Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

---

[a]Compare it with Eq. (21) on p. 144.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}. \tag{23}$$

  - Compare the above formula with Eq. (20) on p. 138.

- The one-period forward rate:[a]

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

[a]Compare it with Eq. (22) on p. 144.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now, the (instantaneous) forward rate curve is:

$$f(T) \stackrel{\Delta}{=} \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T\,\frac{\partial S}{\partial T}. \tag{24}$$

- So $f(T) > S(T)$ if and only if $\partial S / \partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S / \partial T)$, then $f(T) < 0$.[a]

[a] Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

An Example

- Let the interest rates be continuously compounded.
- Suppose the spot rate curve is[a]

$$S(T) \stackrel{\Delta}{=} 0.08 - 0.05\, e^{-0.18T}.$$

- Then by Eq. (24) on p. 151, the forward rate curve is

$$\begin{aligned} & f(T) \\ = \quad & S(T) + T S'(T) \\ = \quad & 0.08 - 0.05\, e^{-0.18T} + 0.009T\, e^{-0.18T}. \end{aligned}$$

[a]Hull & White (1994).

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\, S(a, b)\,]. \tag{25}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on average.

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (20) on p. 138.
  - So $E[\,S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A "Bad" Expectations Theory

- The expected returns[a] on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1))\, E[\, 1 + S(1, 2)\,] \tag{26}$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[\, 1 + S(1, 2)\,]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

[a]More precisely, the one-plus returns.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is

$$E[\,(1 + S(1,2))^{-1}\,]/(1 + S(2))^{-2}.$$

  - Strategy two buys a one-period bond with a return of $1 + S(1)$.

A "Bad" Expectations Theory (continued)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (26) on p. 155 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

A "Bad" Expectations Theory (concluded)

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use

$$g(x) \stackrel{\Delta}{=} (1+x)^{-1}$$

    to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \quad \text{for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration in Practice

- To handle more general types of spot rate curve changes, define a vector $[\,c_1, c_2, \ldots, c_n\,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\,1, 1, \ldots, 1\,]$.
  - Twist: $[\,1, 1, \ldots, 1, -1, \ldots, -1\,]$, $[\,1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]$, etc.
  - $\ldots$.
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let
$$P(y) \stackrel{\Delta}{=} \sum_i C_i/(1+S(i)+yc_i)^i$$
be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as
$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y)-P(-\Delta y)}{2P(0)\Delta y}.$$
- Modified duration equals the above when
$$[\,c_1, c_2, \ldots, c_n\,] = [\,1, 1, \ldots, 1\,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \stackrel{\Delta}{=} E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \stackrel{\Delta}{=} E\,[\,(X - \mu_X)(Y - \mu_Y)\,],$$

where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \triangleq \sqrt{\mathrm{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \triangleq \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

[a]Wilmott (2009), "the correlations between financial quantities are notoriously unstable." It may even break down "at high-frequency time intervals" (Budish, Cramton, & Shim, 2015).

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\, X_i, X_j \,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\, X_i \,]$$

when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation

$$E[\, X \mid I \,]$$

is the expected value of X conditional on I; it is a random variable.

- The law of iterated conditional expectations[a] says

$$E[\, X \,] = E[\, E[\, X \mid I \,]\,].$$

[a] Or the tower law.

Conditional Expectation (concluded)

- If I_2 contains at least as much information as I_1, then

$$E[\,X \mid I_1\,] = E[\,E[\,X \mid I_2\,] \mid I_1\,]. \tag{27}$$

  - I_1 contains price information up to time t_1, and I_2 contains price information up to a later time $t_2 > t_1$.

- In general,

$$I_1 \subseteq I_2 \subseteq \cdots$$

  means the players never forget past data so the information sets are increasing over time.[a]

[a] Hirsa & Neftci (2014). This idea is used in sigma fields and filtration.

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$

- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function

$$\text{Prob}[\, X \leq z \,] = N(z) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is defined as

$$\theta_X(t) \stackrel{\Delta}{=} E[\, e^{tX} \,].$$

- The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{28}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then
$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$
- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose
$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\, X_i, X_j\,]\right)$$
for every linear combination $\sum_{i=1}^n t_i X_i$.[a]
- X_i are said to have a multivariate normal distribution.

[a] Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1\,]$ so that

$$\text{Prob}[\,X \leq x\,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \stackrel{\Delta}{=} (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \stackrel{\Delta}{=} \sqrt{-2(\ln \omega)/\omega}\,.$$

[a]As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0,1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i\right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are 1/2 and 1/12, respectively.

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- The general formula is

$$\frac{(\sum_{i=1}^{n} \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

---

[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated as follows.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$
\begin{aligned}
U &\stackrel{\Delta}{=} aX_1, \\
V &\stackrel{\Delta}{=} a\rho X_1 + a\sqrt{1-\rho^2}\, X_2.
\end{aligned}
$$

Generation of Bivariate Normal Distributions (continued)

- U and V are the desired random variables with

$$\begin{aligned}\mathrm{Var}[\,U\,] &= \mathrm{Var}[\,V\,] = a^2,\\ \mathrm{Cov}[\,U, V\,] &= \rho a^2.\end{aligned}$$

- Note that the mapping from (X_1, X_2) to (U, V) is a one-to-one correspondence for $a \neq 0$.

Generation of Bivariate Normal Distributions (concluded)

- It is convenient to write the mapping in matrix form:

$$\begin{bmatrix} U \\ V \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{29}$$

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \stackrel{\Delta}{=} e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu + \sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu + \sigma^2} \left(e^{\sigma^2} - 1 \right), \tag{30}$$

  respectively.
  - They follow from $E[\, Y^n \,] = e^{n\mu + n^2 \sigma^2 / 2}$.

The Lognormal Distribution (continued)

- Conversely, suppose Y is lognormally distributed with mean μ and variance σ^2.
- Then

$$
\begin{aligned}
E[\ln Y] &= \ln(\mu/\sqrt{1+(\sigma/\mu)^2}), \\
\text{Var}[\ln Y] &= \ln(1+(\sigma/\mu)^2).
\end{aligned}
$$

- If X and Y are joint-lognormally distributed, then

$$
\begin{aligned}
E[XY] &= E[X]\,E[Y]\,e^{\text{Cov}[\ln X, \ln Y]}, \\
\text{Cov}[X, Y] &= E[X]\,E[Y]\left(e^{\text{Cov}[\ln X, \ln Y]} - 1\right).
\end{aligned}
$$

The Lognormal Distribution (concluded)

- Let Y be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then

$$\int_a^\infty y f(y)\, dy = e^{\mu+\sigma^2/2}\, N\left(\frac{\mu - \ln a}{\sigma} + \sigma \right).$$

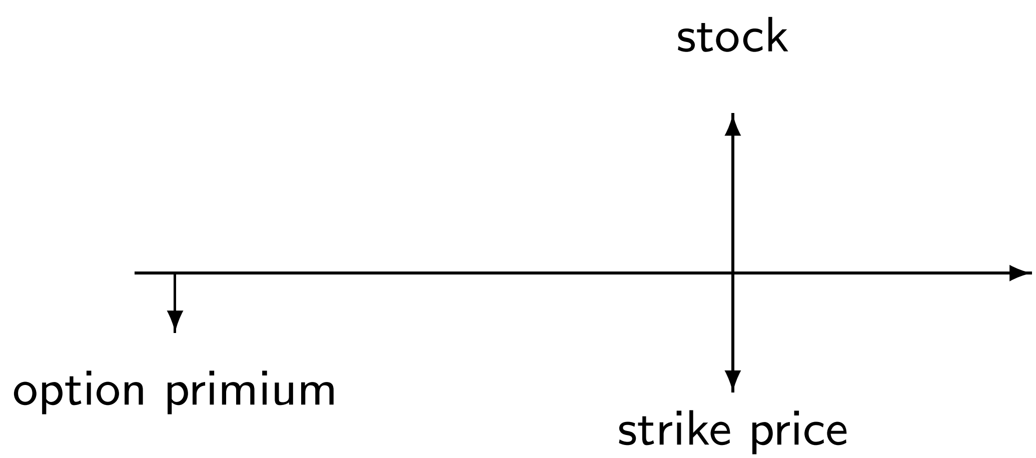
Option Basics

The shift toward options as
the center of gravity of finance [...]
— Merton H. Miller (1923–2000)

Too many potential physicists and engineers
spend their careers shifting money around
in the financial sector,
instead of applying their talents to
innovating in the real economy.
— Barack Obama (2016)

Calls and Puts

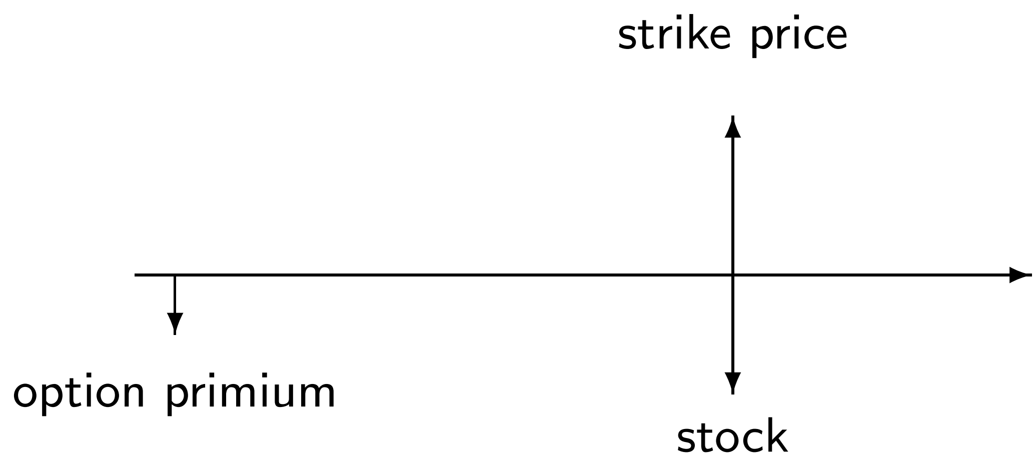
- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.[a]



[a]The cash flow at expiration is contingent.

Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
  - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

Payoff, Mathematically Speaking

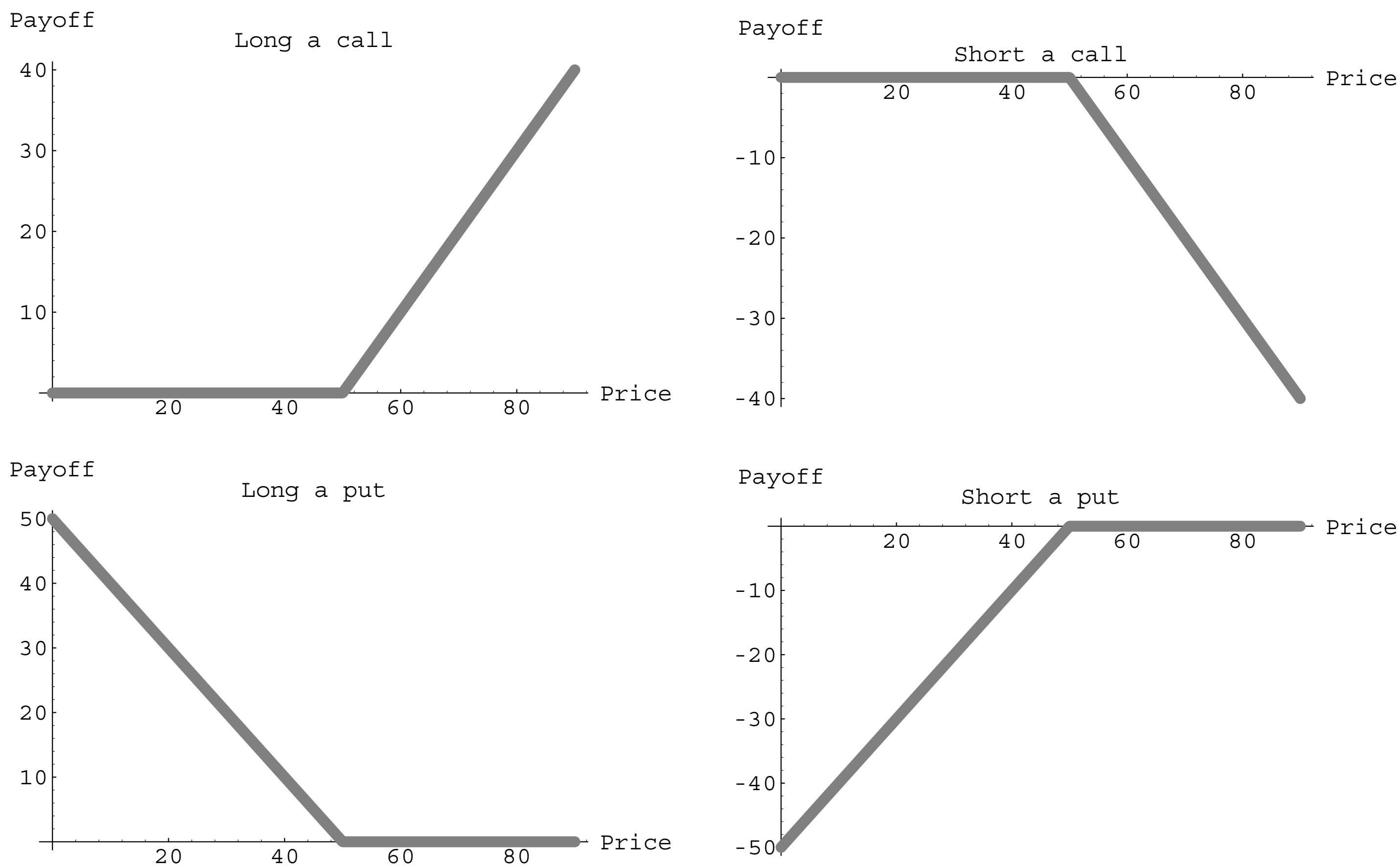
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

the intrinsic value of a call.

- At any time t before the expiration date, we call

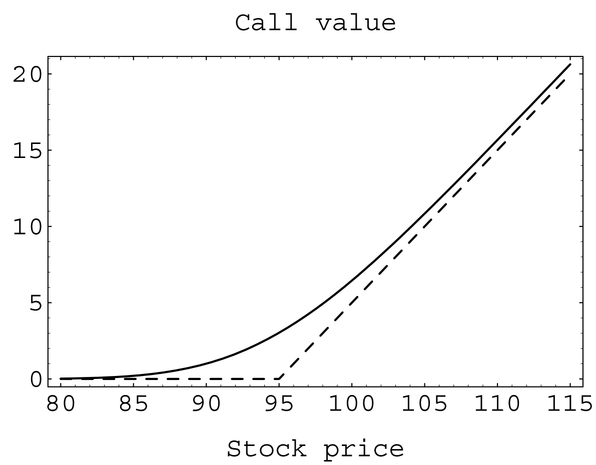
$$\max(0, X - S_t)$$

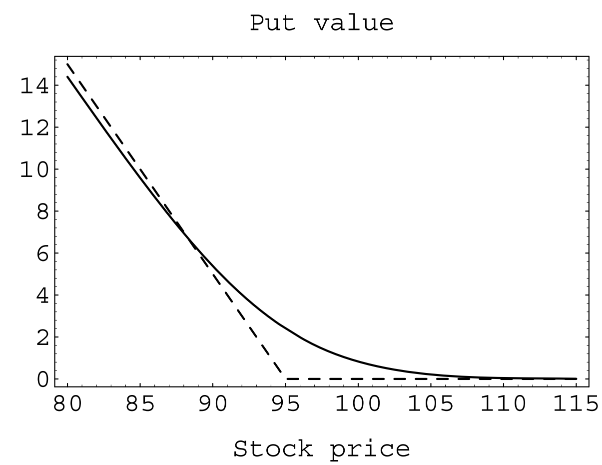
the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time *before* expiration is a major intellectual breakthrough.

[a] 11% of option holders let in-the-money options expire worthless.


Call value
0
5
10
15
20
80 85 90 95 100 105 110 115
Stock price


Put value
0
2
4
6
8
10
12
14
80 85 90 95 100 105 110 115
Stock price

Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for *cash* dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n-for-m stock split, m shares become n shares.
- Accordingly, the strike price is only m/n times its previous value, and the number of shares covered by one option becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- We assume options are unprotected.

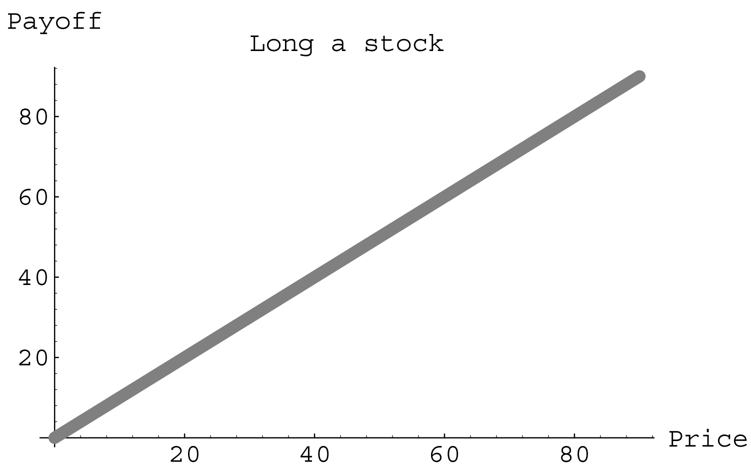
Example

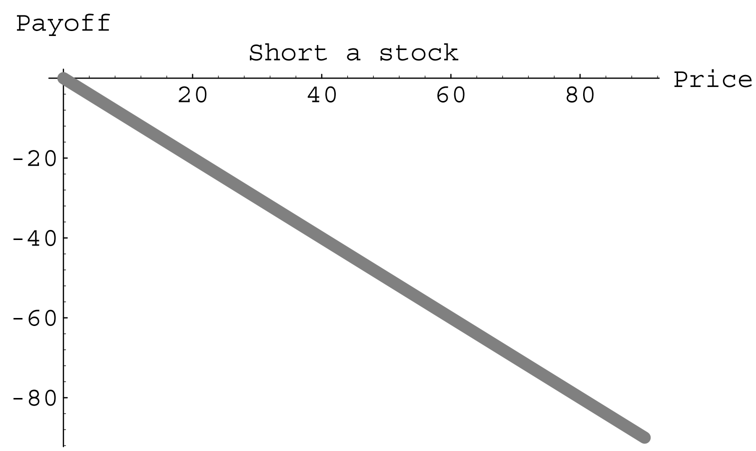
- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.

Payoff of Stock





Short Selling (concluded)

- Not all assets can be shorted.
- In reality, short selling is not simply the opposite of going long.[a]

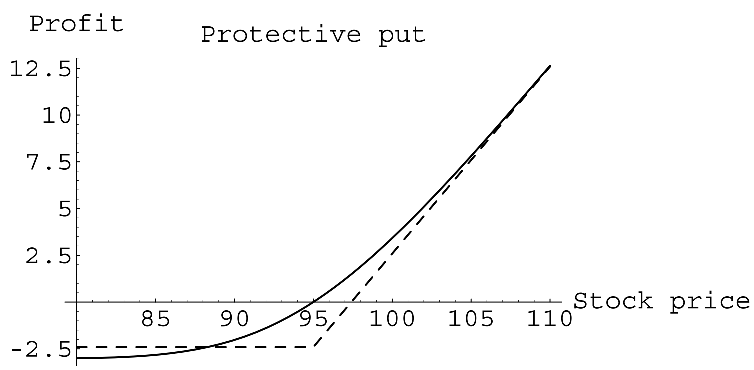
---

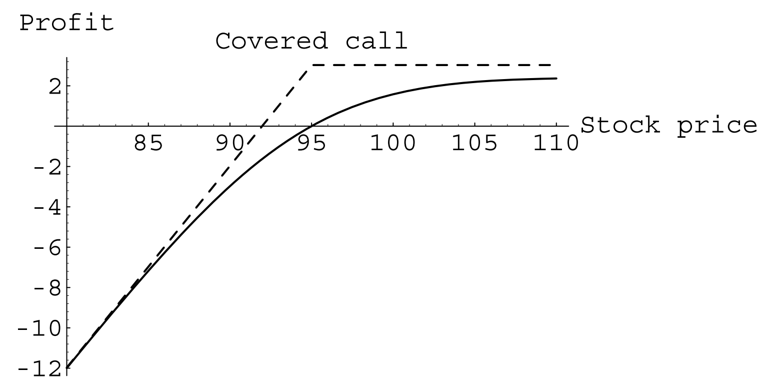
[a] Kosowski & Neftci (2015). See `https://tw.news.appledaily.com/headline/daily/20180307/37950481/` for an example in Taiwan on February 6, 2018.

Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Covered call: A long position in stock with a short call.[a]
  - It is "covered" because the stock can be delivered to the buyer of the call if the call is exercised.
- Protective put: A long position in stock with a long put.
- Both strategies break even only if the stock price rises above a certain level, so they are bullish.

[a] A short position has a payoff opposite in sign to that of a long position.





Solid lines are profits of the portfolio one month before maturity, assuming the portfolio is set up when $S = 95$ then.

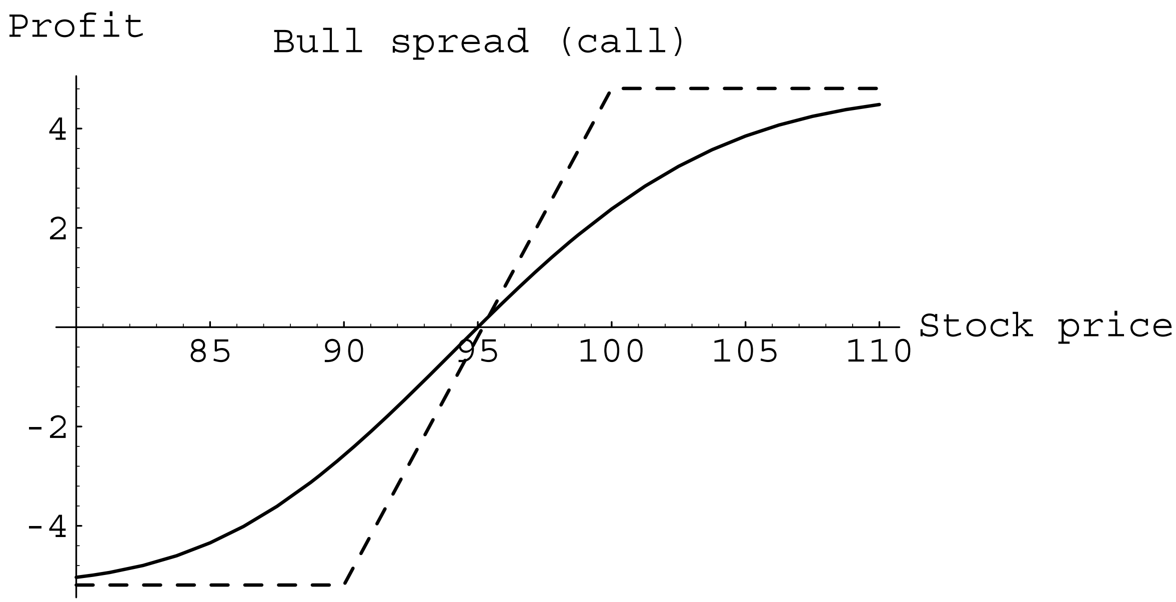
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with

$$X_L < X_M < X_H.$$

Covered Position: Spread (continued)

- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
    * When both are exercised at expiration.
  - The maximum loss is $C_L - C_H$.
    * When neither is exercised at expiration.
  - If we buy $(X_H - X_L)^{-1}$ units of the bull call spread and $X_H - X_L \to 0$, a (Heaviside) step function emerges as the payoff.



Profit
Bull spread (call)
4
2
-2
-4
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.[a]
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short *two* X_M calls.

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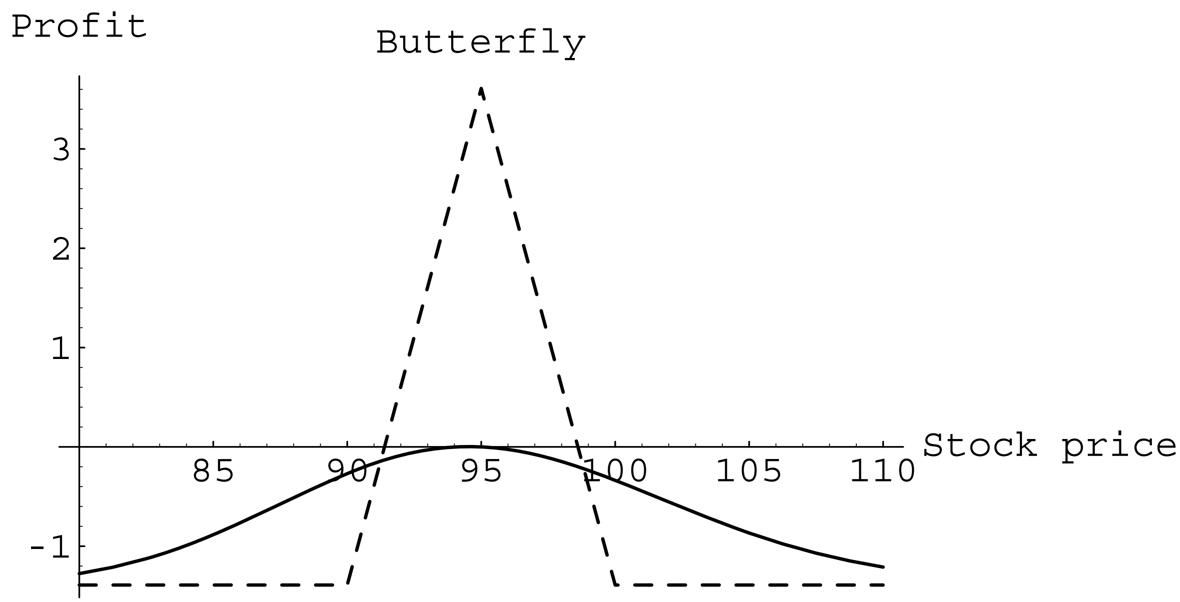
[a] See `https://www.businesstoday.com.tw/article/category/80392/post/201803070` for a sad example in Taiwan on February 6, 2018.


Profit
Butterfly
3
2
1
-1
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- A butterfly spread pays a positive amount at expiration only if the asset price falls between X_L and X_H.
- Assume $X_M = (X_H + X_L)/2$.
- Take a position in $(X_M - X_L)^{-1}$ units of the butterfly spread.
- When $X_H - X_L \to 0$, it approximates a state contingent claim,[a] which pays \$1 only when the state $S = X_M$ happens.[b]

---

[a] Alternatively, Arrow security.

[b] See Exercise 7.4.5 of the textbook.

Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The state price equals

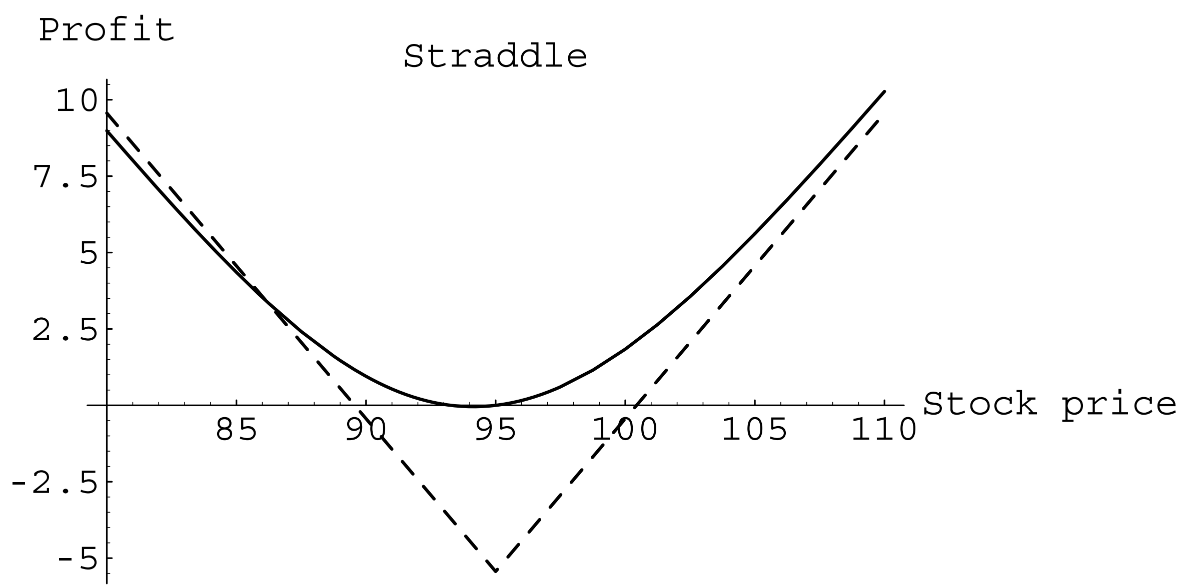
$$\frac{\partial^2 C}{\partial X^2}.$$

  - Recall that C is the call's price.[a]
- In fact, the FV of $\partial^2 C/\partial X^2$ is the probability density of the stock price $S_T = X$ at option's maturity.[b]

[a] One can also use the put (see Exercise 9.3.6 of the textbook).
[b] Breeden & Litzenberger (1978). This formula is model-independent!

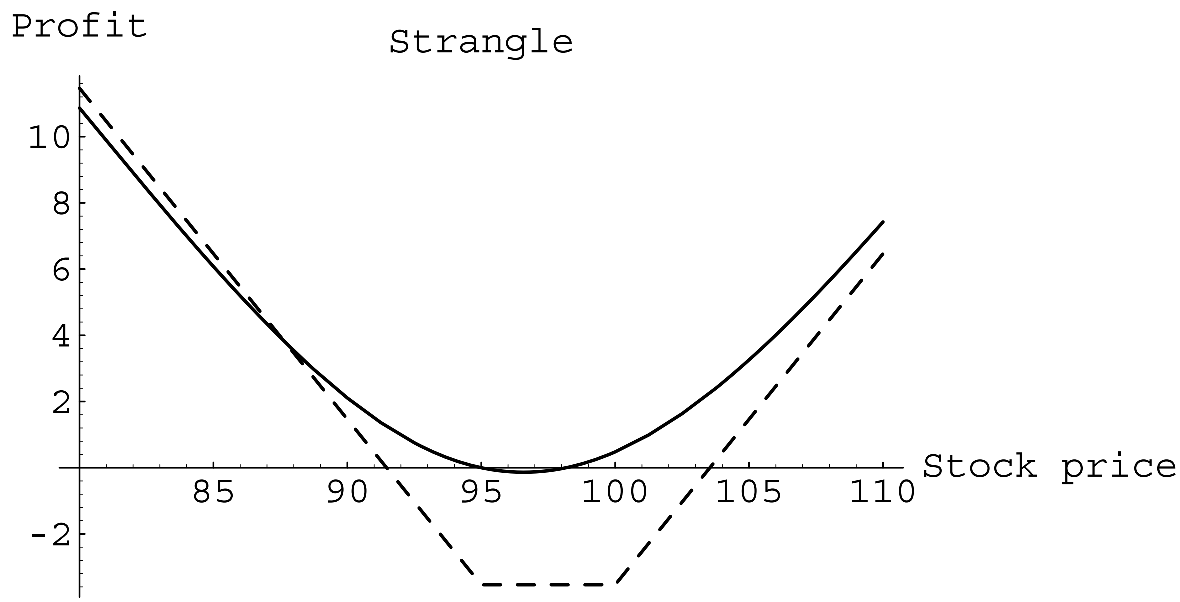
Covered Position: Combination

- A combination consists of options of different types on the same underlying asset.
  - These options must be either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
  - Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
  - Selling a straddle benefits from low volatility.


Profit
Straddle
10
7.5
5
2.5
-2.5
-5
85
90
95
100
105
110
Stock price

Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.


Profit
Strangle
10
8
6
4
2
-2
85
90
95
100
105
110
Stock price