

# Principles of Financial Computing

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# *Introduction*

You must go into finance, Amory.  
— F. Scott Fitzgerald (1896–1940),  
*This Side of Paradise* (1920)

The two most dangerous words in Wall Street  
vocabulary are “financial engineering.”  
— Wilbur Ross (2007)

## Class Information

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press, 2002.
- Official Web page is

`www.csie.ntu.edu.tw/~lyuu/finance1.html`

- Lecture notes will be uploaded before class.
- Homeworks and teaching assistants will also be announced there.
- Do not mistake last year's homeworks for this year's!

## Class Information (continued)

- Check

`www.csie.ntu.edu.tw/~lyuu/capitals.html`

for some of the software.

- Check the CBIS (Convertible Bond Information System) at

`http://www.cb.idv.tw/cb/`

for a convertible-bond software system.<sup>a</sup>

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<sup>a</sup>Thanks to Prof. Gow-Hsing King and his strong team.

## Class Information (concluded)

- Please ask many questions in class.
  - This is the best way for me to remember you in a large class.<sup>a</sup>

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<sup>a</sup> “[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name.” (*New York Times*, September 3, 2003.)

## Grading

- Programming assignments.
- Treat each homework as an examination.
- You are expected to write your own codes and turn in your source code.
- Do not copy or collaborate with fellow students.
- Never ask your friends to write programs for you.
- Never give your code to other students or publish your code.

## What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Help in finding your thesis directions.



## What This Course Is *Not* About

- How to program.<sup>a</sup>
  - A software bug cost Knight Capital Group, Inc. US\$457.6 million on August 1, 2012.<sup>b</sup>
- Basic calculus, probability, combinatorics, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.
- Professional behavior.

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<sup>a</sup><http://www.csie.ntu.edu.tw/train/>

<sup>b</sup>Kirilenko & Lo (2013).

## Useful Journals

- *Applied Mathematical Finance.*
- *Applied Mathematics and Computations.*
- *Communications on Pure and Applied Mathematics.*
- *Computers & Mathematics with Applications.*
- *European Journal of Finance.*
- *European Journal of Operational Research.*
- *Finance and Stochastics.*
- *Finance Research Letters.*
- *Financial Analysts Journal.*
- *Financial Management.*

## Useful Journals (continued)

- *Insurance: Mathematics and Economics.*
- *International Journal of Finance & Economics.*
- *International Journal of Theoretical and Applied Finance.*
- *Journal of Banking & Finance.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Empirical Finance.*

## Useful Journals (continued)

- *Journal of Finance.*
- *Journal of Financial Economics.*
- *Journal of Financial Markets.*
- *Journal of Fixed Income.*
- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*

## Useful Journals (concluded)

- *Journal of Risk and Uncertainty.*
- *Management Science.*
- *Mathematical Finance.*
- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Review of Finance.*
- *Risk Magazine.*
- *SIAM Journal on Financial Mathematics.*
- *Stochastics and Stochastics Reports.*

## The Modelers' Hippocratic Oath<sup>a</sup>

- I will remember that I didn't make the world, and it doesn't satisfy my equations.
- Though I will use models boldly to estimate value, I will not be overly impressed by mathematics.
- I will never sacrifice reality for elegance without explaining why I have done so.
- Nor will I give the people who use my model false comfort about its accuracy. Instead, I will make explicit its assumptions and oversights.
- I understand that my work may have enormous effects on society and the economy, many of them beyond my comprehension.

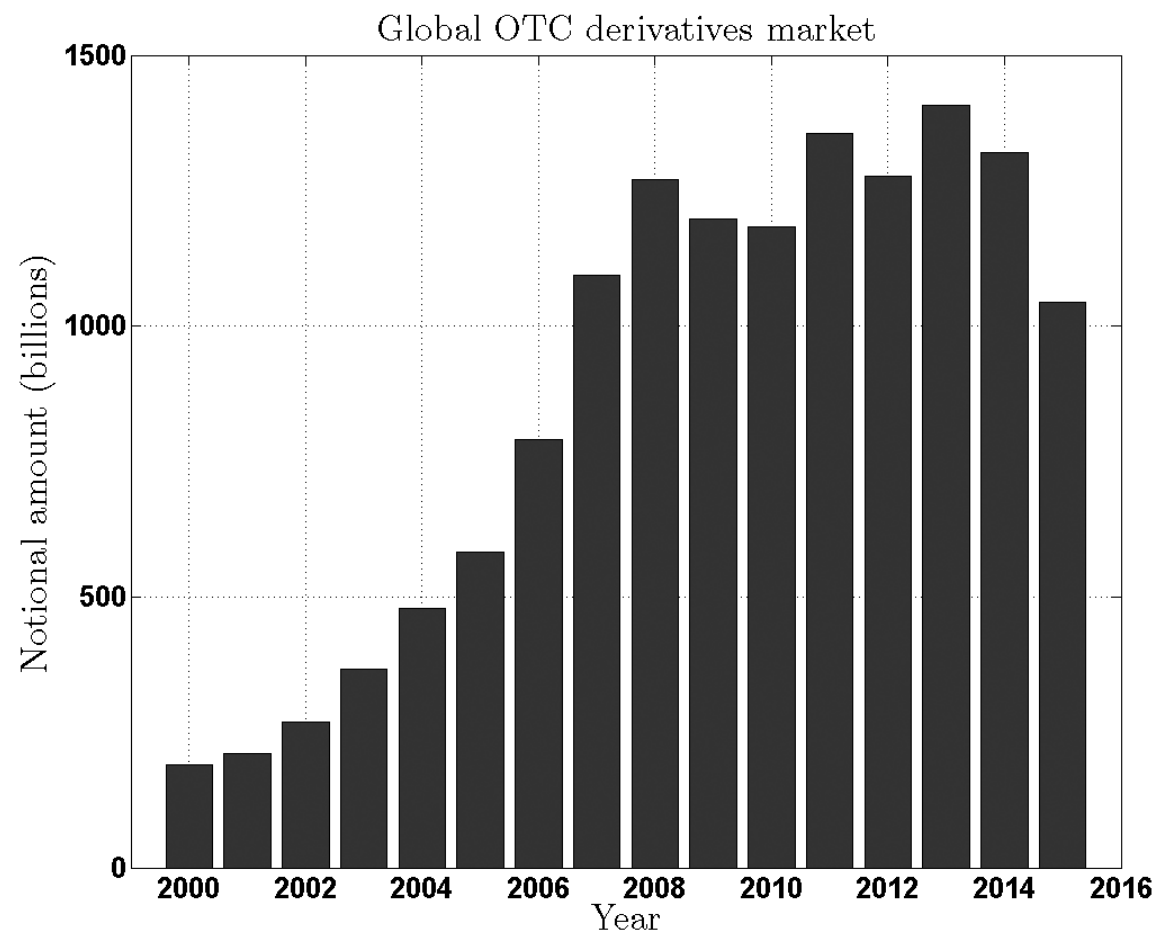
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<sup>a</sup>Emanuel Derman & Paul Wilmott, January 7, 2009.

## Outstanding U.S. Debts (bln)

| Year      | Municipal | Treasury | Mortgage—<br>related | U.S.<br>corporate | Fed<br>agencies | Money<br>market | Asset—<br>backed | Total    |
|-----------|-----------|----------|----------------------|-------------------|-----------------|-----------------|------------------|----------|
| <b>85</b> | 859.5     | 1,437.7  | 372.1                | 776.5             | 293.9           | 847.0           | 0.9              | 4,537.6  |
| <b>86</b> | 920.4     | 1,619.0  | 534.4                | 959.6             | 307.4           | 877.0           | 7.2              | 5,225.0  |
| <b>87</b> | 1,010.4   | 1,724.7  | 672.1                | 1,074.9           | 341.4           | 979.8           | 12.9             | 5,816.2  |
| <b>88</b> | 1,082.3   | 1,821.3  | 772.4                | 1,195.7           | 381.5           | 1,108.5         | 29.3             | 6,391.0  |
| <b>89</b> | 1,135.2   | 1,945.4  | 971.5                | 1,292.5           | 411.8           | 1,192.3         | 51.3             | 7,000.0  |
| <b>90</b> | 1,184.4   | 2,195.8  | 1,333.4              | 1,350.4           | 434.7           | 1,156.8         | 89.9             | 7,745.4  |
| <b>91</b> | 1,272.2   | 2,471.6  | 1,636.9              | 1,454.7           | 442.8           | 1,054.3         | 129.9            | 8,462.4  |
| <b>92</b> | 1,302.8   | 2,754.1  | 1,937.0              | 1,557.0           | 484.0           | 994.2           | 163.7            | 9,192.8  |
| <b>93</b> | 1,377.5   | 2,989.5  | 2,144.7              | 1,674.7           | 570.7           | 971.8           | 199.9            | 9,928.8  |
| <b>94</b> | 1,341.7   | 3,126.0  | 2,251.6              | 1,755.6           | 738.9           | 1,034.7         | 257.3            | 10,505.8 |
| <b>95</b> | 1,293.5   | 3,307.2  | 2,352.1              | 1,937.5           | 844.6           | 1,177.3         | 316.3            | 11,228.5 |
| <b>96</b> | 1,296.0   | 3,459.7  | 2,486.1              | 2,122.2           | 925.8           | 1,393.9         | 404.4            | 12,088.1 |
| <b>97</b> | 1,367.5   | 3,456.8  | 2,680.2              | 2,346.3           | 1,022.6         | 1,692.8         | 535.8            | 13,102.0 |
| <b>98</b> | 1,464.3   | 3,355.5  | 2,955.2              | 2,666.2           | 1,296.5         | 1,978.0         | 731.5            | 14,447.2 |
| <b>99</b> | 1,532.5   | 3,281.0  | 3,334.2              | 3,022.9           | 1,616.5         | 2,338.2         | 900.8            | 16,026.4 |
| <b>00</b> | 1,567.8   | 2,966.9  | 3,564.7              | 3,372.0           | 1,851.9         | 2,661.0         | 1,071.8          | 17,066.1 |
| <b>01</b> | 1,688.4   | 2,967.5  | 4,125.5              | 3,817.4           | 2,143.0         | 2,542.4         | 1,281.1          | 18,565.3 |
| <b>02</b> | 1,783.8   | 3,204.9  | 4,704.9              | 3,997.2           | 2,358.5         | 2,577.5         | 1,543.3          | 20,170.1 |

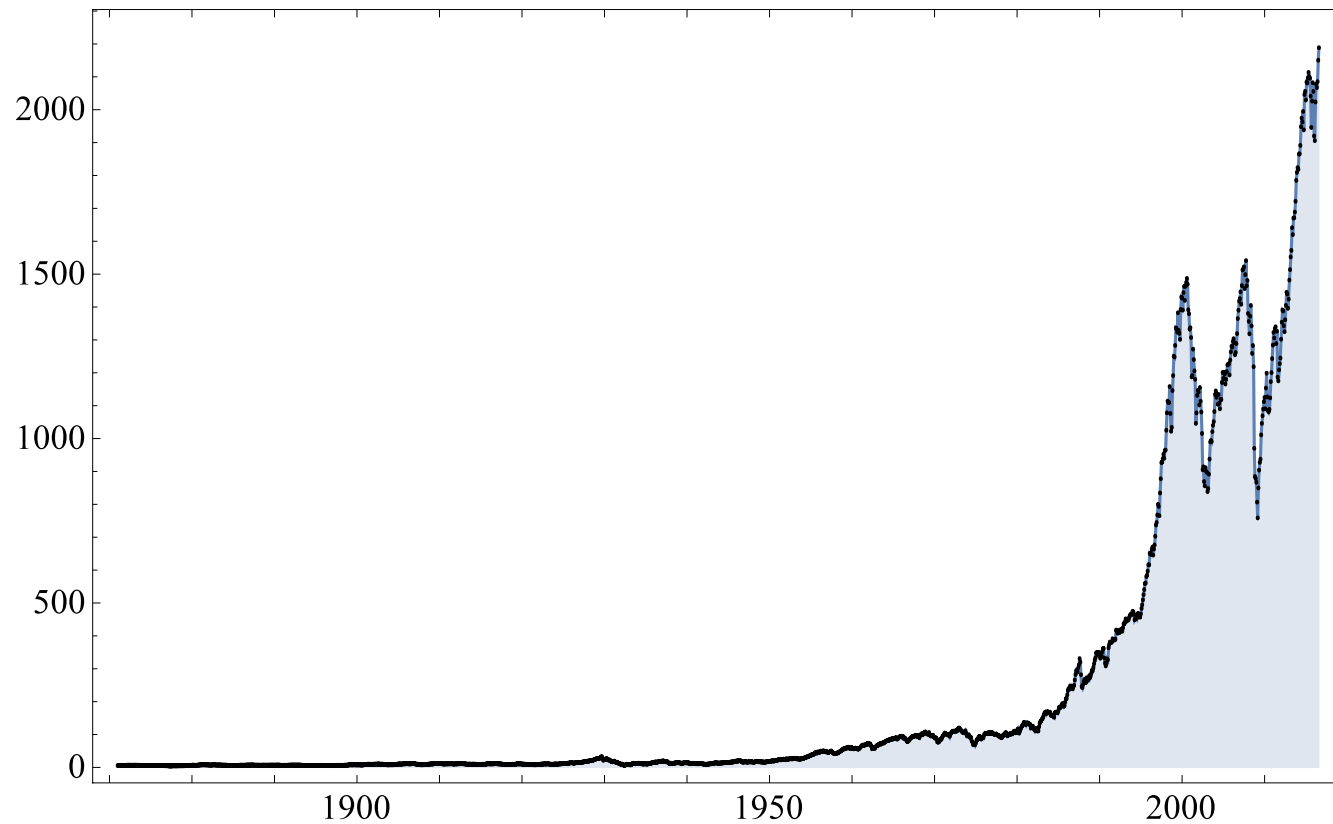
# Global OTC Derivatives Market





## Standard and Poor's (S&P) 500 Index (by Robert Shiller<sup>a</sup>)

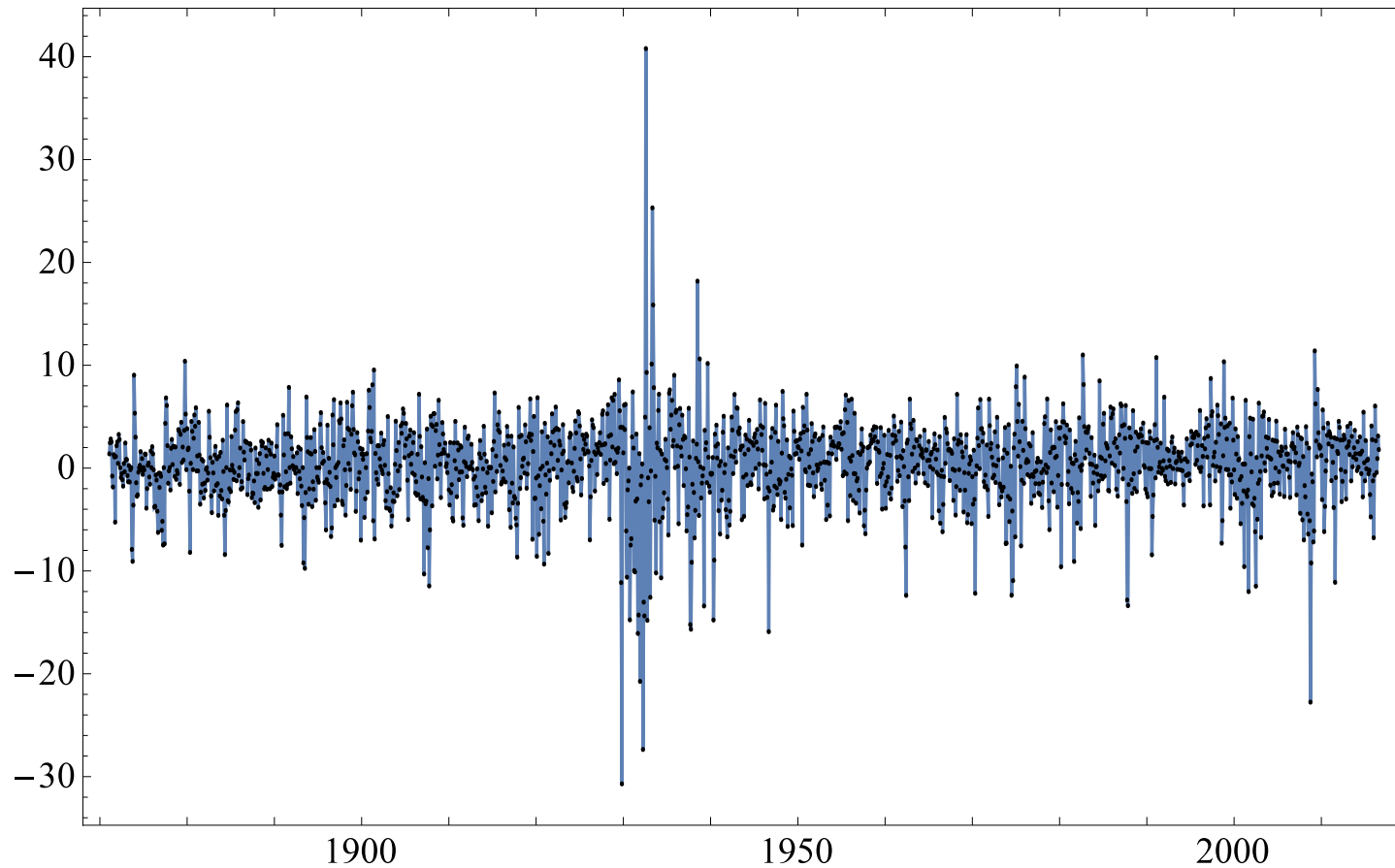
Monthly S&amp;P 500 Index (Jan 1, 1871–Aug 1, 2016)



<sup>a</sup>Co-winner of the 2013 Nobel Prize in Economic Sciences.

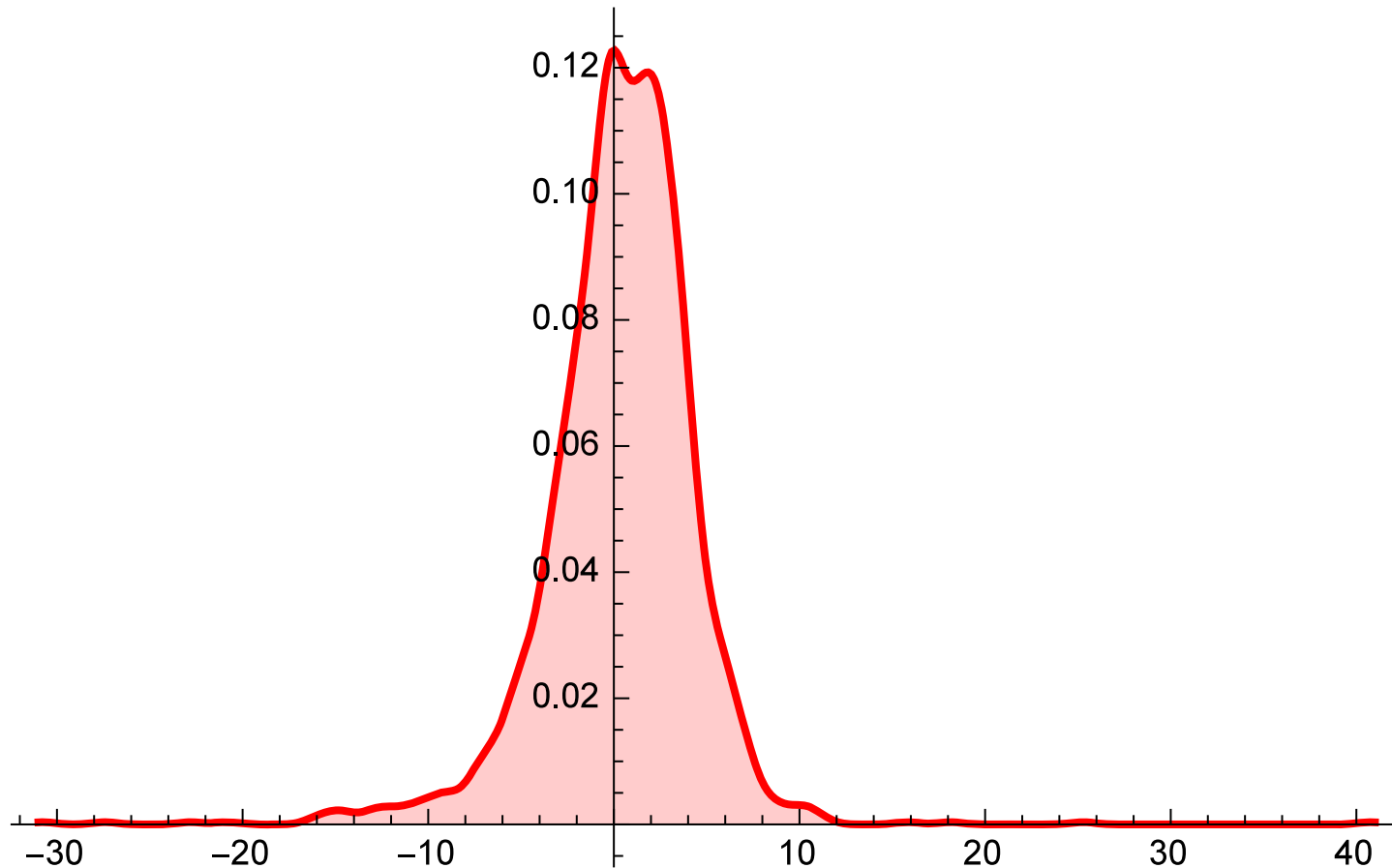
# Returns of S&P 500 Index

Monthly log returns (%) of S&P 500 Index (Feb 1, 1871–Aug 1, 2016)



## Distribution of Returns of S&P 500 Index

Monthly log returns (%) of S&P 500 Index (Feb 1, 1871–Aug 1, 2016)



# *Analysis of Algorithms*

I can calculate the motions  
of the heavenly bodies,  
but not the madness of people.  
— Isaac Newton (1642–1727)

It is unworthy of excellent men  
to lose hours like slaves  
in the labor of computation.  
— Gottfried Wilhelm Leibniz (1646–1716)

## Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

## Complexity

- A set of basic operations are assumed to take one unit of time  $(+, -, \times, /, \log, x^y, e^x, \dots)$ .
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

## Common (Asymptotic) Complexities

- Let  $n$  stand for the “size” of the problem.
  - Number of elements, number of cash flows, number of time periods, etc.
- Linear time if the complexity is  $O(n)$ .
- Quadratic time if the complexity is  $O(n^2)$ .
- Cubic time if the complexity is  $O(n^3)$ .
- Superpolynomial if the complexity is higher than polynomials, say  $2^{O(\sqrt{n})}$ .<sup>a</sup>
- Exponential time if the complexity is  $2^{O(n)}$ .

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<sup>a</sup>E.g., Dai (B82506025, R86526008, D8852600) & Lyuu (2007); Lyuu & C. Wang (F95922018) (2011); H. Chiu (R98723059) (2012).

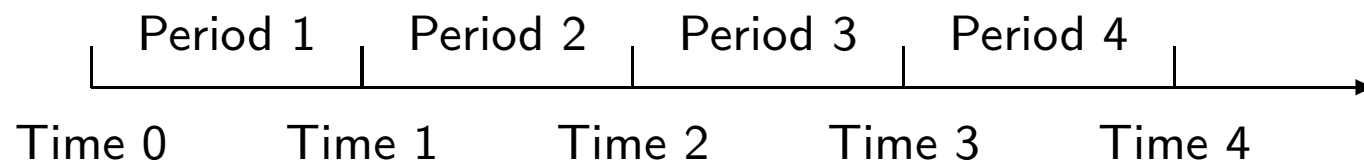


# *Basic Financial Mathematics*

In the fifteenth century  
mathematics was mainly concerned with  
questions of commercial arithmetic and  
the problems of the architect.  
— Joseph Alois Schumpeter (1883–1950)

I'm more concerned about  
the return of my money than  
the return on my money.  
— Will Rogers (1879–1935)

## The Time Line



## Time Value of Money<sup>a</sup>

$$FV = PV(1 + r)^n, \quad (1)$$

$$PV = FV \times (1 + r)^{-n}.$$

- FV (future value).
- PV (present value).
- $r$ : interest rate.

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<sup>a</sup>Fibonacci (1170–1240); Irving Fisher (1867–1947).

## Periodic Compounding

- Suppose the annual interest rate  $r$  is compounded  $m$  times per annum.
- Then

$$1 \rightarrow \left(1 + \frac{r}{m}\right) \rightarrow \left(1 + \frac{r}{m}\right)^2 \rightarrow \left(1 + \frac{r}{m}\right)^3 \rightarrow \dots$$

- Hence, after  $n$  years,

$$\text{FV} = \text{PV} \left(1 + \frac{r}{m}\right)^{nm}. \quad (2)$$

## Common Compounding Methods

- Annual compounding:  $m = 1$ .
- Semiannual compounding:  $m = 2$ .
- Quarterly compounding:  $m = 4$ .
- Monthly compounding:  $m = 12$ .
- Weekly compounding:  $m = 52$ .
- Daily compounding:  $m = 365$ .
- Continuous compounding:  $m = \infty$ .

## Easy Translations

- An annual interest rate of  $r$  compounded  $m$  times a year is “equivalent to” an interest rate of  $r/m$  per  $1/m$  year.
- If a loan asks for a return of 1% per month, the annual interest rate will be 12% *with monthly compounding*.

## Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[1 + (0.1/2)]^2 = 1.1025$$

one year from now.

- The rate is “equivalent to” an interest rate of 10.25% compounded once *per annum*,

$$1 + 0.1025 = 1.1025$$



## Rule of 72

- Let the annual interest rate be  $r$ .
- How many years  $T$  will it take for your money to double?
- The identity to solve is

$$(1 + r)^T = 2.$$

- So

$$T = \frac{\ln 2}{\ln(1 + r)}.$$

- Is there an easier way?

## Rule of 72 (concluded)

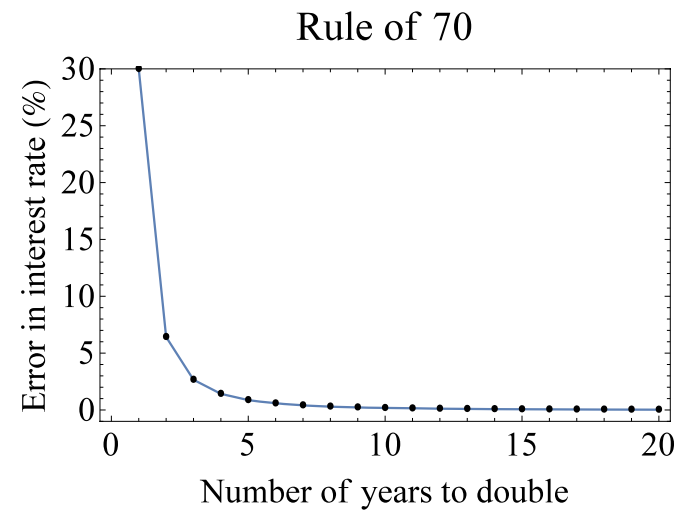
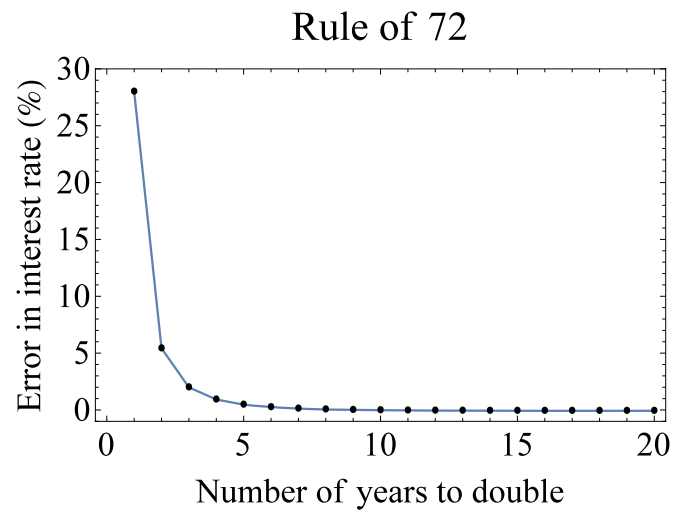
- The rule of 72 is a heuristic to estimate  $T$ .
- It says

$$T \approx \frac{72}{r (\%)}$$

- So it takes about  $72/12 = 6$  years to double the GDP if the annual growth rate is 12%.
- Reason:

$$\frac{\ln 2}{\ln(1 + r)} \approx \frac{0.693}{r} + 0.3466.$$

## How Good Is the Rule of 72?<sup>a</sup>



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<sup>a</sup>True interest rate subtracted by the approximation (in %).

## Continuous Compounding<sup>a</sup>

- Let  $m \rightarrow \infty$  so that

$$\left(1 + \frac{r}{m}\right)^m \rightarrow e^r$$

in Eq. (2) on p. 29.

- Then

$$\text{FV} = \text{PV} \times e^{rn},$$

where  $e = 2.71828 \dots$

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<sup>a</sup>Jacob Bernoulli (1654–1705) in 1685.

## Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is  $r_1$  for  $n_1$  years and  $r_2$  for the following  $n_2$  years.
- Then the FV of one dollar will be

$$e^{r_1 n_1 + r_2 n_2}$$

after  $n_1 + n_2$  years.

## Conversion between Compounding Methods

- Suppose  $r_1$  is the annual rate with continuous compounding.
- Suppose  $r_2$  is the equivalent rate compounded  $m$  times per annum.
- How are they related?

## Conversion between Compounding Methods (continued)

- Principle: Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1 + \frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,<sup>a</sup>

$$\begin{aligned} r_1 &= m \ln \left(1 + \frac{r_2}{m}\right), \\ r_2 &= m \left(e^{r_1/m} - 1\right). \end{aligned}$$

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<sup>a</sup>Are they really equivalent? In what sense are they equivalent?

## Conversion between Compounding Methods (concluded)

- Suppose  $r_1$  is the annual rate compounded  $m_1$  times per annum.
- Suppose  $r_2$  is the equivalent rate compounded  $m_2$  times per annum.
- Then

$$\left(1 + \frac{r_1}{m_1}\right)^{m_1} = \left(1 + \frac{r_2}{m_2}\right)^{m_2}. \quad (3)$$



## The PV Formula

- The PV of the cash flow  $C_1, C_2, \dots, C_n$  at times  $1, 2, \dots, n$  is

$$\text{PV} = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \dots + \frac{C_n}{(1+y)^n}. \quad (4)$$

- This formula and its variations are the engine behind most of financial calculations.<sup>a</sup>
  - What is  $y$ ?
  - What are  $C_i$ ?
  - What is  $n$ ?
- It will be justified on p. 219.

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<sup>a</sup>Cochrane (2005), “Asset pricing theory all stems from one simple concept [...]: price equals expected discounted payoff.”

## An Algorithm for Evaluating PV in Eq. (4)

```
1:  $x := 0$ ;  
2: for  $i = 1, 2, \dots, n$  do  
3:    $x := x + C_i / (1 + y)^i$ ;  
4: end for  
5: return  $x$ ;
```

- The algorithm takes time proportional to  $\sum_{i=1}^n i = O(n^2)$ .<sup>a</sup>
- Can improve it to  $O(n)$  if you apply  $a^b = e^{b \ln a}$  in step 3.<sup>b</sup>

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<sup>a</sup>If only  $+$ ,  $-$ ,  $\times$ , and  $/$  are allowed.

<sup>b</sup>Recall that we count  $x^y$  as taking one unit of time.

## Another Algorithm for Evaluating PV

```
1:  $x := 0$ ;  
2:  $d := 1 + y$ ;  
3: for  $i = n, n - 1, \dots, 1$  do  
4:    $x := (x + C_i)/d$ ;  
5: end for  
6: return  $x$ ;
```

## Horner's Rule: The Idea Behind p. 43

- This idea is

$$\left( \cdots \left( \left( \frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

– Due to Horner (1786–1837) in 1819.

- The algorithm takes  $O(n)$  time.
- It is the most efficient possible in terms of the absolute number of arithmetic operations.<sup>a</sup>

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<sup>a</sup>A. Borodin & Munro (1975).

## Annuities<sup>a</sup> (Certain)

- An annuity pays out the same  $C$  dollars at the end of each year for  $n$  years.
- With a rate of  $r$ , the FV at the end of the  $n$ th year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C \frac{(1+r)^n - 1}{r}. \quad (5)$$

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<sup>a</sup>Jan de Witt (1625–1672) in 1671; Nicholas Bernoulli (1687–1759) in 1709.

## General Annuities

- Suppose that  $m$  payments of  $C$  dollars each are received per year (the general annuity).
- Let  $r$  be compounded  $m$  times per annum.<sup>a</sup>
- Then Eq. (5) becomes

$$C \frac{\left(1 + \frac{r}{m}\right)^{nm} - 1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\text{PV} = \sum_{i=1}^{nm} C \left(1 + \frac{r}{m}\right)^{-i} = C \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \quad (6)$$

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<sup>a</sup>If your  $r$  has a compounding frequency different from  $m$ , convert it with Eq. (3) on p. 40 first.

## Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

## Amortization (concluded)

- From Eq. (6) on p. 46, the regular payment equals

$$C = \text{loan amount} \times \frac{\frac{r}{m}}{1 - \left(1 + \frac{r}{m}\right)^{-nm}}. \quad (7)$$

- For monthly payments, take  $m = 12$ .



## Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, only the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.<sup>a</sup>

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<sup>a</sup> *The Economist* (2016), “In most countries banks minimize their risk by offering short-term or floating-rate mortgages. American borrowers get a better deal: cheap 30-year fixed-rate mortgages that can be repaid early free.”

## A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solve Eq. (6) on p. 46 with  $PV = 250000$ ,  $n = 15$ ,  $m = 12$ , and  $r = 0.08$ .
- That is,

$$250000 = C \frac{1 - \left(1 + \frac{0.08}{12}\right)^{-15 \times 12}}{\frac{0.08}{12}}.$$

- This gives a monthly payment of  $C = 2389.13$ .

## The Amortization Schedule

| Month | Payment     | Interest    | Principal   | Remaining<br>principal |
|-------|-------------|-------------|-------------|------------------------|
|       |             |             |             | 250,000.000            |
| 1     | 2,389.13    | 1,666.667   | 722.464     | 249,277.536            |
| 2     | 2,389.13    | 1,661.850   | 727.280     | 248,550.256            |
| 3     | 2,389.13    | 1,657.002   | 732.129     | 247,818.128            |
| ...   |             |             |             |                        |
| 178   | 2,389.13    | 47.153      | 2,341.980   | 4,730.899              |
| 179   | 2,389.13    | 31.539      | 2,357.591   | 2,373.308              |
| 180   | 2,389.13    | 15.822      | 2,373.308   | 0.000                  |
| Total | 430,043.438 | 180,043.438 | 250,000.000 |                        |

## A Numerical Example (continued)

In every month:

- The principal and interest parts add up to \$2,389.13.
- The remaining principal is reduced by the amount indicated under the Principal heading.<sup>a</sup>
  - The Principal column forms a geometric sequence.<sup>b</sup>
- The interest is computed by multiplying the remaining balance of the previous month by  $0.08/12$ .

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<sup>a</sup>This column varies with  $r$ . Thanks to a lively class discussion on Feb 24, 2010. In fact, every column varies with  $r$ . Contributed by Ms. Wu, Japie (R01921056) on February 20, 2013.

<sup>b</sup>See p. 1283. Contributed by Mr. Sun, Ao (R05922147) on February 22, 2017.

## A Numerical Example (concluded)

Note that:

- The Principal column adds up to \$250,000.
- The Payment column adds up to \$430,043.438!
- If the borrower plans to pay off the mortgage right after the 178th month's monthly payment, he needs to pay another \$4,730.899.<sup>a</sup>

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<sup>a</sup>Contributed by Ms. Wu, Japie (R01921056) on February 20, 2013.

## Method 1 of Calculating the Remaining Principal

- A month's principal payment = monthly payment – (previous month's remaining principal)  $\times$  (monthly interest rate).
- A month's remaining principal = previous month's remaining principal – principal payment calculated above.
- Generate the amortization schedule until you reach the particular month you are interested in.

## Method 1 of Calculating the Remaining Principal (concluded)

- This method is relatively slow but is universal in its applicability.
- It can, for example, accommodate prepayments and variable interest rates.

## Method 2 of Calculating the Remaining Principal

- Right after the  $k$ th payment, the remaining principal is the PV of the future  $nm - k$  cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}. \quad (8)$$

- This method is much faster.
- But it is more limited in applications because it makes more assumptions.



## Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- Recall Eq. (2) on p. 29:  $FV = PV \left(1 + \frac{r}{m}\right)^{nm}$ .
- BEY corresponds to the  $r$  above that equates PV with FV when  $m = 2$ .
- MEY corresponds to the  $r$  above that equates PV with FV when  $m = 12$ .
- Again, although BEY and MEY might be different, they nevertheless mean the *same* yield.

## Internal Rate of Return (IRR)

- It is the yield  $y$  which equates an investment's PV with its price  $P$ ,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

- IRR assumes all cash flows are reinvested at the *same* rate as the internal rate of return because:

$$FV = C_1(1+y)^{n-1} + C_2(1+y)^{n-2} + \cdots + C_n.$$

- So it must be used with extreme caution.

## Numerical Methods for IRRs

- Define

$$f(y) \triangleq \sum_{t=1}^n \frac{C_t}{(1+y)^t} - P.$$

–  $P$  is the market price.

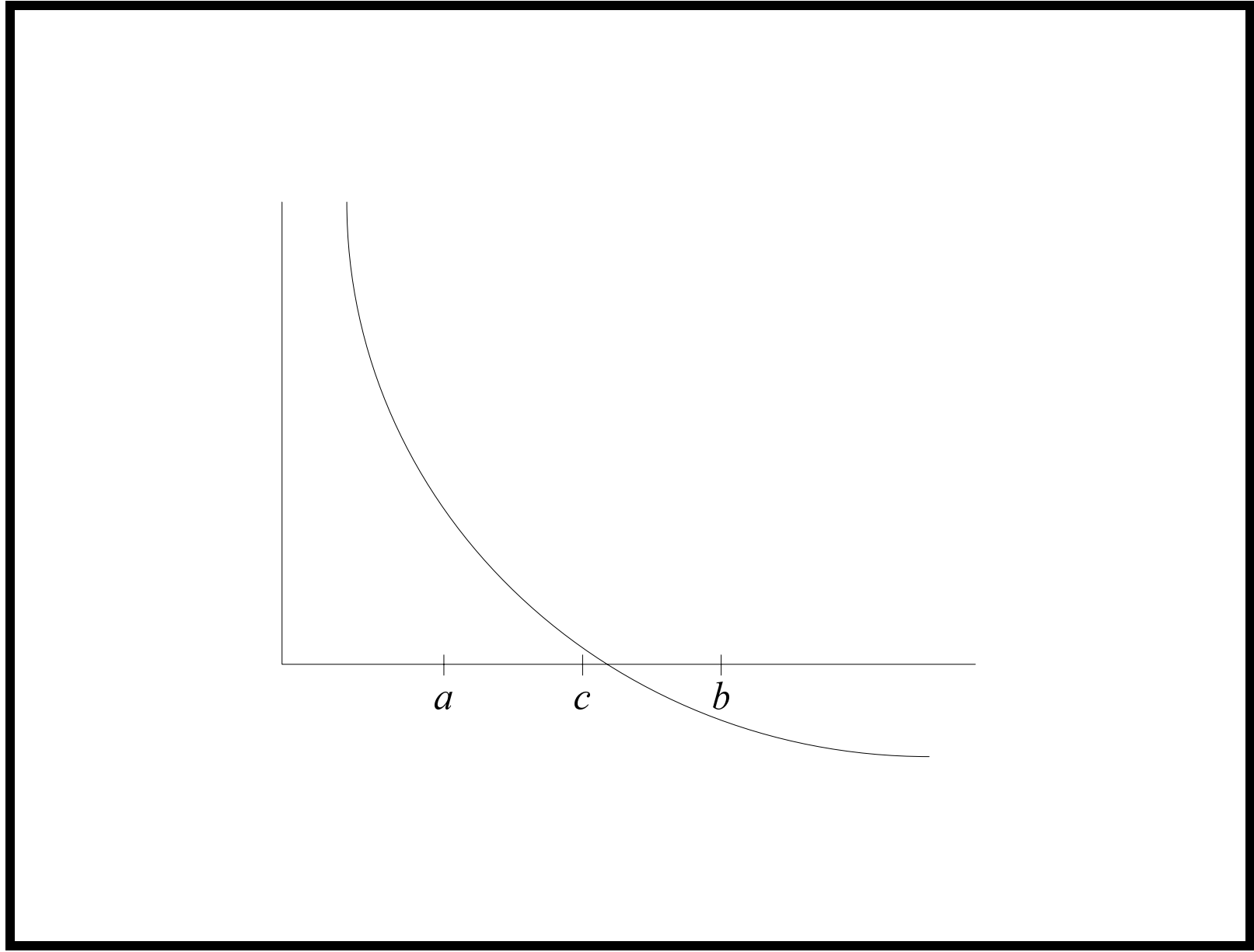
- Solve  $f(y) = 0$  for a real  $y \geq -1$ .<sup>a</sup>
- $f(y)$  is monotonically decreasing in  $y$  if  $C_t > 0$  for all  $t$ .
- So a unique real-number solution exists for this  $f(y)$ .

---

<sup>a</sup>Negative interest rates became a reality for German and Swiss bonds in 2015. In 2016, Sweden, Denmark, and Japan imposed negative interest rates on excess reserves. As many as 355 corporate bonds were issued with negative yields as of June of 2016. Even so,  $-100\%$  should be a natural lower bound because why would anyone or financial institutions want to have every cent confiscated?

## The Bisection Method

- Start with  $a$  and  $b$  where  $a < b$  and  $f(a)f(b) < 0$ .
- Then  $f(\xi)$  must be zero for some  $\xi \in [a, b]$ .
- If we evaluate  $f$  at the midpoint  $c \triangleq (a + b)/2$ , either (1)  $f(c) = 0$ , (2)  $f(a)f(c) < 0$ , or (3)  $f(c)f(b) < 0$ .
- In the first case we are done, in the second case we continue the process with the new bracket  $[a, c]$ , and in the third case we continue with  $[c, b]$ .
- The bracket is halved in the latter two cases.
- After  $n$  steps, we will have confined  $\xi$  within a bracket of length  $(b - a)/2^n$ .



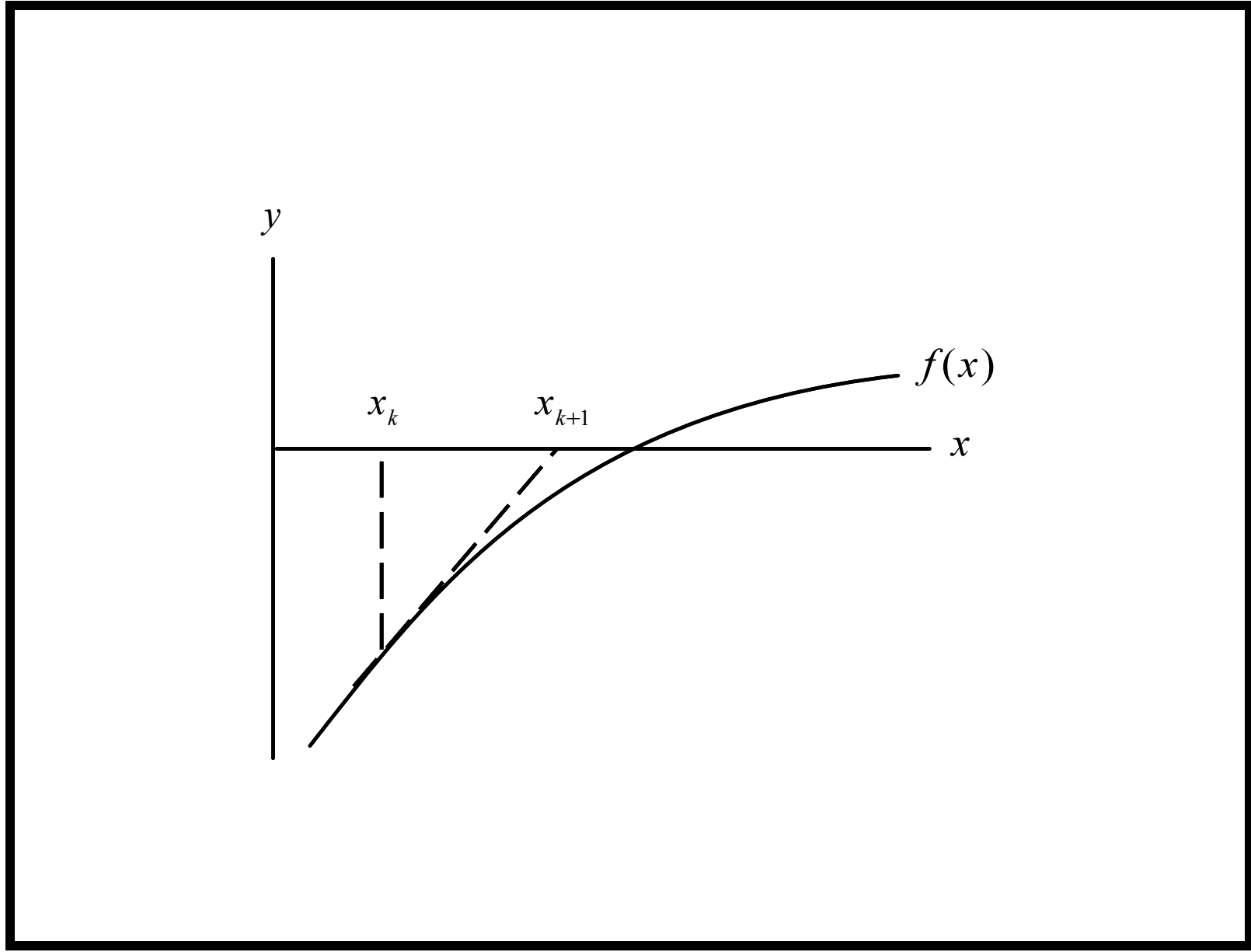
## The Newton-Raphson Method

- It converges faster than the bisection method.
- Start with a first approximation  $x_0$  to a root of  $f(x) = 0$ .
- Then

$$x_{k+1} \triangleq x_k - \frac{f(x_k)}{f'(x_k)}.$$

- When computing yields,

$$f'(x) = - \sum_{t=1}^n \frac{tC_t}{(1+x)^{t+1}}. \quad (9)$$



## The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations  $x_0$  and  $x_1$ .
- Then compute the  $(k + 1)$ st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

- Note that it is easier to calculate  $[f(x_k) - f(x_{k-1})]/(x_k - x_{k-1})$  than the  $f'(x_k)$  on p. 62.



## The Secant Method (concluded)

- Its convergence rate is 1.618.
- This is slightly worse than the Newton-Raphson method's 2.
- But the secant method does not need to evaluate  $f'(x_k)$  needed by the Newton-Raphson method.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

## Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let  $(x_k, y_k)$  be the  $k$ th approximation to the solution of the two simultaneous equations,

$$f(x, y) = 0,$$

$$g(x, y) = 0.$$

## Solving Systems of Nonlinear Equations (continued)

- The  $(k + 1)$ st approximation  $(x_{k+1}, y_{k+1})$  satisfies the following linear equations,

$$\begin{bmatrix} \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\ \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix},$$

with unknowns

$$\begin{aligned} \Delta x_{k+1} &\triangleq x_{k+1} - x_k, \\ \Delta y_{k+1} &\triangleq y_{k+1} - y_k. \end{aligned}$$

## Solving Systems of Nonlinear Equations (concluded)

- The above has a unique solution for  $(\Delta x_{k+1}, \Delta y_{k+1})$  when the  $2 \times 2$  matrix is invertible.
- Finally, set

$$x_{k+1} = x_k + \Delta x_{k+1},$$

$$y_{k+1} = y_k + \Delta y_{k+1}.$$

## Zero-Coupon Bonds (Pure Discount Bonds)

- By Eq. (1) on p. 28, the price of a zero-coupon bond that pays  $F$  dollars in  $n$  periods is

$$F/(1 + r)^n, \quad (10)$$

where  $r$  is the interest rate per period.

- Can be used to meet future obligations as there is no reinvestment risk.<sup>a</sup>

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<sup>a</sup>Recall p. 58.

## Example

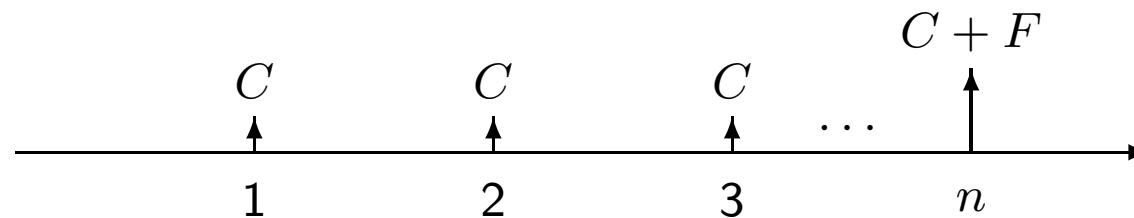
- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at  $1/(1.04)^{40}$ , or 20.83%, of its par value.
- It will be quoted as 20.83.<sup>a</sup>
- If the bond matures in 10 years instead of 20, its price would be 45.64.

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<sup>a</sup>Only one fifth of the par value!

## Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- $F$  denotes the par value, and  $C$  denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.<sup>a</sup>

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<sup>a</sup> “You see, Daddy didn’t bake the cake, and Daddy isn’t the one who gets to eat it. But he gets to slice the cake and hand it out. And when he does, little golden crumbs fall off the cake. And Daddy gets to eat those,” wrote Tom Wolfe (1931–2018) in *Bonfire of the Vanities* (1987).

## Pricing Formula

$$\begin{aligned} P &= \sum_{i=1}^n \frac{C}{\left(1 + \frac{r}{m}\right)^i} + \frac{F}{\left(1 + \frac{r}{m}\right)^n} \\ &= C \frac{1 - \left(1 + \frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1 + \frac{r}{m}\right)^n}. \end{aligned} \quad (11)$$

- $n$ : number of cash flows.
- $m$ : number of payments per year.
- $r$ : annual rate compounded  $m$  times per annum.
- Note  $C = Fc/m$  when  $c$  is the annual coupon rate.
- Price  $P$  can be computed in  $O(1)$  time.



## Yields to Maturity

- It is the  $r$  that satisfies Eq. (11) on p. 72 with  $P$  being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$5 \times \frac{1 - [1 + (0.15/2)]^{-2 \times 10}}{0.15/2} + \frac{100}{[1 + (0.15/2)]^{2 \times 10}} \\ = 74.5138$$

percent of par.

- So 15% is the yield to maturity if the bond sells for 74.5138.<sup>a</sup>

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<sup>a</sup>Note that the yield 15% exceeds the coupon rate 10%.

## Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- “Only 24 percent answered the question correctly.”<sup>a</sup>

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<sup>a</sup>CNN, December 21, 2001.

## Price Behavior (2)<sup>a</sup>

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate  $c$  is above the market interest rate  $r$ ;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

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<sup>a</sup>Consult the text for proofs.

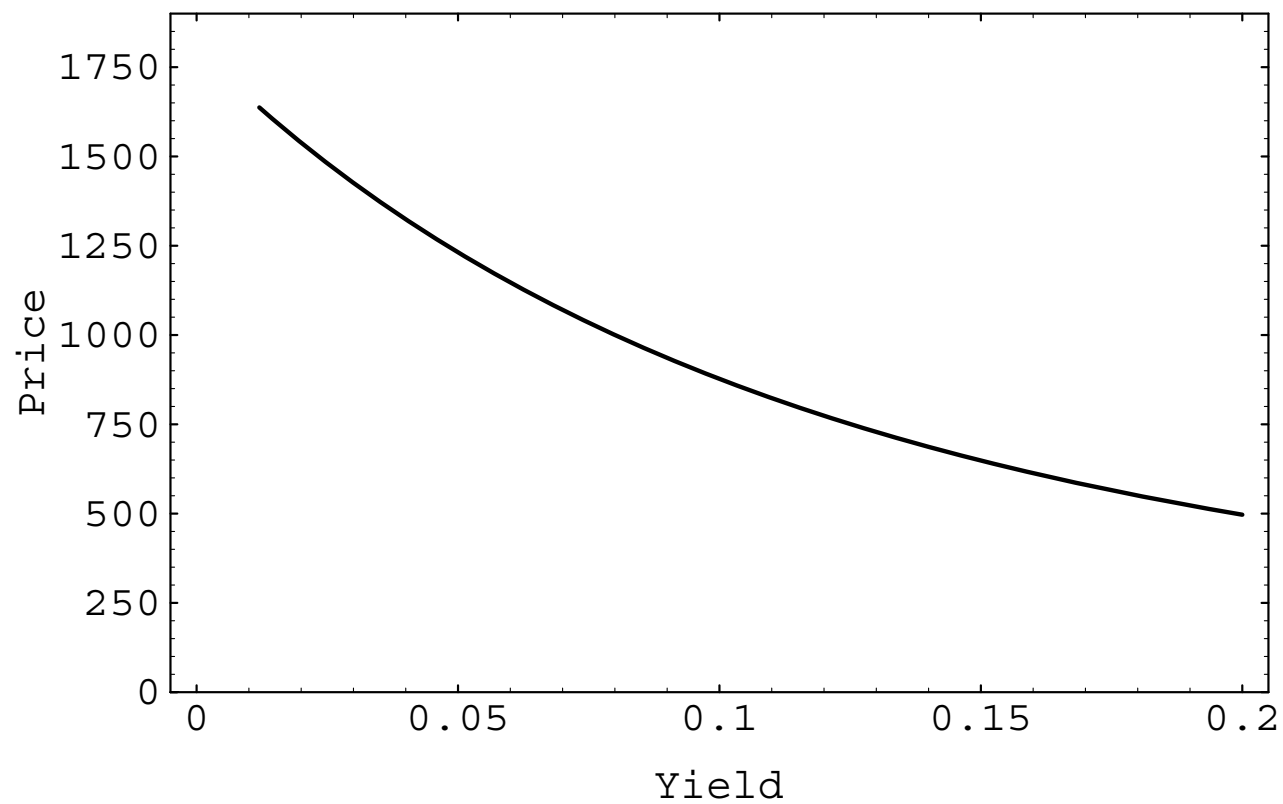
## 9% Coupon Bond

| Yield (%) | Price<br>(% of par) |
|-----------|---------------------|
| 7.5       | 113.37              |
| 8.0       | 108.65              |
| 8.5       | 104.19              |
| 9.0       | 100.00              |
| 9.5       | 96.04               |
| 10.0      | 92.31               |
| 10.5      | 88.79               |

## Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

### Price Behavior (3): Convexity



## Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

## Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date  $(y_1, m_1, d_1)$  to date  $(y_2, m_2, d_2)$  is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1). \quad (12)$$



## Day Count Conventions: 30/360 (continued)

- If  $d_1$  or  $d_2$  is 31, we must change it to 30 *before* applying formula (12).<sup>a</sup>
- Hence:
  - There are 3 days between February 28 and March 1.
  - There are 2 days between February 29 and March 1.
  - There are 29 days between March 1 and March 31.

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<sup>a</sup>The simplest of all the “30/360” variations, this is called the “30E/360” convention, used mainly in the Eurobond market (Kosowski & Neftci, 2015).

## Day Count Conventions: 30/360 (concluded)

- An equivalent formula to (12) on p. 80 without any adjustment is (check it)

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\ + \max(30 - d_1, 0) + \min(d_2, 30).$$

- There are many variations on the “30/360” convention regarding 31, February 28, and February 29.<sup>a</sup>

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<sup>a</sup>Kosowski & Neftci (2015).