Spot and Forward Rates under Continuous Compounding

• The pricing formula:

\[ P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}. \]

• The market discount function:

\[ d(n) = e^{-nS(n)}. \]

• The spot rate is an arithmetic average of forward rates,\(^a\)

\[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}. \]

\(^a\)Compare it with Eq. (21) on p. 144.
Spot and Forward Rates under Continuous Compounding (continued)

• The formula for the forward rate:

\[
f(i, j) = \frac{jS(j) - iS(i)}{j - i}.
\]  

(23)

– Compare the above formula with Eq. (20) on p. 138.

• The one-period forward rate:\(^a\)

\[
f(j, j + 1) = - \ln \frac{d(j + 1)}{d(j)}.
\]

\(^a\)Compare it with Eq. (22) on p. 144.
Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

\[ f(T) \triangleq \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}. \] (24)

- So \( f(T) > S(T) \) if and only if \( \partial S/\partial T > 0 \) (i.e., a normal spot rate curve).

- If \( S(T) < -T(\partial S/\partial T) \), then \( f(T) < 0 \).\(^a\)

\(^a\)Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is\(^a\)
  \[
  S(T) \overset{\Delta}{=} 0.08 - 0.05 e^{-0.18T}.
  \]
- Then by Eq. (24) on p. 151, the forward rate curve is
  \[
  f(T) = S(T) + TS'(T) \\
  = 0.08 - 0.05 e^{-0.18T} + 0.009T e^{-0.18T}.
  \]

\(^a\)Hull & White (1994).
Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

\[ f(a, b) = E[S(a, b)]. \]  \hspace{1cm} (25)

- It does not imply that the forward rate is an accurate predictor for the future spot rate.

- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on average.
Unbiased Expectations Theory and Spot Rate Curve

• It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  
  \[ f(j, j+1) > S(j + 1) \text{ if and only if } S(j + 1) > S(j) \]
  
  from Eq. (20) on p. 138.

  \[ \text{So } E[S(j, j+1)] > S(j + 1) > \cdots > S(1) \text{ if and only if } S(j + 1) > \cdots > S(1). \]

• Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
A “Bad” Expectations Theory

• The expected returns\(^a\) on all possible riskless bond strategies are equal for all holding periods.

• So

\[
(1 + S(2))^2 = (1 + S(1)) E[1 + S(1, 2)] \tag{26}
\]

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

• After rearrangement,

\[
\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]

\(^a\)More precisely, the one-plus returns.
A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  - The expected return is
    \[
    E[(1 + S(1, 2))^{-1}] / (1 + S(2))^{-2}.
    \]
  - Strategy two buys a one-period bond with a return of \(1 + S(1)\).
A “Bad” Expectations Theory (continued)

- The theory says the returns are equal:

\[
\frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
\]

- Combine this with Eq. (26) on p. 155 to obtain

\[
E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.
\]
A “Bad” Expectations Theory (concluded)

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that $E[g(X)] > g(E[X])$
    for any nondegenerate random variable $X$ and
    strictly convex function $g$ (i.e., $g''(x) > 0$).
  – Use
    
    \[ g(x) \equiv (1 + x)^{-1} \]

    to prove our point.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1, n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \text{ for all } n > 1.
\]

- This theory is the basis of many interest rate models.
Duration in Practice

• To handle more general types of spot rate curve changes, define a vector \([c_1, c_2, \ldots, c_n]\) that characterizes the perceived type of change.
  
  – Parallel shift: \([1, 1, \ldots, 1]\).
  
  – Twist: \([1, 1, \ldots, 1, -1, \ldots, -1]\), \([1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]\), etc.
  
  – ....

• At least one \(c_i\) should be 1 as the reference point.


Duration in Practice (concluded)

1. Let

\[ P(y) \triangleq \sum_{i} C_i / (1 + S(i) + y c_i)^i \]

be the price associated with the cash flow \( C_1, C_2, \ldots \).

2. Define duration as

\[ -\frac{\partial P(y)}{\partial y} \bigg|_{y=0} \quad \text{or} \quad - \frac{P(\Delta y) - P(-\Delta y)}{2P(0)\Delta y}. \]

3. Modified duration equals the above when

\[ [c_1, c_2, \ldots, c_n] = [1, 1, \ldots, 1], \]

\[ S(1) = S(2) = \cdots = S(n). \]
Some Loose Ends on Dates

• Holidays.
• Weekends.
• Business days ($T + 2$, etc.).
• Shall we treat a year as 1 year whether it has 365 or 366 days?
Fundamental Statistical Concepts
There are three kinds of lies: lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli (1804–1881)

If 50 million people believe a foolish thing, it’s still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy, but a million deaths are a statistic.
— Josef Stalin (1879–1953)
Moments

• The variance of a random variable $X$ is defined as

$$\text{Var}[X] \triangleq E\left[ (X - E[X])^2 \right].$$

• The covariance between random variables $X$ and $Y$ is

$$\text{Cov}[X,Y] \triangleq E\left[ (X - \mu_X)(Y - \mu_Y) \right],$$

where $\mu_X$ and $\mu_Y$ are the means of $X$ and $Y$, respectively.

• Random variables $X$ and $Y$ are uncorrelated if

$$\text{Cov}[X,Y] = 0.$$
Correlation

- The standard deviation of $X$ is the square root of the variance,
  \[ \sigma_X \triangleq \sqrt{\text{Var}[X]} . \]

- The correlation (or correlation coefficient) between $X$ and $Y$ is
  \[ \rho_{X,Y} \triangleq \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y} , \]
  provided both have nonzero standard deviations.\(^a\)

---
\(^a\) Wilmott (2009), “the correlations between financial quantities are notoriously unstable.” It may even break down “at high-frequency time intervals” (Budish, Cramton, & Shim, 2015).
Variance of Sum

- Variance of a weighted sum of random variables equals

\[
\text{Var} \left( \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \text{Cov}[X_i, X_j].
\]

- It becomes

\[
\sum_{i=1}^{n} a_i^2 \text{Var}[X_i]
\]

when \( X_i \) are uncorrelated.
Conditional Expectation

- “$X \mid I$” denotes $X$ conditional on the information set $I$.
- The information set can be another random variable’s value or the past values of $X$, say.
- The conditional expectation
  \[ E[ X \mid I ] \]
  is the expected value of $X$ conditional on $I$; it is a random variable.
- The law of iterated conditional expectations\(^a\) says
  \[ E[ X ] = E[ E[ X \mid I ] ]. \]

\(^a\)Or the tower law.
Conditional Expectation (concluded)

- If $I_2$ contains at least as much information as $I_1$, then

$$E[X | I_1] = E[E[X | I_2] | I_1].$$  \hspace{1cm} (27)

- $I_1$ contains price information up to time $t_1$, and $I_2$ contains price information up to a later time $t_2 > t_1$.

- In general,

$$I_1 \subseteq I_2 \subseteq \cdots$$

means the players never forget past data so the information sets are increasing over time.$^a$

---

$^a$Hirsa & Neftci (2014). This idea is used in sigma fields and filtration.
The Normal Distribution

- A random variable $X$ has the normal distribution with mean $\mu$ and variance $\sigma^2$ if its probability density function is
  \[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}. \]

- This is expressed by $X \sim N(\mu, \sigma^2)$.

- The standard normal distribution has zero mean, unit variance, and the following distribution function
  \[ \text{Prob}[X \leq z] = N(z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dx. \]
Moment Generating Function

- The moment generating function of random variable $X$ is defined as
  \[ \theta_X(t) \triangleq E[e^{tX}] . \]
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
  \[ \theta_X(t) = \exp \left[ \mu t + \frac{\sigma^2 t^2}{2} \right] . \] (28)
The Multivariate Normal Distribution

- If \( X_i \sim N(\mu_i, \sigma^2_i) \) are independent, then
  \[
  \sum_i X_i \sim N \left( \sum_i \mu_i, \sum_i \sigma^2_i \right).
  \]

- Let \( X_i \sim N(\mu_i, \sigma^2_i) \), which may not be independent.

- Suppose
  \[
  \sum_{i=1}^n t_i X_i \sim N \left( \sum_{i=1}^n t_i \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \text{Cov}[X_i, X_j] \right)
  \]
  for every linear combination \( \sum_{i=1}^n t_i X_i \).\(^a\)

- \( X_i \) are said to have a multivariate normal distribution.

\(^a\)Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.
Generation of Univariate Normal Distributions

- Let $X$ be uniformly distributed over $(0, 1]$ so that
  $$\text{Prob}[X \leq x] = x, \quad 0 < x \leq 1.$$  

- Repeatedly draw two samples $x_1$ and $x_2$ from $X$ until
  $$\omega \overset{\Delta}{=} (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$  

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where\(^{a}\)
  $$c \overset{\Delta}{=} \sqrt{-2(\ln \omega)/\omega}.$$  

\(^{a}\)As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.
A Dirty Trick and a Right Attitude

- Let \( \xi_i \) are independent and uniformly distributed over \((0, 1)\).

- A simple method to generate the standard normal variable is to calculate\(^a\)

\[
\left( \sum_{i=1}^{12} \xi_i \right) - 6.
\]

- But why use 12?

- Recall the mean and variance of \( \xi_i \) are 1/2 and 1/12, respectively.

\(^a\)Jäckel (2002), “this is not a highly accurate approximation and should only be used to establish ballpark estimates.”
A Dirty Trick and a Right Attitude (concluded)

- The general formula is
  \[
  \frac{\left(\sum_{i=1}^{n} \xi_i\right) - (n/2)}{\sqrt{n/12}}.
  \]

- Choosing \( n = 12 \) yields a formula without the need of division and square-root operations.\(^a\)

- Always blame your random number generator last.\(^b\)

- Instead, check your programs first.

\(^a\)Contributed by Mr. Chen, Shih-Hang (R02723031) on March 5, 2014.
\(^b\)“The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings.” William Shakespeare (1564–1616), \textit{Julius Caesar}. 
Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation $\rho$ can be generated as follows.
- Let $X_1$ and $X_2$ be independent standard normal variables.
- Set

\[
U \triangleq aX_1, \\
V \triangleq a\rho X_1 + a\sqrt{1 - \rho^2} X_2.
\]
Generation of Bivariate Normal Distributions (continued)

• $U$ and $V$ are the desired random variables with

$$
\text{Var}[U] = \text{Var}[V] = a^2,
\text{Cov}[U, V] = \rho a^2.
$$

• Note that the mapping from $(X_1, X_2)$ to $(U, V)$ is a one-to-one correspondence for $a \neq 0$.  

Generation of Bivariate Normal Distributions (concluded)

- It is convenient to write the mapping in matrix form:

\[
\begin{bmatrix}
U \\
V
\end{bmatrix} = a \begin{bmatrix}
1 & 0 \\
\rho & \sqrt{1 - \rho^2}
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}.
\]  

(29)
The Lognormal Distribution

- A random variable $Y$ is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \triangleq e^X$.
- The mean and variance of $Y$ are
  \[
  \mu_Y = e^{\mu + \sigma^2/2} \quad \text{and} \quad \sigma_Y^2 = e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right),
  \]
  respectively.
  - They follow from $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$.
The Lognormal Distribution (continued)

- Conversely, suppose \( Y \) is lognormally distributed with mean \( \mu \) and variance \( \sigma^2 \).

- Then

\[
E[\ln Y] = \ln\left(\frac{\mu}{\sqrt{1 + (\sigma/\mu)^2}}\right),
\]
\[
\text{Var}[\ln Y] = \ln(1 + (\sigma/\mu)^2).
\]

- If \( X \) and \( Y \) are joint-lognormally distributed, then

\[
E[XY] = E[X] E[Y] e^{\text{Cov}[\ln X, \ln Y]},
\]
\[
\text{Cov}[X, Y] = E[X] E[Y] \left( e^{\text{Cov}[\ln X, \ln Y]} - 1 \right).
\]
The Lognormal Distribution (concluded)

- Let $Y$ be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then

$$\int_{a}^{\infty} y f(y) \, dy = e^{\mu + \sigma^2 / 2} N \left( \frac{\mu - \ln a}{\sigma} + \sigma \right).$$
Option Basics
The shift toward options as the center of gravity of finance [...]  
— Merton H. Miller (1923–2000)

Too many potential physicists and engineers spend their careers shifting money around in the financial sector, instead of applying their talents to innovating in the real economy.  
— Barack Obama (2016)
Calls and Puts

- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.\(^a\)

\[^a\text{The cash flow at expiration is contingent.}\]
Calls and Puts (continued)

- A put gives its holder the right to sell a unit of the underlying asset for the strike price.
Calls and Puts (concluded)

• An embedded option has to be traded along with the underlying asset.

• How to price options?
  – It can be traced to Aristotle’s (384 B.C.–322 B.C.) *Politics*, if not earlier.
Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.
American and European

• American options can be exercised at any time up to the expiration date.
• European options can only be exercised at expiration.
• An American option is worth at least as much as an otherwise identical European option.
Convenient Conventions

- $C$: call value.
- $P$: put value.
- $X$: strike price.
- $S$: stock price.
- $D$: dividend.
Payoff, Mathematically Speaking

- The payoff of a call at expiration is
  \[ C = \max(0, S - X). \]

- The payoff of a put at expiration is
  \[ P = \max(0, X - S). \]

- A call will be exercised only if the stock price is higher than the strike price.

- A put will be exercised only if the stock price is less than the strike price.
Payoff, Mathematically Speaking (continued)

- At any time \( t \) before the expiration date, we call
  
  \[
  \max(0, S_t - X)
  \]

  the intrinsic value of a call.

- At any time \( t \) before the expiration date, we call
  
  \[
  \max(0, X - S_t)
  \]

  the intrinsic value of a put.
Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.

- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.

- Options that are in the money at expiration should be exercised.\(^a\)

- Finding an option’s value at any time before expiration is a major intellectual breakthrough.

\(^a\)11\% of option holders let in-the-money options expire worthless.
Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for *cash* dividends.

- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.

- Cash dividends are detrimental for calls.

- The opposite is true for puts.
Stock Splits and Stock Dividends

• Options are adjusted for stock splits.
• After an $n$-for-$m$ stock split, $m$ shares become $n$ shares.
• Accordingly, the strike price is only $m/n$ times its previous value, and the number of shares covered by one option becomes $n/m$ times its previous value.
• Exchange-traded stock options are adjusted for stock dividends.
• We assume options are unprotected.
Example

- Consider an option to buy 100 shares of a company for $50 per share.

- A 2-for-1 split changes the term to a strike price of $25 per share for 200 shares.
Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.
Payoff of Stock

Long a stock:

- Payoff increases as price increases.

Short a stock:

- Payoff decreases as price increases.

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Short Selling (concluded)

- Not all assets can be shorted.
- In reality, short selling is not simply the opposite of going long.\(^a\)

Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.

- Covered call: A long position in stock with a short call.
  - It is “covered” because the stock can be delivered to the buyer of the call if the call is exercised.

- Protective put: A long position in stock with a long put.

- Both strategies break even only if the stock price rises above a certain level, so they are bullish.

\(^a\) A short position has a payoff opposite in sign to that of a long position.
Solid lines are profits of the portfolio one month before maturity, assuming the portfolio is set up when $S = 95$ then.
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.

- We use \( X_L, X_M, \) and \( X_H \) to denote the strike prices with

\[ X_L < X_M < X_H. \]
Covered Position: Spread (continued)

- A bull call spread consists of a long $X_L$ call and a short $X_H$ call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
    * When both are exercised at expiration.
  - The maximum loss is $C_L - C_H$.
    * When neither is exercised at expiration.
  - If we buy $(X_H - X_L)^{-1}$ units of the bull call spread and $X_H - X_L \to 0$, a (Heaviside) step function emerges as the payoff.
Bull spread (call)

Profit

Stock price
Covered Position: Spread (continued)

- Writing an $X_H$ put and buying an $X_L$ put with identical expiration date creates the bull put spread.\(^a\)
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one $X_L$ call, long one $X_H$ call, and short two $X_M$ calls.

\(^a\)See https://www.businesstoday.com.tw/article/category/80392/post/20180307 for a sad example in Taiwan on February 6, 2018.
Covered Position: Spread (continued)

- A butterfly spread pays a positive amount at expiration only if the asset price falls between $X_L$ and $X_H$.
- Assume $X_M = (X_H + X_L)/2$.
- Take a position in $(X_M - X_L)^{-1}$ units of the butterfly spread.
- When $X_H - X_L \to 0$, it approximates a state contingent claim\(^a\), which pays $1$ only when the state $S = X_M$ happens.\(^b\)

---

\(^a\)Alternatively, Arrow security.
\(^b\)See Exercise 7.4.5 of the textbook.
Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The state price equals
  \[ \frac{\partial^2 C}{\partial X^2}. \]
    - Recall that \( C \) is the call’s price.\(^a\)
- In fact, the FV of \( \frac{\partial^2 C}{\partial X^2} \) is the probability density of the stock price \( S_T = X \) at option’s maturity.\(^b\)

\(^a\)One can also use the put (see Exercise 9.3.6 of the textbook).
\(^b\)Breeden & Litzenberger (1978). This formula is model-independent!
Covered Position: Combination

• A combination consists of options of different types on the same underlying asset.
  – These options must be either all bought or all written.

• Straddle: A long call and a long put with the same strike price and expiration date.
  – Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
  – Selling a straddle benefits from low volatility.
A graph illustrating the profit from a straddle strategy as a function of stock price.

- The profit is maximized at the current stock price of 100.
- The profit decreases as the stock price moves away from 100 in either direction.
- The profit can be positive if the stock price stays close to 100, or negative if it moves significantly away from 100.

The graph shows two curves, one for the call option and one for the put option, both converging at the strike price of 100.
Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call’s strike price is higher than the put’s.