Arbitrage in Option Pricing

All general laws are
attended with inconveniences,
when applied to particular cases.
— David Hume (1711–1776)

The problem with QE is
it works in practice,
but it doesn't work in theory.
— Ben Bernanke (2014)

Arbitrage

- The no-arbitrage principle says there is no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).[a]

---

[a] Forbes (2013), "In the real world of investments, however, there are obvious arguments against the EMH [efficient market hypothesis]. There are investors who have beaten the market—Warren Buffett."

Portfolio Dominance Principle

- Consider two portfolios A and B.
- Suppose A's payoff is at least as good as B's under *all* circumstances and better under *some*.
- Then A should be more valuable than B.

Two Simple Corollaries

- A portfolio yielding a zero return in every possible scenario must have a zero PV.[a]
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.
- Two portfolios that yield the same return in every possible scenario must have the same price.[b]

---

[a] "We have incurred net losses each year since our inception and we may not be able to achieve or maintain profitability in the future." (Lyft, Inc, 2019).

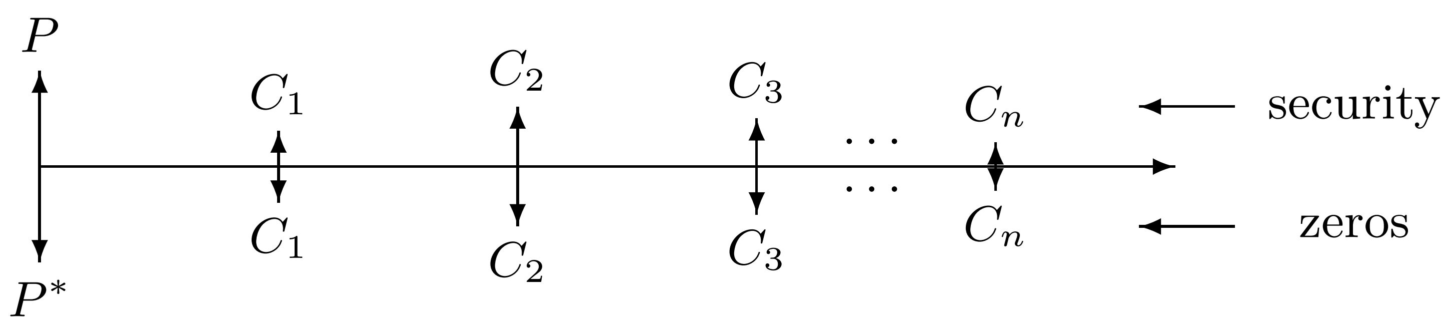
[b] Aristotle, "those who are equal should have everything alike."

The PV Formula (p. 41) Justified

Theorem 1 *For a certain cash flow* $C_1, C_2, \ldots, C_n$,

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Suppose the price $P^* < P$.
- Short the n zeros that match the security's n cash flows.
- The proceeds are P dollars.


P
P^*
C_1
C_1
C_2
C_2
C_3
C_3
$\cdots$
$\cdots$
C_n
C_n
security
zeros

The Proof (concluded)

- Then use P^* of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of $P - P^*$ dollars has been realized now.
- If $P^* > P$, just reverse the trades.

One More Example

Theorem 2 *A put or a call must have a nonnegative value.*

- Suppose otherwise and the option has a negative price.
- Buy the option for a positive cash flow now.
- It will end up with a nonnegative amount at expiration.
- So an arbitrage profit is realized now.

Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs[a] or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.

[a]Schwab cut the fees of online trades of stocks and ETFs to zero on October 7, 2019!

Relative Option Prices (concluded)

- Let the current time be time zero.
- $\mathrm{PV}(x)$ stands for the PV of x dollars at expiration.
- Hence

$$\mathrm{PV}(x) = xd(\tau),$$

where τ is the time to expiration.

Put-Call Parity[a]

$$C = P + S - \mathrm{PV}(X). \tag{31}$$

- Consider the portfolio of:
  - One short European call;
  - One long European put;
  - One share of stock;
  - A loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price X and time to expiration, τ.
- The initial cash flow is therefore

$$C - P - S + \mathrm{PV}(X).$$

[a]Castelli (1877).

The Proof (continued)

- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.
- The loan is now X.
- The net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- The net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle (p. 218) implies that the initial investment to set up the portfolio must be nil as well.

Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.[a]
- $S = C - P + \mathrm{PV}(X)$: A stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$.
- $C - P = S - \mathrm{PV}(X)$: A long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

[a] Recall the synthetic bonds on p. 148.

Intrinsic Value

Lemma 3 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- An American call cannot be worth less than its intrinsic value.[a]
- For European options, the put-call parity implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

- Recall $C \geq 0$ (p. 222).
- It follows that $C \geq \max(S - X, 0)$, the intrinsic value.

[a]See Lemma 8.3.1 of the textbook.

Intrinsic Value (concluded)

A European *put* on a non-dividend-paying stock may be worth less than its intrinsic value $X - S$.

Lemma 4 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$*.*

- Prove it with the put-call parity.[a]
- Can explain the right figure on p. 194 why $P < X - S$ when S is small.

[a] See Lemma 8.3.2 of the textbook.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends!

Theorem 5 (Merton, 1973) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By Exercise 8.3.2 of the text, $C \geq \max(S - \mathrm{PV}(X), 0)$.
- If the call is exercised, the value is $S - X$.
- But
$$\max(S - \mathrm{PV}(X), 0) \geq S - X.$$

Remarks

- The above theorem does *not* mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock, however.
  - Stock price declines as the stock goes ex-dividend.
  - And recall that we assume options are unprotected.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.[a]

Theorem 6 *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

[a]See Theorem 8.4.2 of the textbook.

A General Result[a]

Theorem 7 (Cox & Rubinstein, 1985) *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

Corollary 8 *Any sufficiently well-behaved payoff function can be approximated by a portfolio of calls and puts.*

---

[a]See Exercise 8.3.6 of the textbook.

Option Pricing Models

Black insisted that anything one could do
with a mouse could be done better
with macro redefinitions
of particular keys on the keyboard.
— Emanuel Derman (2004),
My Life as a Quant

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[a]
  - Known as the Black-Scholes option pricing model.

[a]The results were obtained as early as June 1969. Merton and Scholes were winners of the 1997 Nobel Prize in Economic Sciences.

Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \stackrel{\Delta}{=} e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

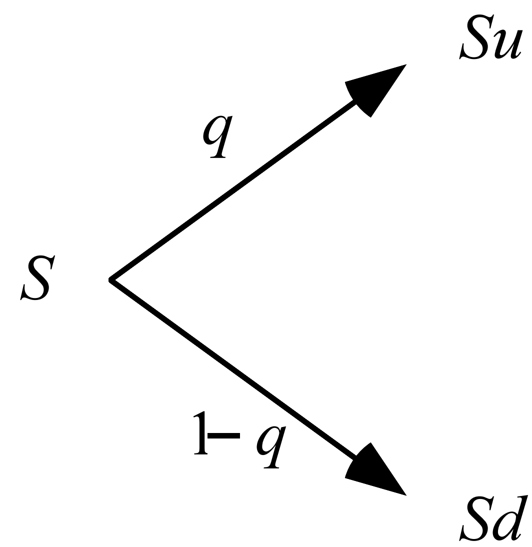
Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.[a]
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:

  S, u, d, X, $\hat{r}$, and the number of periods to expiration.

---

[a]See Exercise 9.2.1 of the textbook.

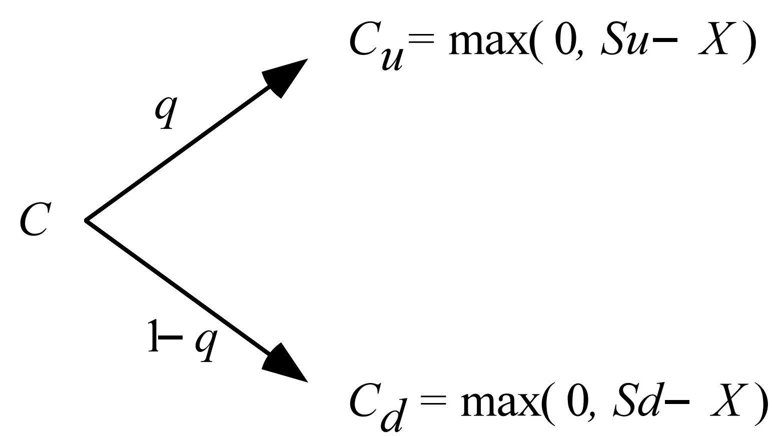


Su
q
S
1 − q
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time 1 if the stock price moves to Su.
- C_d is the call price at time 1 if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is

$$\begin{aligned} &hSu + RB, &&\text{up move,} \\ &hSd + RB, &&\text{down move.} \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose h and B such that the portfolio *replicates* the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{32}$$

$$B = \frac{uC_d - dC_u}{(u - d)\, R}. \tag{33}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,[a]

$$C = hS + B.$$

- As $uC_d - dC_u < 0$, the equivalent portfolio is a *levered* long position in stocks.

[a] Or the replicating portfolio, as it replicates the option.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 5 (p. 231).
- So
$$C = hS + B.$$

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q.[a]
- Hence it is independent of the expected value of the stock,

$$qSu + (1 - q)\, Sd.$$

- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Then the set of possible stock prices is the same whatever q is.

[a]More precisely, not directly dependent on q. Thanks to a lively class discussion on March 16, 2011.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \triangleq \frac{R-d}{u-d}. \tag{34}$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as

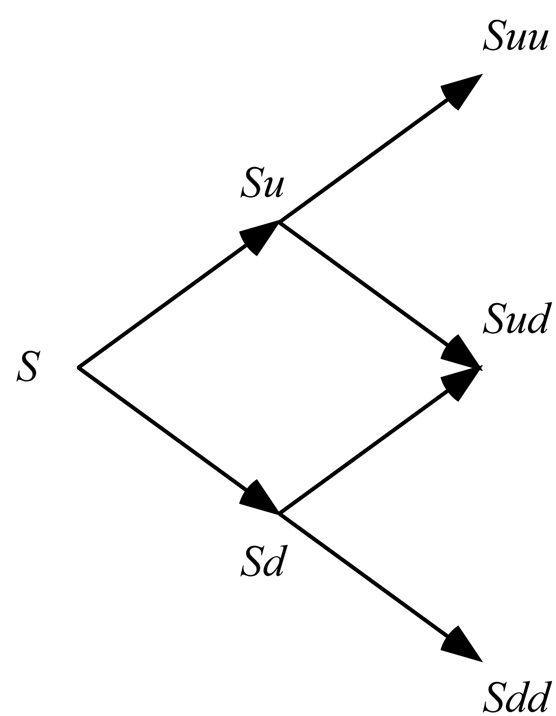
$$pSu + (1-p)\, Sd = RS. \tag{35}$$

- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate *in a risk-neutral economy.*

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree *combines* or *recombines*.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.[a]

[a] It is Markovian.


Suu
Su
Sud
S
Sd
Sdd

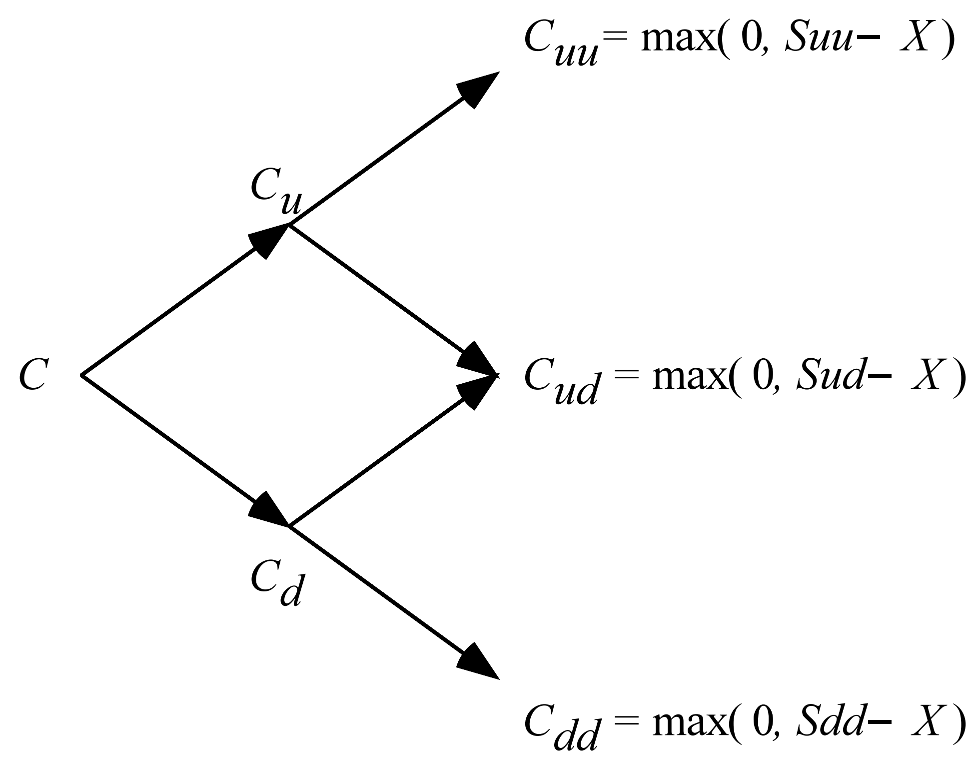
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$



$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, && (36)\\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (32) on p. 245.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

$$\frac{pC_u + (1-p)\, C_d}{R}$$

  as the option price.

- The values of delta h and B can be derived from Eqs. (32)–(33) on p. 245.

Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\,C_d}{R} \geq \frac{[\,pu + (1-p)\,d\,]\,S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  - The call again will not be exercised at present.[a]
- So

$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

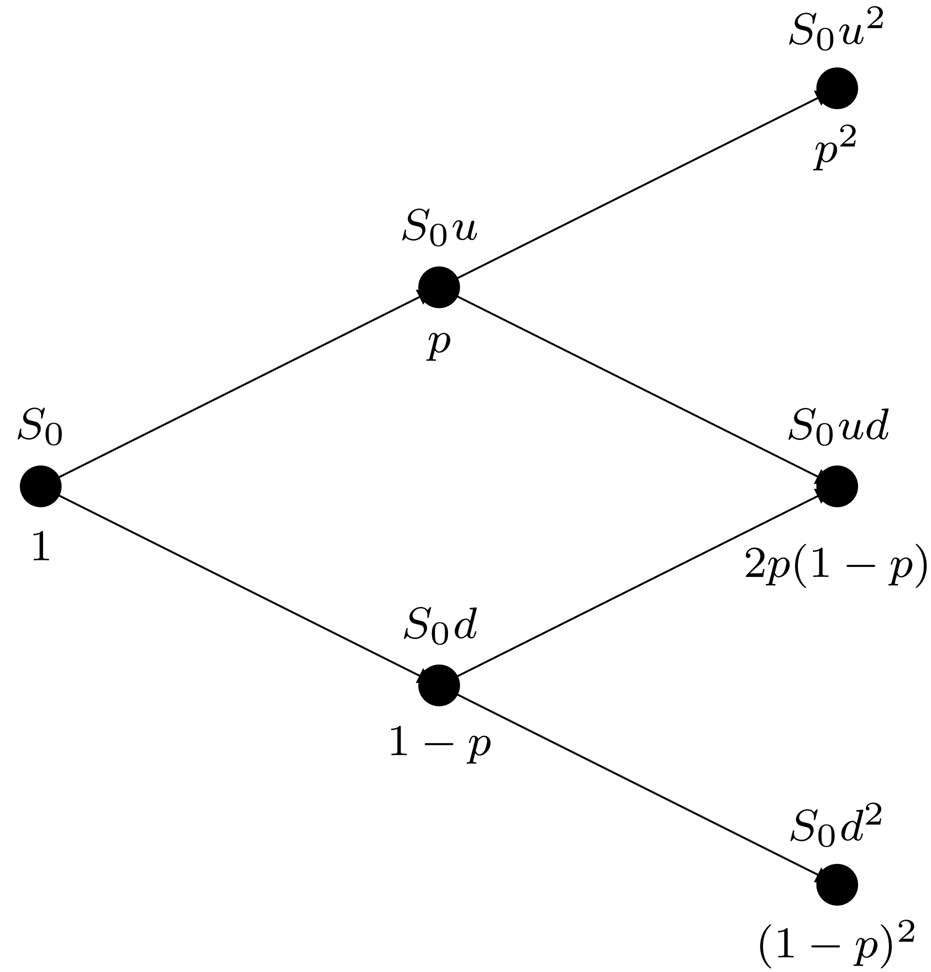
[a] Consistent with Theorem 5 (p. 231).

Backward Induction[a]

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (36) on p. 255.
- This recursive procedure is called backward induction.
- C equals

$$
\begin{aligned}
& [\, p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \; & [\, p^2 \max\left(0, Su^2 - X\right) + 2p(1-p) \max\left(0, Sud - X\right) \\
& +(1-p)^2 \max\left(0, Sd^2 - X\right)]/R^2.
\end{aligned}
$$

[a]Ernst Zermelo (1871–1953).


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (continued)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Backward Induction (concluded)

- Note that
$$p_j \triangleq \frac{\binom{n}{j} p^j (1-p)^{n-j}}{R^n}$$
is the state price[a] for the state $Su^j d^{n-j}$, $j = 0, 1, \ldots, n$.

- In general,
$$\text{option price} = \sum_j (p_j \times \text{payoff at state } j).$$

[a]Recall p. 209. One can obtain the undiscounted state price $\binom{n}{j} p^j (1-p)^{n-j}$—the risk-neutral probability—for the state $Su^j d^{n-j}$ with $(X_M - X_L)^{-1}$ units of the butterfly spread where $X_L = Su^{j-1} d^{n-j+1}$, $X_M = Su^j d^{n-j}$, and $X_H = Su^{j-1+1} d^{n-j-1}$. See Bahra (1997).

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\, \mathcal{D} \,]. \tag{37}$$

  – E^{π} means the expectation is taken under the risk-neutral probability.

- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does *not* depend on predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is *self-financing* because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

---

[a]Except at the beginning, of course, when you have to put up the option value C or P before the replication starts.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \triangleq \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you flip a coin n times with p being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting j heads.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

- Hence,

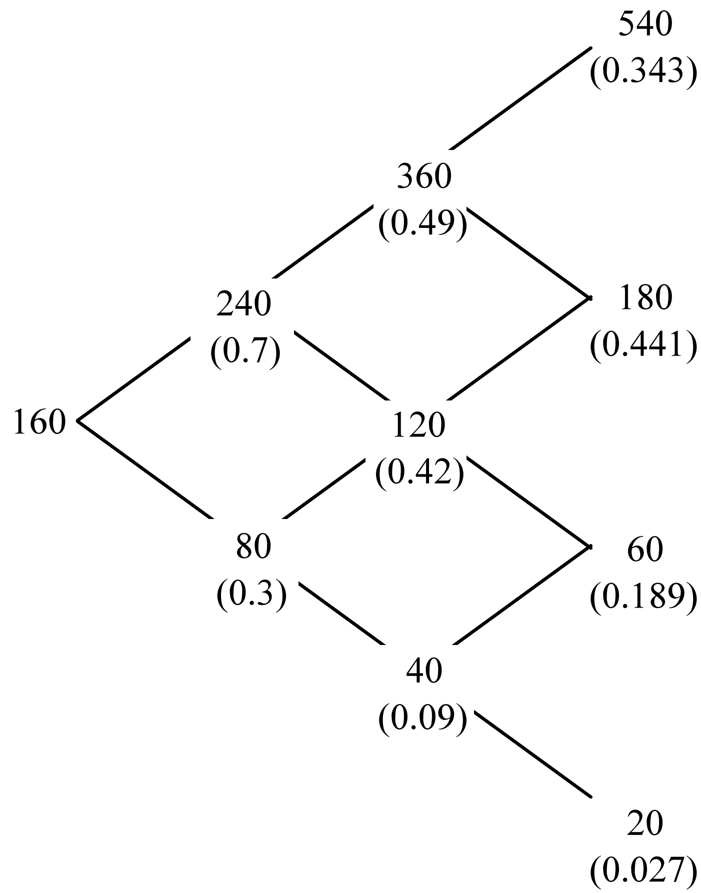
$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} \qquad (38) \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). \qquad (39)
\end{aligned}
$$

Numerical Examples

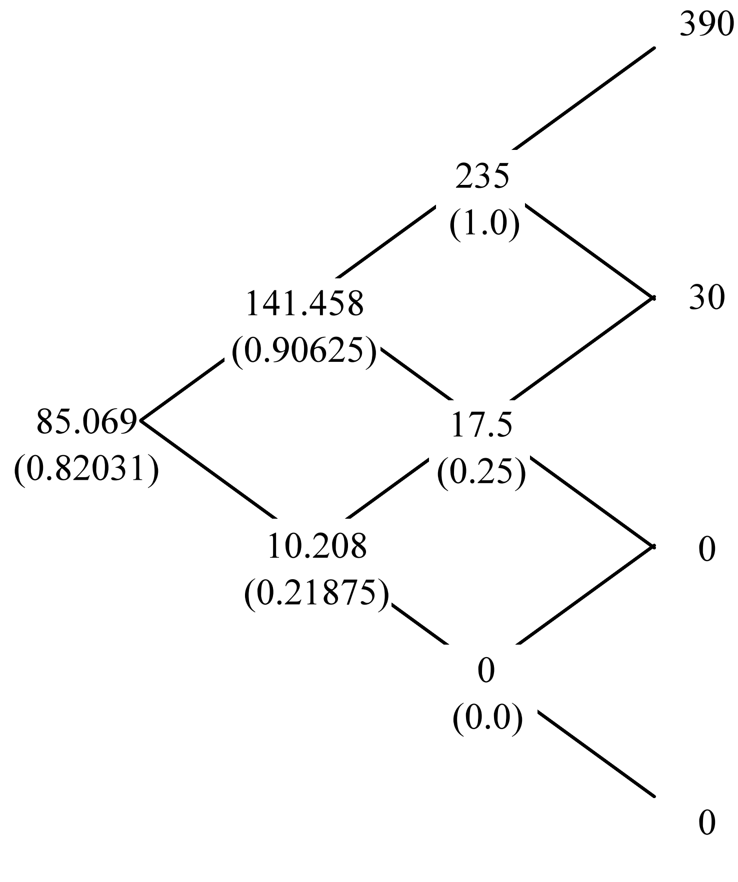
- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

Binomial process for the stock price
(probabilities in parentheses)



Binomial process for the call price
(hedge ratios in parentheses)



Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90.
- Invest \$85.069 in the *replicating* portfolio with 0.82031 shares of stock as required by the delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

is the arbitrage profit, as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy

  $$0.90625 - 0.82031 = 0.08594$$

  more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value by backward induction![a]

[a] See p. 268.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- Close out the call's short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 - 150 = 30$ dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Use \$85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.[b]
- ···
- Without hedge, one may end up forking out \$390 in the worst case (see p. 268)![c]

---

[a] Thanks to a lively class discussion on March 16, 2011.

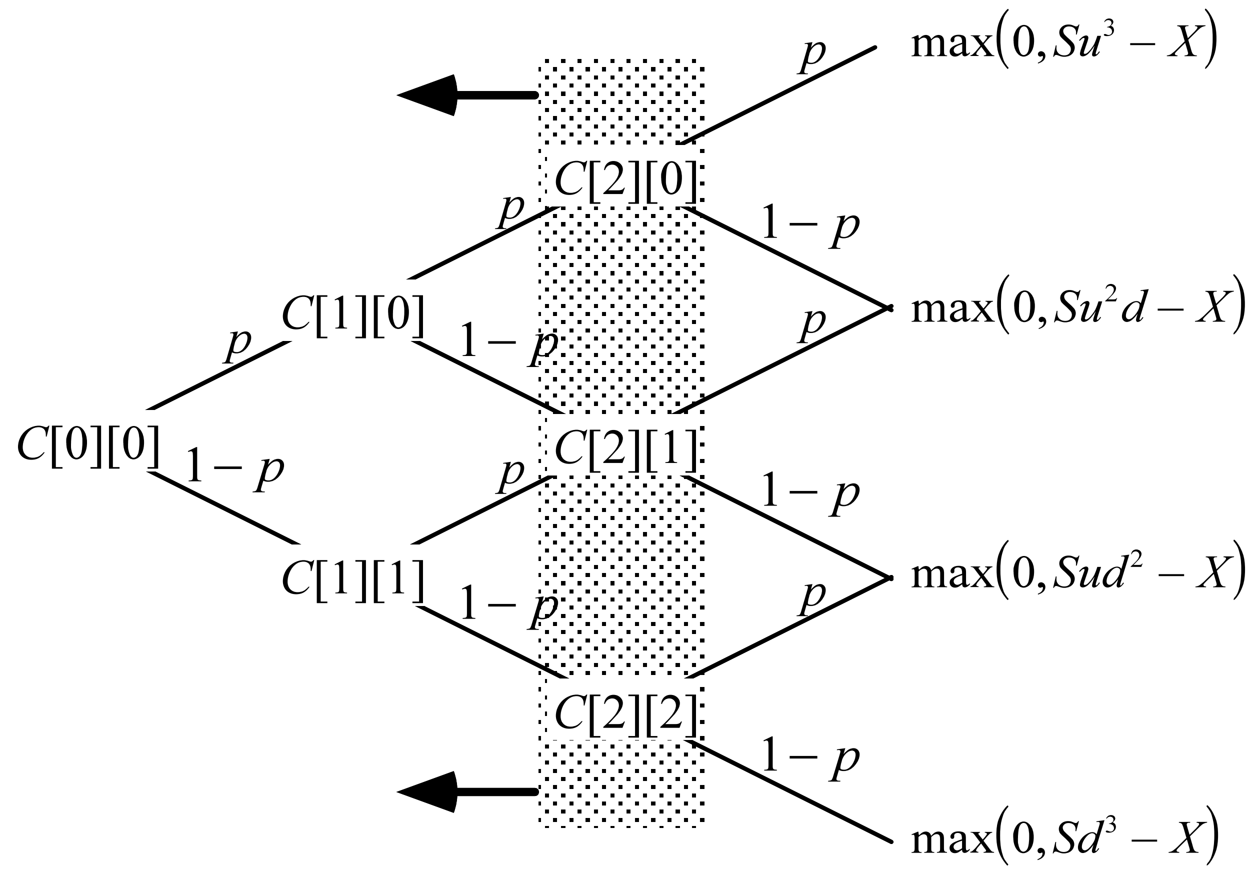
[b] Hedging and replication are mirror images.

[c] Thanks to a lively class discussion on March 16, 2016.

Binomial Tree Algorithms for European Options

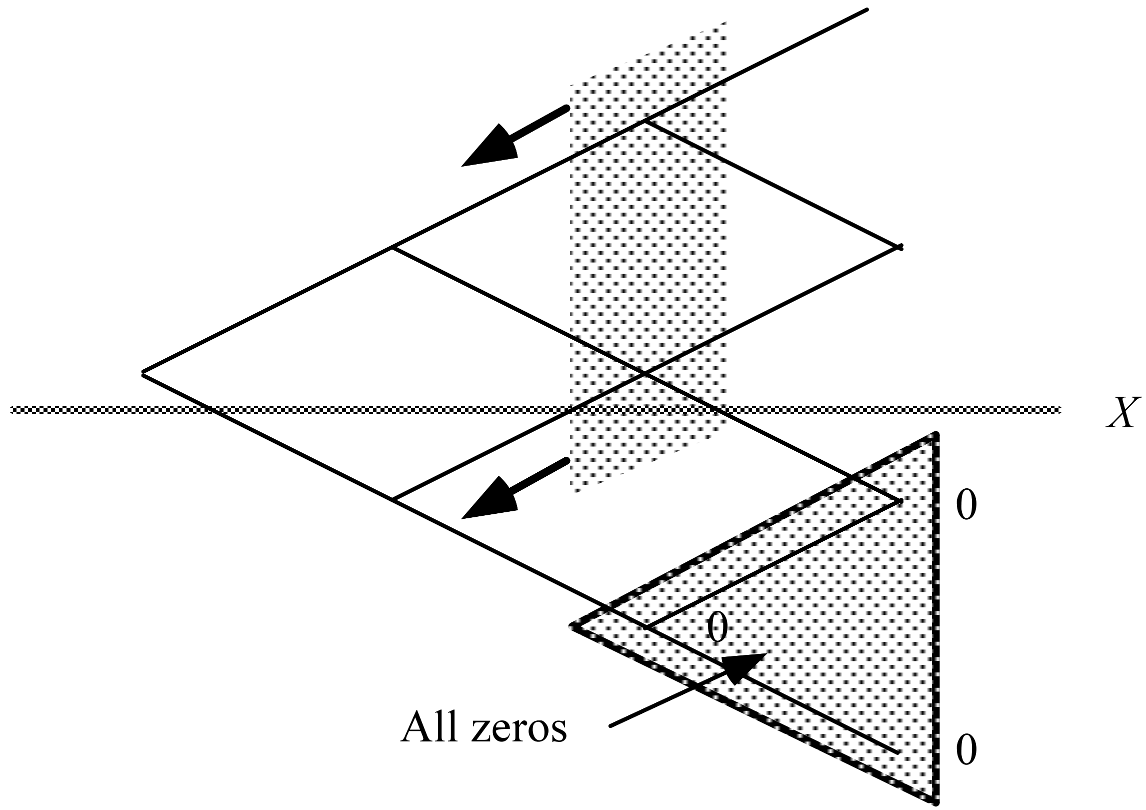
- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To price European puts, simply replace the payoff.

[a] But watch out for the proper updating of array entries.



max(0, Su^3 − X)
p
C[2][0]
p
1 − p
C[1][0]
1 − p
max(0, Su^2d − X)
p
p
C[0][0]
1 − p
C[2][1]
p
1 − p
C[1][1]
1 − p
max(0, Sud^2 − X)
p
C[2][2]
1 − p
max(0, Sd^3 − X)

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a} p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
    $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (38) on p. 266 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree