is τ/n years long.
  - The period gross return $R = e^{\hat{r}}$.
- Let
$$\widehat{\mu} \triangleq \frac{1}{n}\, E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period of the BOPM.
- Let
$$\widehat{\sigma}^2 \triangleq \frac{1}{n}\, \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that[a]

$$\widehat{\mu} = q \ln(u/d) + \ln d,$$
$$\widehat{\sigma}^2 = q(1-q) \ln^2(u/d).$$

- Assume the stock's *true* continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

[a] Recall the Bernoulli distribution.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$n\widehat{\mu} = n[\,q\ln(u/d) + \ln d\,] \to \mu\tau, \tag{40}$$

$$n\widehat{\sigma}^2 = nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \tag{41}$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 278).
  - Other choices are possible (see text).

- Exact solutions for u, d, q are feasible if Eqs. (40)–(41) are replaced by equations: 3 equations for 3 variables.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}. \qquad (42)$$

- With Eqs. (42), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

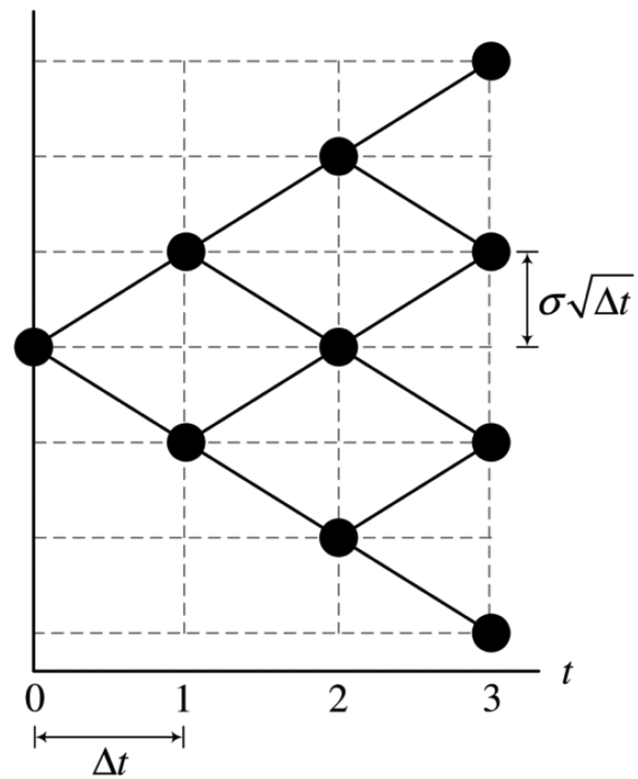
Toward the Black-Scholes Formula (continued)

- The choices (42) result in the CRR binomial model.[a]
- With the above choice, even if σ is not calibrated correctly, the mean is still matched![b]

---

[a]Cox, Ross, & Rubinstein (1979).

[b]Recall Eq. (35) on p. 250. So u and d are related to volatility exclusively in the CRR model. They do not depend on r.


σ√Δt
t
0
1
2
3
Δt

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (42) on p. 289 or Eq. (34) on p. 249.
  - If this happens, the probabilities lie outside $[\,0, 1\,]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

  i.e., when $n > r^2\tau/\sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions can be found in the textbook[b] or will be presented later.

---

[a] Many papers and programs forget to check this condition!

[b] See Exercise 9.3.1 of the textbook.

Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

[a] The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.

Toward the Black-Scholes Formula (continued)

Lemma 9 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \stackrel{\Delta}{=} (e^{r\tau/n}-d)/(u-d).$$

- Let $n \to \infty$.[a]
- Then $\mu = r - \sigma^2/2$.

[a]See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return[b] is thus the riskless rate r.

[a]By Lemma 9 (p. 294) and Eq. (30) on p. 179.

[b]In the sense of $(1/\tau) \ln E[\, S_\tau / S \,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S) \,]$ (geometric average rate of return). In the latter case, it would be $r - \sigma^2/2$ by Lemma 9.

Toward the Black-Scholes Formula (continued)[a]

Theorem 10 (The Black-Scholes Formula, 1973)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

$$x \stackrel{\Delta}{=} \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

[a] On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

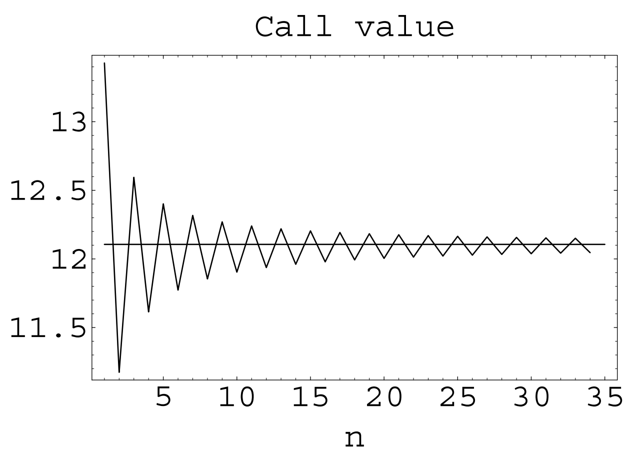
Toward the Black-Scholes Formula (concluded)

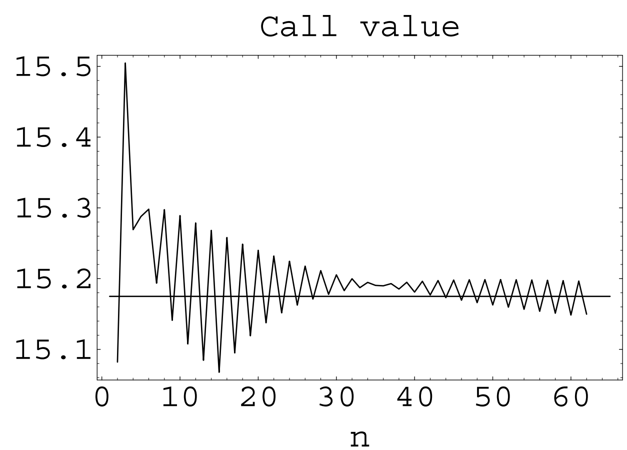
- See Eq. (39) on p. 266 for the meaning of x.
- See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$\begin{aligned} u &= e^{\sigma\sqrt{\tau/n}}, \\ d &= e^{-\sigma\sqrt{\tau/n}}, \\ \hat{r} &= r\tau/n. \end{aligned}$$





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of u and d.[b]

---

[a] L. Chang & Palmer (2007).

[b] See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?[b]

---

[a]Implied volatility is hard to compute when τ is small (why?).

[b]OptionMetrics's (2015) IvyDB uses the CRR binomial tree (see `http://www.ckgsb.com/uploads/report/file/201611/02/1478069847635278.pdf`).

Implied Volatility (concluded)

- Implied volatility is

  > the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Just think of it as an alternative to quoting option prices.
- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

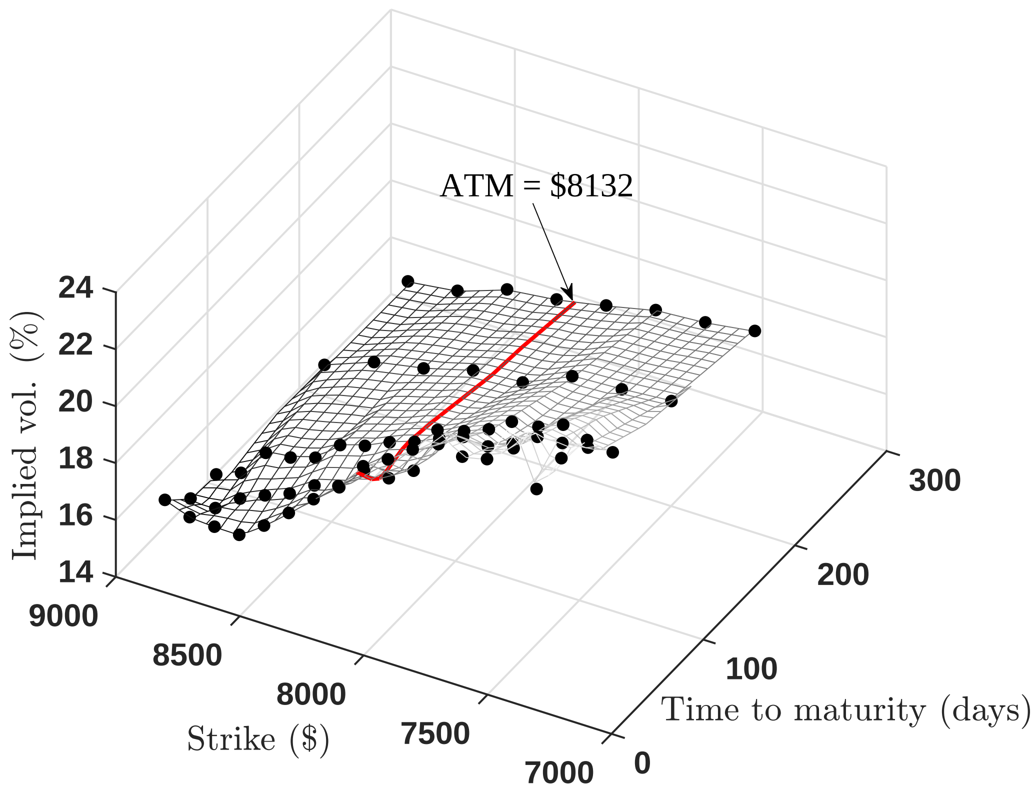
---

[a] Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

TXO Calls (September 25, 2015)[a]



[a]The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

Solutions to the Smile

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

$$\max(0, X - Su^j d^{n-j})$$

  and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the *continuation value*.
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Instantaneous Volatility[a]

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ.
- In the limit, the variance of $\ln(S_\tau/S)$ is

$$\int_0^\tau \sigma^2(t)\,dt$$

rather than $\sigma^2\tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\frac{\int_0^\tau \sigma^2(t)\,dt}{\tau}}\,.$$

[a]Merton (1973).

Time-Dependent Instantaneous Volatility (concluded)

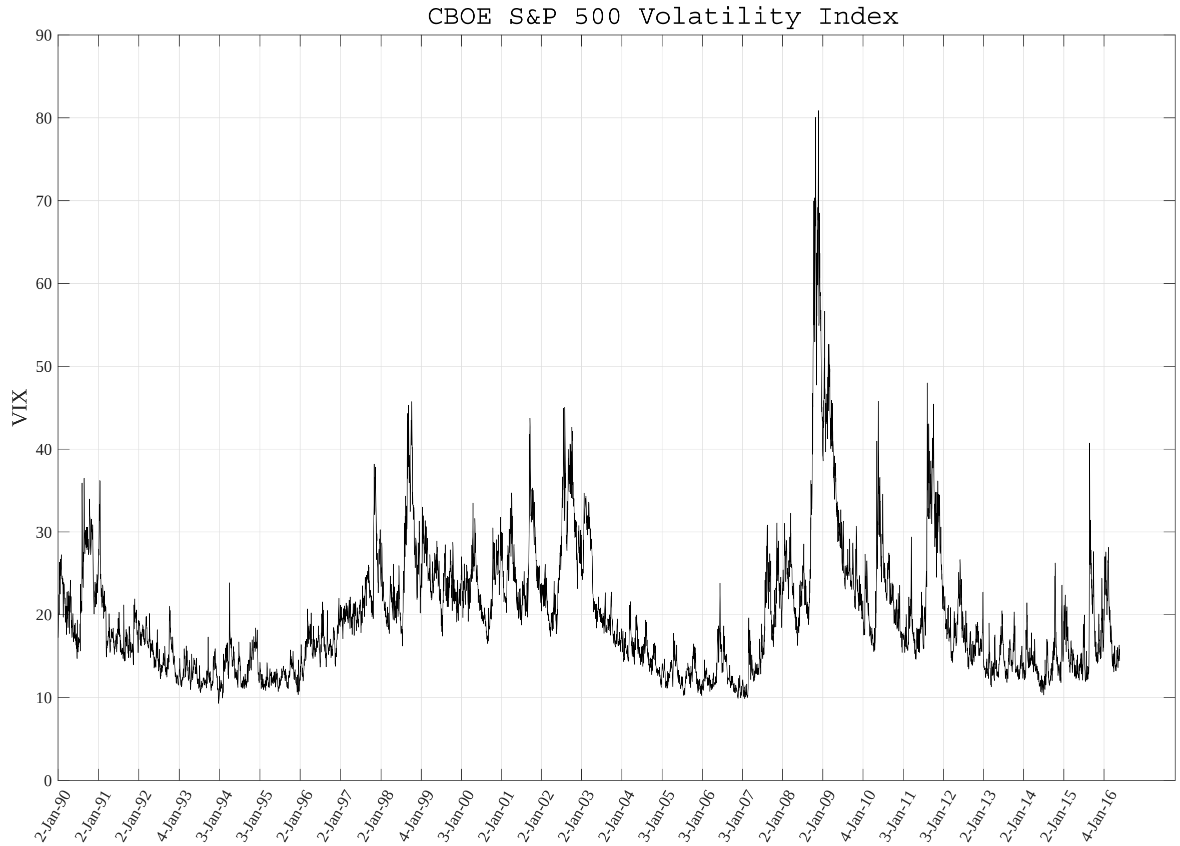
- For the binomial model, u and d depend on time:

$$u = e^{\sigma(t)\sqrt{\tau/n}},$$
$$d = e^{-\sigma(t)\sqrt{\tau/n}}.$$

- But how to make the binomial tree combine?[a]

[a] Amin (1991); C. I. Chen (R98922127) (2011).

Volatility (1990–2016)[a]



[a]Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.
- The annual riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where r_i is the continuously compounded short rate measured in periods for period i.[a]

- Will the binomial tree fail to combine?

[a]That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a] Fama (1965); K. French (1980); K. French & Roll (1986).

[b] Recall p. 162 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

[a]D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

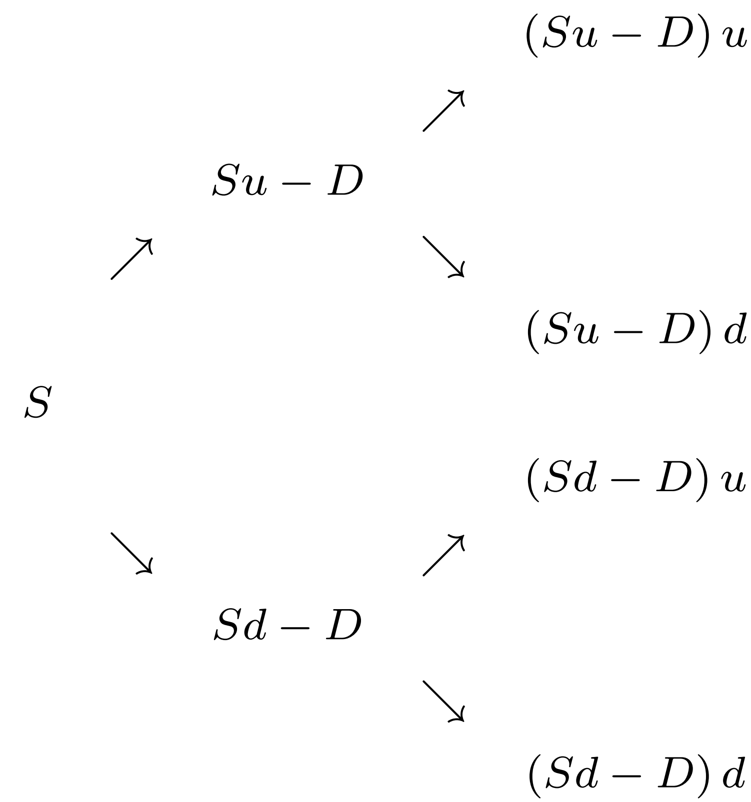
[a] Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.


$(Su - D)\,u$
$Su - D$
$(Su - D)\,d$
S
$(Sd - D)\,u$
$Sd - D$
$(Sd - D)\,d$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

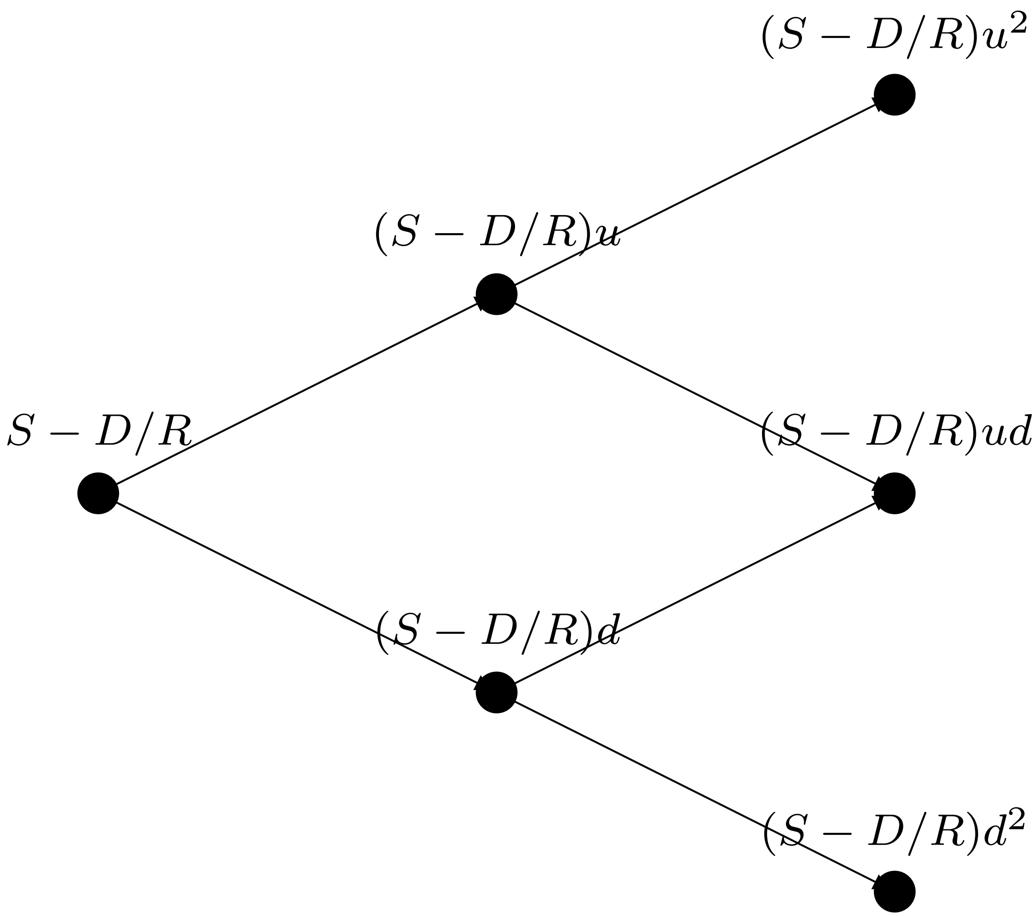
---

[a]Roll (1977); Heath & Jarrow (1988).

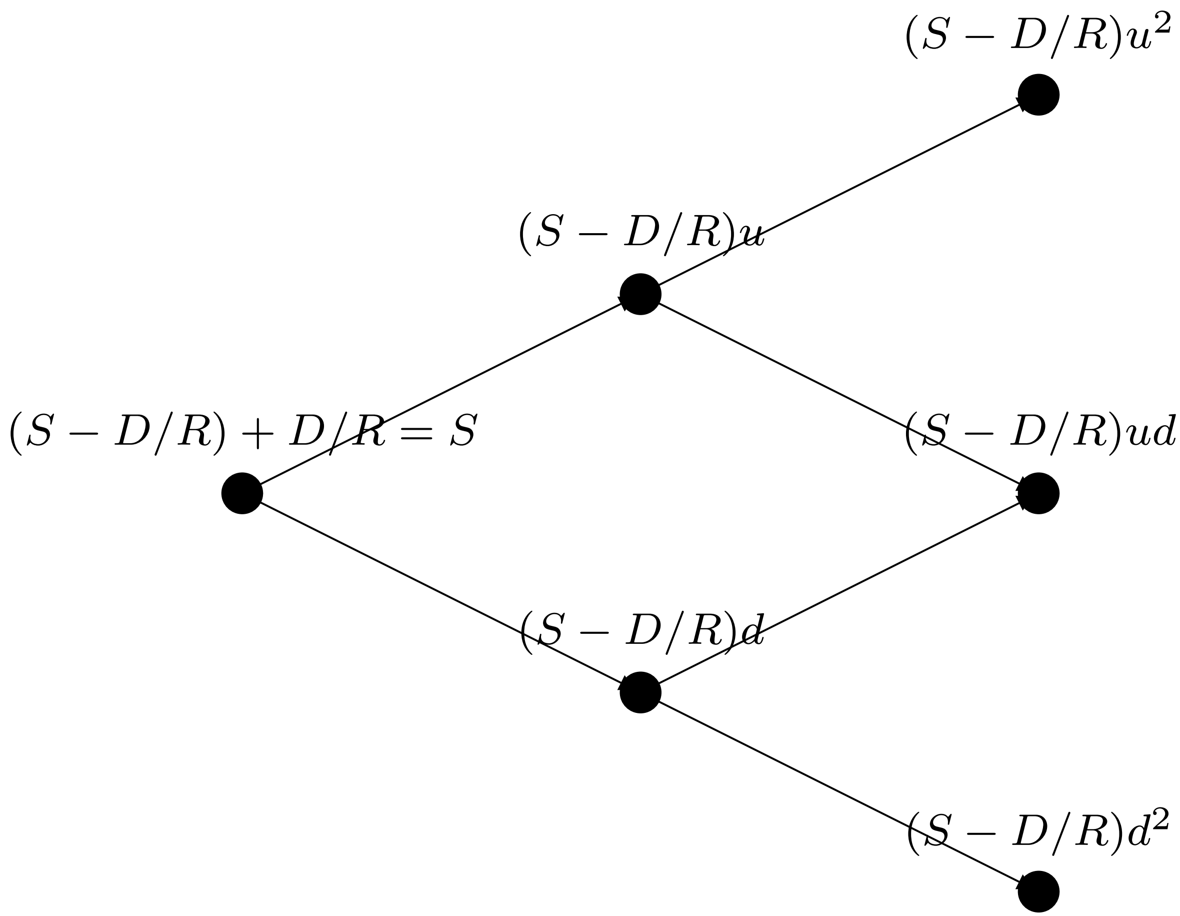
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 317 (Step 1)



The Ad-Hoc Approximation vs. P. 317 (Step 2)



The Ad-Hoc Approximation vs. P. 317[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\,u, (Su - D)\,d, (Sd - D)\,u, (Sd - D)\,d$ (p. 317).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned} (Su - D)\,u &> (S - D/R)u^2, \\ (Sd - D)\,d &< (S - D/R)d^2, \end{aligned}$$

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 317 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually increased when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 769ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

---

[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

[b] Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{43}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{43$'$}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (43) and (43′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \triangleq \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

[a]Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

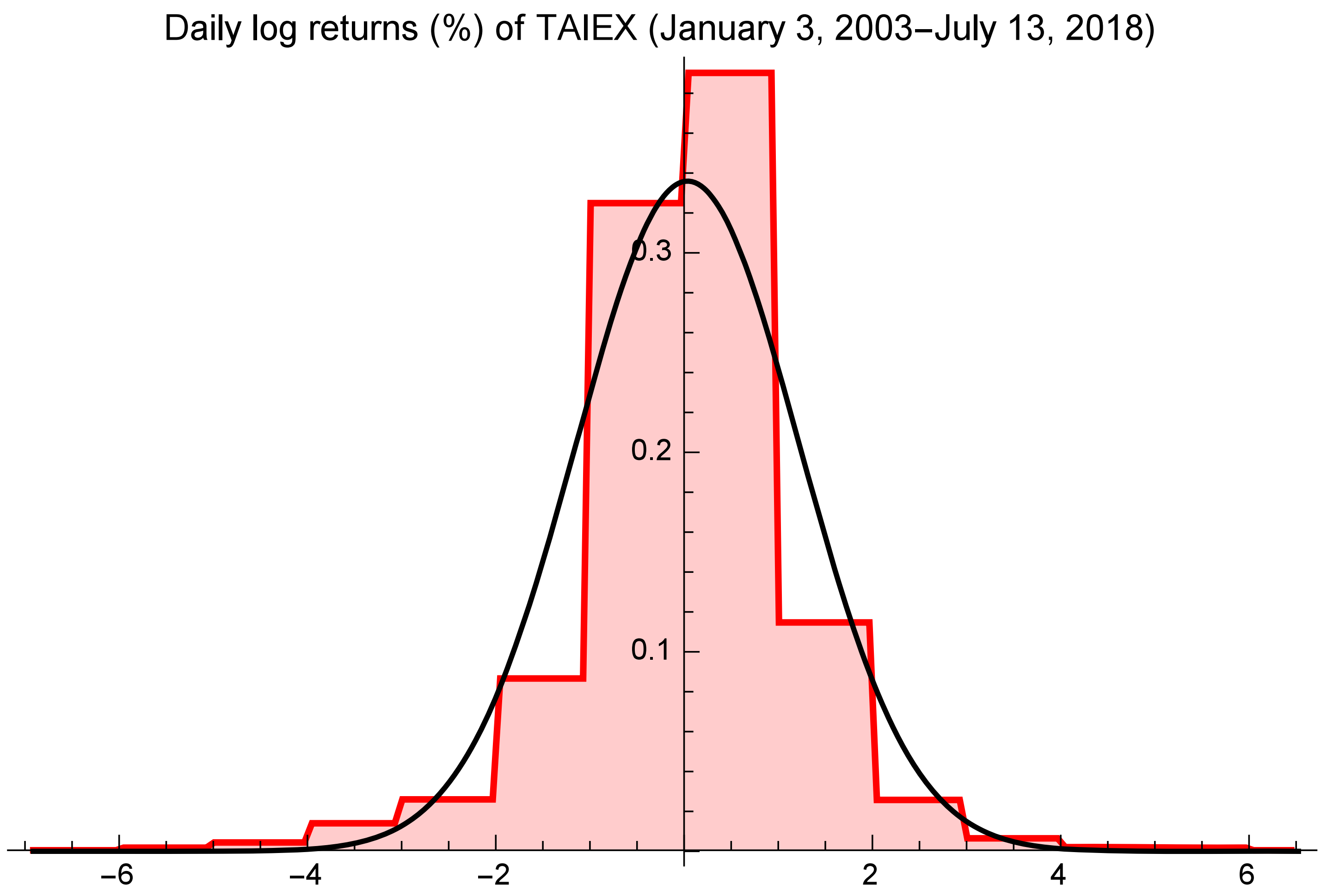
$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{44}$$

where $\Delta t \triangleq \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Other than the change in Eq. (44), binomial tree algorithms stay the same *as if there were no dividends.*

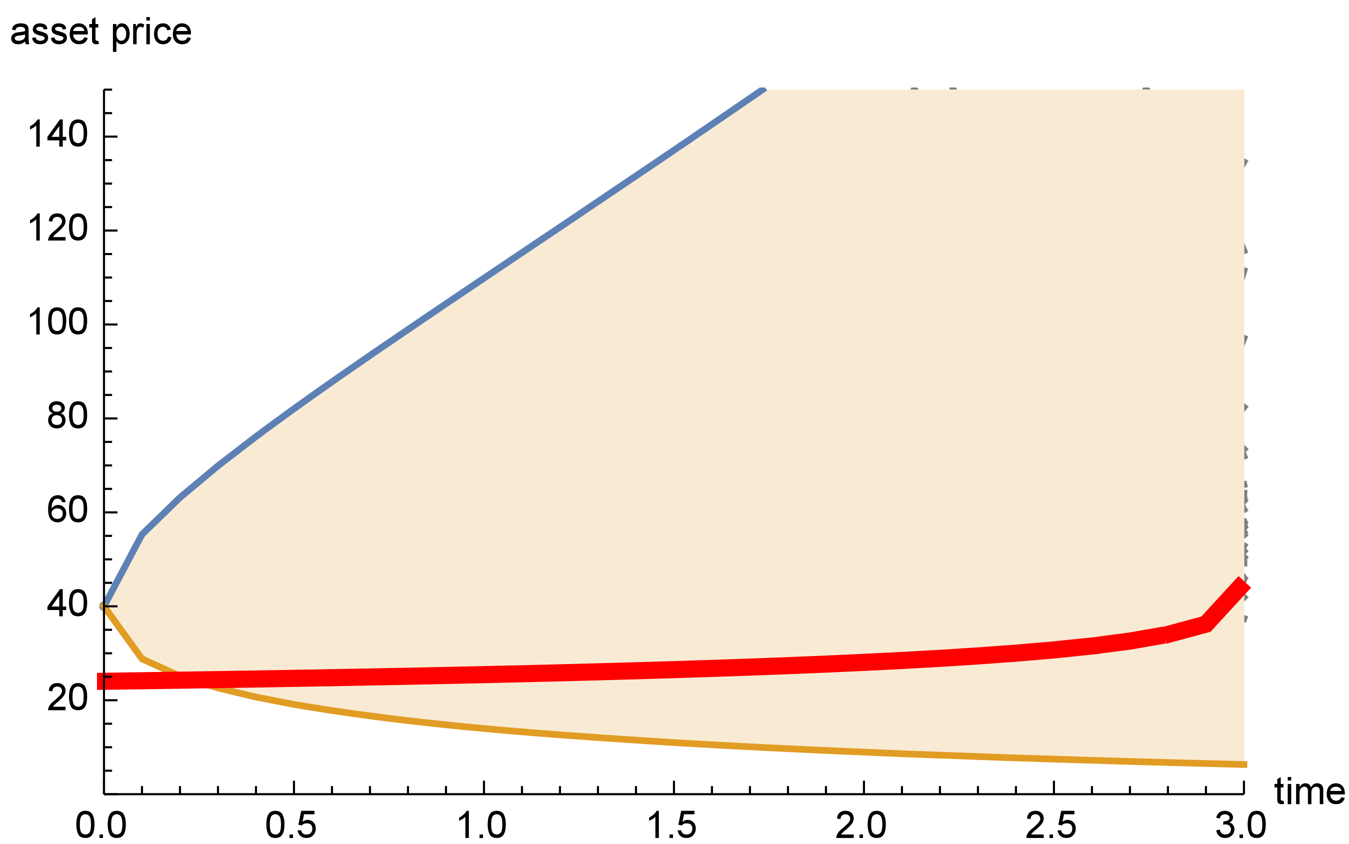
Distribution of Logarithmic Returns of TAIEX



Exercise Boundaries of American Options (in the Continuous-Time Model)[a]

- The exercise boundary is a nondecreasing function of t for American puts (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.

[a]See Section 9.7 of the textbook for the tree analog.


asset price
140
120
100
80
60
40
20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
time

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \stackrel{\Delta}{=} \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 296).
- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

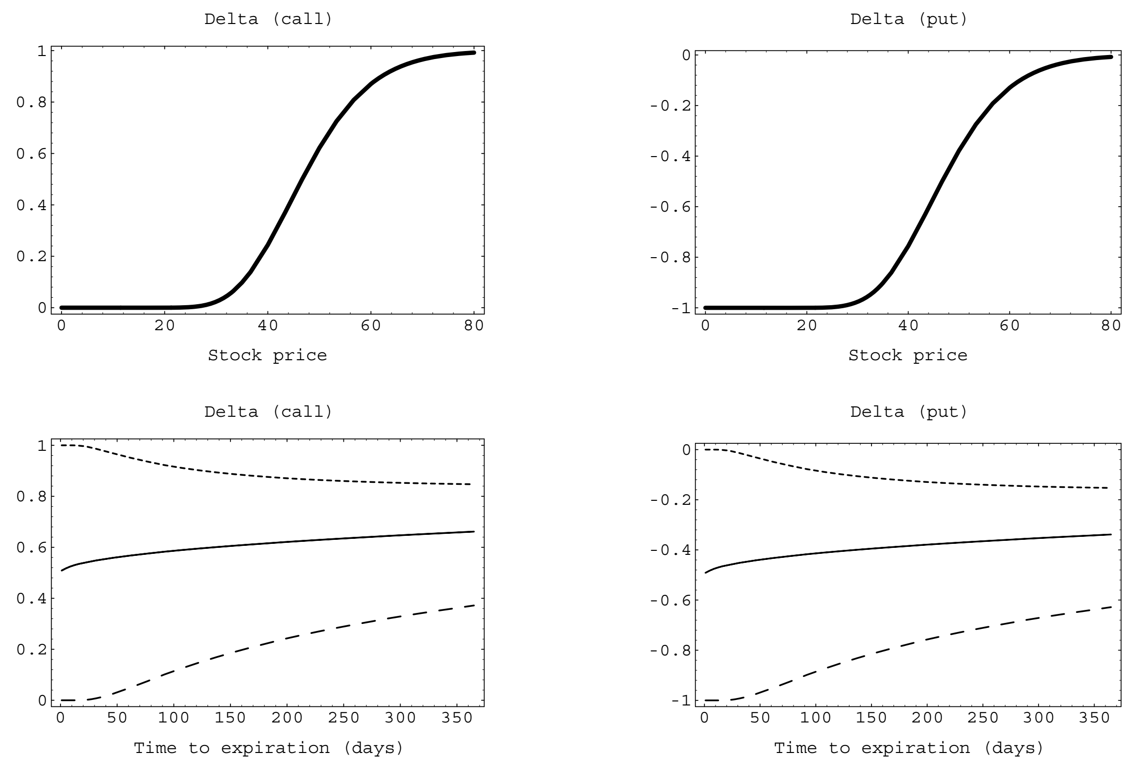
- Defined as
$$\Delta \triangleq \frac{\partial f}{\partial S}.$$
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM (p. 243) is the discrete analog.
- The delta of a long stock is apparently 1.

[a]Elementary calculus.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.$$
- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]
$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{45}$$

[a]The straddle (p. 210) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{46}$$

(recall p. 326).

- Then

$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{47}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r-2q+\sigma^2)\,\tau}. \tag{48}$$

---

[a]The strangle (p. 212) $C + P$ then has zero delta!

Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then
$$X = Se^{(r-q+\sigma^2/2)\,\tau}$$
leads to a straddle with zero delta.
  - This generalizes Eq. (45) on p. 336.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

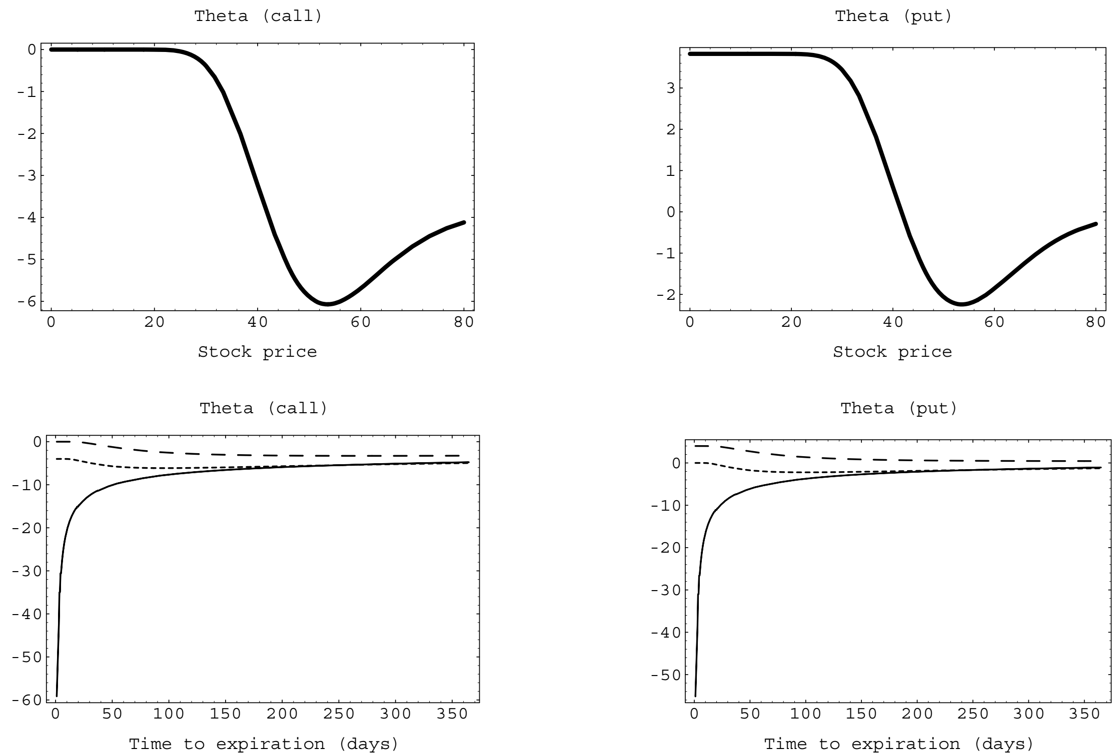
- Defined as the rate of change of a security's value with respect to time, or $\Theta \stackrel{\Delta}{=} -\partial f/\partial\tau = \partial f/\partial t$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x-\sigma\sqrt{\tau}) < 0.$$

  - The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x+\sigma\sqrt{\tau}).$$

  - Can be negative or positive.



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q.
- Define x as in Eq. (46) on p. 338.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) + qSe^{-q\tau}\,N(x).$$

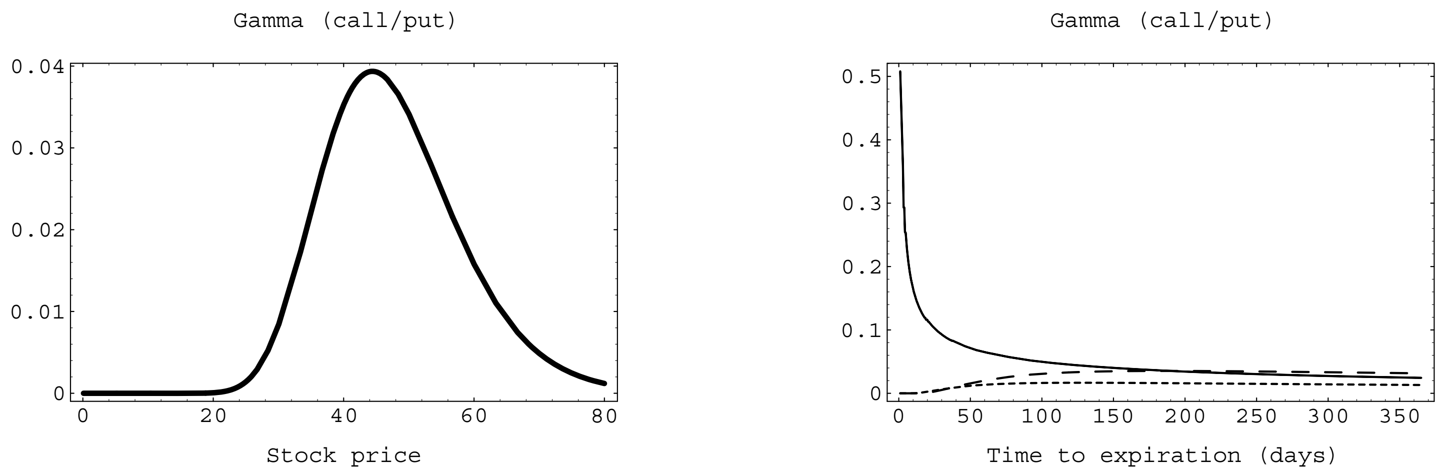
- For a European put, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - qSe^{-q\tau}\,N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \stackrel{\Delta}{=} \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$



Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \triangleq \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - So higher volatility always increases the option value.

---

[a]Vega is not Greek.

Vega (continued)

- Note that[a]

$$\Lambda = \tau\sigma S^2\Gamma.$$

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = Se^{-q\tau}\sqrt{\tau}\, N'(x),$$

where x is defined in Eq. (46) on p. 338.

- Vega is maximized when $x = 0$, i.e., when

$$S = Xe^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

---

[a]Reiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

where x_i are defined in Eq. (47) on p. 339.

- This also leads to Eq. (48) on p. 339.
  - Recall the same condition led to zero delta for the strangle $C + P$ (p. 339).

Vega (concluded)

- Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

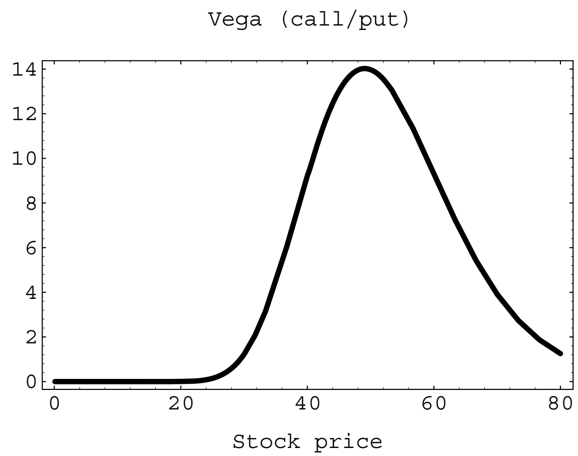
(which answers the question on p. 301 for the Black-Scholes model).

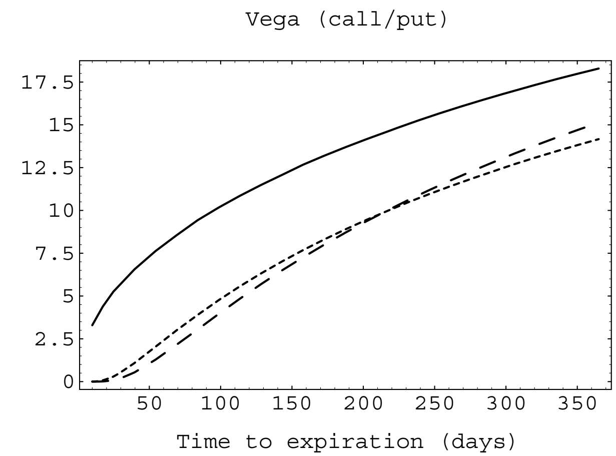
- The Black-Scholes formula (p. 296) implies

$$\begin{aligned} C &\to S, \\ P &\to X e^{-r\tau}, \end{aligned}$$

as $\sigma \to \infty$.

- These boundary conditions may be handy for certain numerical methods.





Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Variance Vega[a]

- Defined as the rate of change of a security's value with respect to the variance (square of volatility) of the underlying asset

$$\text{variance vega} \triangleq \frac{\partial f}{\partial \sigma^2}.$$

  - Note that it is not defined as $\partial^2 f / \partial \sigma^2$!

- It is easy to verify that

$$\text{variance vega} = \frac{\Lambda}{2\sigma}.$$

[a] Demeterfi, Derman, Kamal, & Zou (1999).

Volga (Vomma, Volatility Gamma, Vega Convexity)

- Defined as the rate of change of a security's vega with respect to the volatility of the underlying asset

$$\text{volga} \triangleq \frac{\partial \Lambda}{\partial \sigma} = \frac{\partial^2 f}{\partial \sigma^2}.$$

- It can be shown that

$$\begin{aligned} \text{volga} &= \Lambda \, \frac{x(x - \sigma\sqrt{\tau})}{\sigma} \\ &= \frac{\Lambda}{\sigma} \left[\frac{\ln^2(S/X)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4} \right], \end{aligned}$$

where x is defined in Eq. (46) on p. 338.[a]

[a]Derman & M. B. Miller (2016).

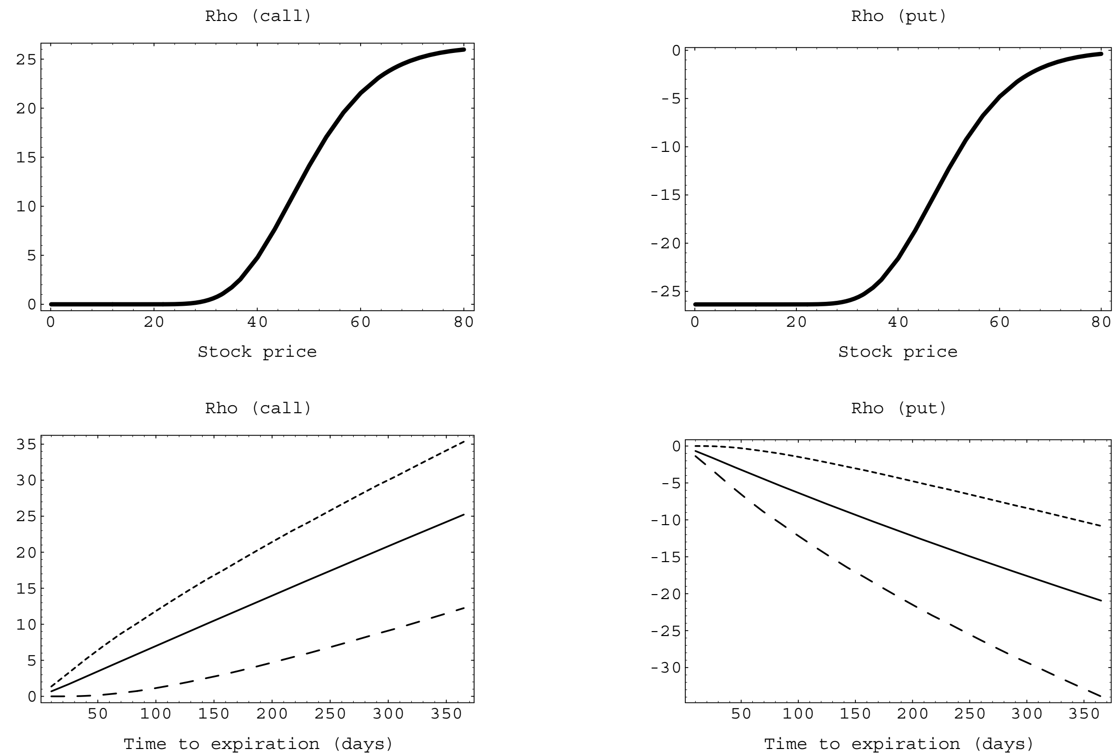
Volga (concluded)

- Volga is zero when $S = Xe^{\pm\sigma^2\tau/2}$.
- For typical values of σ and τ, volga is positive except where $S \approx X$.
- Volga can be used to measure the 4th moment of the underlying asset and the smile of implied volatility at the same maturity.[a]

[a] Bennett (2014).

Rho

- Defined as the rate of change in its value with respect to interest rates
$$\rho \stackrel{\Delta}{=} \frac{\partial f}{\partial r}.$$
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curves: out-of-the-money call or in-the-money put.

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,

$$\frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S}.$$

- The computation time roughly doubles that for evaluating the derivative security itself.

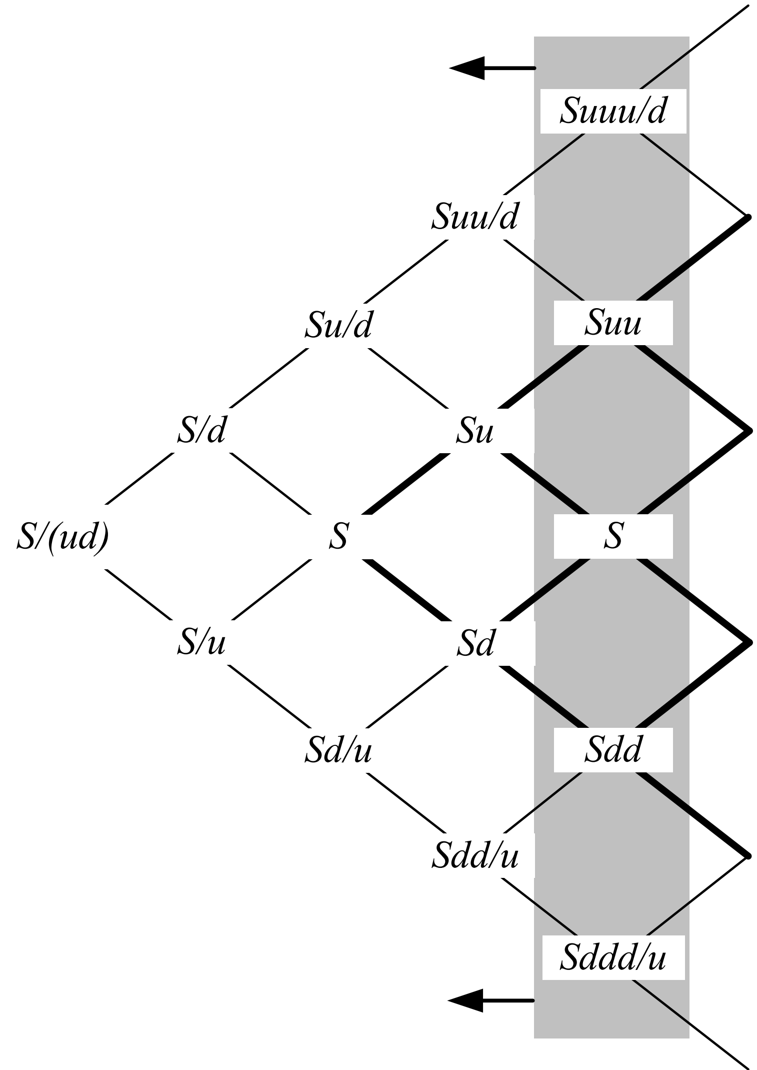
An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}. \tag{49}$$

- Almost zero extra computational effort.

[a]Pelsser & Vorst (1994).



Suuu/d
Suu/d
Suu
Su/d
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}. \tag{50}$$

- Alternative formulas exist (p. 671).

Alternative Numerical Delta and Gamma

- Let $\epsilon \equiv \ln u$.
- Think in terms of $\ln S$.
- Then
$$\left(\frac{f_u - f_d}{2\epsilon}\right)\frac{1}{S}$$
approximates the numerical delta.
- And
$$\left(\frac{f_{uu} - 2f_{ud} + f_{dd}}{\epsilon^2} - \frac{f_{uu} - f_{dd}}{2\epsilon}\right)\frac{1}{S^2}$$
approximates the numerical gamma.

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives
$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$
- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.
- But why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option can be derived from delta and gamma (p. 670).

- The vega of a European option can be derived from gamma (p. 348).

- For rho, there seems no alternative but to run the binomial tree algorithm twice.[a]

[a] But see p. 850 and pp. 1040ff.