Risk-Neutral Pricing

• Assume the local expectations theory.

• The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  
  – For all $t + 1 < T$,

  $$\frac{E_t[P(t+1,T)]}{P(t,T)} = 1 + r(t). \tag{143}$$

  – Relation (143) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^a\)Theorem 16 on p. 550.
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability \( \pi \).

- Equation (143) on p. 1055 can also be expressed as

\[
E_t[P(t+1, T)] = F(t, t+1, T).
\]

- Verify that with, e.g., Eq. (138) on p. 1050.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.\(^a\)

\(^a\)Under the local expectations theory. But the forward rate is not an unbiased estimator of the expected future short rate (p. 1006).
Risk-Neutral Pricing (continued)

- Rewrite Eq. (143) on p. 1055 as

$$E_{t}^{\pi} \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = P(t, T).$$  \hspace{1cm} (144)

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (concluded)

- Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right] = E_t^\pi \left[ \frac{E_{t+1}^\pi [ P(t+2, T) ]}{(1 + r(t))(1 + r(t+1))} \right] = \ldots
\]
\[
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t+1)) \cdots (1 + r(T-1))} \right].
\]
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \]  \hspace{1cm} (145)

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

• Consider an interest rate swap made at time \( t \) (now) with payments to be exchanged at times \( t_1, t_2, \ldots, t_n \).

• For simplicity, assume \( t_{i+1} - t_i \) is a fixed constant \( \Delta t \) for all \( i \), and the notional principal is one dollar.

• The fixed rate is \( c \) per annum.

• The floating-rate payments are based on the future annual rates \( f_0, f_1, \ldots, f_{n-1} \) at times \( t_0, t_1, \ldots, t_{n-1} \).

• The amount to be paid out at time \( t_{i+1} \) is \( (f_i - c) \Delta t \) for the floating-rate payer.
Interest Rate Swaps (continued)

- Simple rates are adopted here.
- Hence $f_i$ satisfies
  \[ P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}. \]
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$. 
Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$
\sum_{i=1}^{n} E_{t}^{\pi} \left[ e^{-\int_{t}^{t_{i}} r(s) ds} (f_{i-1} - c) \Delta t \right] \\
= \sum_{i=1}^{n} E_{t}^{\pi} \left[ e^{-\int_{t}^{t_{i}} r(s) ds} \left( \frac{1}{P(t_{i-1}, t_{i})} - (1 + c\Delta t) \right) \right] \\
= \sum_{i=1}^{n} E_{t}^{\pi} \left[ e^{-\int_{t}^{t_{i}} r(s) ds} \left( e^{\int_{t_{i-1}}^{t_{i}} r(s) ds} - (1 + c\Delta t) \right) \right] \\
= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_{i}) \right] \\
= P(t, t_{0}) - P(t, t_{n}) - c\Delta t \sum_{i=1}^{n} P(t, t_{i}).
$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present-value calculations.
Swap Rate

• The swap rate, which gives the swap zero value, equals

\[ S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \]  \hspace{1cm} (146)

• The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

• For an ordinary swap, \( P(t, t_0) = 1 \).

• The swap rate is called a forward swap rate if \( t_0 > t \).
The Term Structure Equation$^a$

- Let us start with the zero-coupon bonds and the money market account.

- Let the zero-coupon bond price $P(r, t, T)$ follow

\[
\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.
\]

- At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$.

$^a$Vasicek (1977).
The Term Structure Equation (continued)

• The net wealth change follows

\[-dP(r, t, s_1) + \alpha dP(r, t, s_2)\]

\[= (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) dt\]

\[+ (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) dW.\]

• Pick

\[\alpha \triangleq \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}.\]
The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

\[
-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2) = r.
\]

- Simplify the above to obtain

\[
\frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r.
\]

- This becomes

\[
\frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)}
\]

after rearrangement.
The Term Structure Equation (continued)

- Since the above equality holds for any $s_1$ and $s_2$,

$$
\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \triangleq \lambda(r, t)
$$

(147)

for some $\lambda$ independent of the bond maturity $s$.

- As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

- The term $\lambda(r, t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

• Assume a Markovian short rate model,

\[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]

• Then the bond price process is also Markovian.

• By Eq. (14.15) on p. 202 of the textbook,

\[
\mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right)/P, \]

(148)

\[
\sigma_p = \left( \sigma(r, t) \frac{\partial P}{\partial r} \right)/P, \quad (148')
\]

subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

- Substitute $\mu_p$ and $\sigma_p$ into Eq. (147) on p. 1068 to obtain

$$- \frac{\partial P}{\partial T} + \left[ \mu(r, t) - \lambda(r, t) \sigma(r, t) \right] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP. \tag{149}$$

- This is called the term structure equation.

- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.

- Once $P$ is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$
Numerical Examples

• Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

• Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

4%  8%  2%
Numerical Examples (continued)

- *No* real-world probabilities are specified.

- The prices of one- and two-year zero-coupon bonds are, respectively,

\[
\begin{align*}
100/1.04 & = 96.154, \\
100/(1.05)^2 & = 90.703.
\end{align*}
\]

- They follow the binomial processes on p. 1073.
Numerical Examples (continued)

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

• The pricing of derivatives can be simplified by assuming investors are risk-neutral.

• Suppose all securities have the same expected one-period rate of return, the riskless rate.

• Then

\[(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\% ,\]

where \( p \) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,
  $$C = 0.000 \quad \text{and} \quad 3.039 = 98.039 - 95$$

• To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580 \).

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth
  \[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]
  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

$$F = \begin{cases} 92 \ (= 100 - 8) \\ 98 \ (= 100 - 2) \end{cases}$$

- As the futures price $F$ is the expected future payoff,\(^a\)

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$ 

\(^a\)See Exercise 13.2.11 of the textbook or p. 551.
Numerical Examples: Futures and Forward Prices (concluded)

• The forward price for a one-year forward contract on a one-year zero-coupon bond is\textsuperscript{a}

\[
\frac{90.703}{96.154} = 94.331\%.
\]

• The forward price exceeds the futures price.\textsuperscript{b}

\textsuperscript{a}By Eq. (138) on p. 1050.
\textsuperscript{b}Unlike the nonstochastic case on p. 493.
The nature of modern trade is to give to those who have much and take from those who have little.
— Walter Bagehot (1867), The English Constitution

8. What’s your problem? Any moron can understand bond pricing models.
— Top Ten Lies Finance Professors Tell Their Students
Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy
  \[ r(t, T) = -\frac{\ln P(t, T)}{T - t} \]

  by Eq. (137) on p. 1049.
- Hence the discount function \( P(t, T) \) suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model\textsuperscript{a}

- The short rate follows
  \[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).

- Since the process is an Ornstein-Uhlenbeck process,
  \[ E[r(T) \mid r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)} \]
  from Eq. (83) on p. 614.

\textsuperscript{a}Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody’s for USD$210 million in 2002.
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[
P(t, T) = A(t, T) e^{-B(t, T) r(t)},
\]

where

\[
A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t, T) - T + t)(\beta^2 \mu - \sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t, T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0.
\end{cases}
\]

and

\[
B(t, T) = \begin{cases} 
\frac{1 - e^{-\beta(T - t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0.
\end{cases}
\]
The Vasicek Model (continued)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- But even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The long rate $r(t, \infty)$ is the constant

$$\mu - \frac{\sigma^2}{2\beta^2},$$

independent of the current short rate.
The Vasicek Model (concluded)

• The spot rate volatility structure is the curve

\[ \sigma \frac{\partial r(t, T)}{\partial r} = \frac{\sigma B(t, T)}{T - t}. \]

• As it depends only on \( T - t \) not on \( t \) by itself, the same curve is maintained for any future time \( t \).

• When \( \beta > 0 \), the curve tends to decline with maturity.
  – The long rate’s volatility is zero unless \( \beta = 0 \).

• The speed of mean reversion, \( \beta \), controls the shape of the curve.

• Higher \( \beta \) leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\(^a\)

- Consider a European call with strike price \(X\) expiring at time \(T\) on a zero-coupon bond with par value $1 and maturing at time \(s > T\).

- Its price is given by

\[
P(t, s) N(x) - XP(t, T) N(x - \sigma_v).
\]

\(^a\)Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

• Above

\[ x \triangleq \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]

\[ \sigma_v \equiv v(t, T) B(T, s), \]

\[ v(t, T)^2 \triangleq \begin{cases} \frac{\sigma^2 [1 - e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2 (T - t), & \text{if } \beta = 0 \end{cases}. \]

• By the put-call parity, the price of a European put is

\[ XP(t, T) \, N(-x + \sigma_v) - P(t, s) \, N(-x). \]
Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.

- Let $\Delta t = T/n$ and

$$p(r) \triangleq \frac{1}{2} + \frac{\beta(\mu - r) \sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,$^a$

$$r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.$$

---

Binomial Vasicek (continued)

• Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\xi(k) = 1] = \begin{cases} 
p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\
0, & \text{if } p(r(k)) < 0, \\
1, & \text{if } 1 < p(r(k)). \end{cases}$$

• Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

• This is consistent with mean reversion.
Binomial Vasicek (concluded)

• The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.

• The binomial tree combines.

• The key feature of the model that makes it happen is its constant volatility, $\sigma$. 
The Cox-Ingersoll-Ross Model\textsuperscript{a}

- It is the following square-root short rate model:

\[ dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \tag{151} \]

- The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).

- The parameter \( \beta \) determines the speed of adjustment.

- If \( r(0) > 0 \), then the short rate can reach zero \textit{only if}

\[ 2\beta \mu < \sigma^2. \]

- This is called the Feller (1951) condition.

- See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, & Ross (1985).
Binomial CIR

- We want to approximate the short rate process in the time interval \([0, T]\).
- Divide it into \(n\) periods of duration \(\Delta t = T/n\).
- Assume \(\mu, \beta \geq 0\).
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

- Instead, consider the transformed process\(^{a}\)

\[
x(r) \triangleq 2\sqrt{r}/\sigma.
\]

- By Ito’s lemma (p. 591),

\[
dx = m(x) \, dt + dW,
\]

where

\[
m(x) \triangleq 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).
\]

- This new process has a *constant* volatility.

- Thus its binomial tree combines.

\(^{a}\)See pp. 1106ff for justification.
Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$ 

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.\(^{a}\)

\(^{a}\)Nawalkha & Beliaeva (2007).
Binomial CIR (continued)

• The probability of an up move at each node $r$ is

$$p(r) \triangleq \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}.$$ 

- $r^+ \triangleq f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.
- $r^- \triangleq f(x - \sqrt{\Delta t})$ the result of a down move.

• Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Binomial CIR (concluded)

• It can be shown that

\[ p(r) = \left( \beta \mu - \frac{\sigma^2}{4} \right) \sqrt{\frac{\Delta t}{r}} - B \sqrt{r \Delta t} + C, \]

for some \( B \geq 0 \) and \( C > 0 \).\(^a\)

• If \( \beta \mu - (\sigma^2/4) \geq 0 \), the up-move probability \( p(r) \) decreases if and only if short rate \( r \) increases.

• Even if \( \beta \mu - (\sigma^2/4) < 0 \), \( p(r) \) tends to decrease as \( r \) increases and decrease as \( r \) declines.

• This phenomenon agrees with mean reversion.

\(^a\)Thanks to a lively class discussion on May 28, 2014.
Numerical Examples

- Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1\sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

- We shall use \(\Delta t = 0.2\) (year) for the binomial approximation.

- See p. 1103(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (concluded)

• Consider the node which is the result of an up move from the root.

• Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

• Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$ 

• Once the short rates are in place, computing the probabilities is easy.

• Convergence is quite good.\(^a\)

\(^a\)See p. 369 of the textbook.
Trinomial CIR

- The binomial CIR tree does not have the degree of freedom to match the mean and variance exactly.
- It actually fails to match them at very low $x$.
- A trinomial tree for the CIR model with $O(n^{1.5})$ nodes that matches the mean and variance exactly is recently obtained using the ideas on pp. 761ff.$^a$

---

$^a$Lu (D00922011) & Lyuu (2018); Huang, H. (R03922103) (2019).
A General Method for Constructing Binomial Models\textsuperscript{a}

- We are given a continuous-time process,
  
  \[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]
  
- Need to make sure the binomial model’s drift and diffusion converge to the above process.
  
- Set the probability of an up move to
  
  \[ \frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}. \]
  
- Here \( y_u \overset{\Delta}{=} y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \overset{\Delta}{=} y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

\textsuperscript{a}Nelson & Ramaswamy (1990).
A General Method (continued)

• The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

• But the binomial tree may not combine as

\[
\sigma(y, t) \sqrt{\Delta t} - \sigma(y_u, t + \Delta t) \sqrt{\Delta t} \\
\neq -\sigma(y, t) \sqrt{\Delta t} + \sigma(y_d, t + \Delta t) \sqrt{\Delta t}
\]

in general.

• When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.
A General Method (continued)

- To achieve this, define the transformation

\[ x(y, t) \overset{\Delta}{=} \int_{y}^{y} \sigma(z, t)^{-1} \, dz. \]

- Then \( x \) follows

\[ dx = m(y, t) \, dt + dW \]

for some \( m(y, t) \).\(^{a}\)

- The diffusion term is now a constant, and the binomial tree for \( x \) combines.

\(^{a}\)See Exercise 25.2.13 of the textbook.
A General Method (concluded)

- The transformation is unique.a

- The probability of an up move remains

\[
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
\]

where \(y(x, t)\) is the inverse transformation of \(x(y, t)\) from \(x\) back to \(y\).

- Note that

\[
y_u(x, t) \triangleq y(x + \sqrt{\Delta t}, t + \Delta t),
\]

\[
y_d(x, t) \triangleq y(x - \sqrt{\Delta t}, t + \Delta t).
\]

---

aH. Chiu (R98723059) (2012).
Examples

• The transformation is
  \[ \int_{r}^{r} (\sigma \sqrt{z})^{-1} \, dz = \frac{2\sqrt{r}}{\sigma} \]
  for the CIR model.

• The transformation is
  \[ \int_{S}^{S} (\sigma z)^{-1} \, dz = \frac{\ln S}{\sigma} \]
  for the Black-Scholes model \( dS = \mu S \, dt + \sigma S \, dW \).

• The familiar BOPM and CRR discretize \( \ln S \) not \( S \).
On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate levels only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

• Multifactor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.

• But they are much harder to think about and work with.

• They also take much more computer time—the curse of dimensionality.

• These practical concerns limit the use of multifactor models to two- or three-factor ones.a

Options on Coupon Bonds

- Assume the market discount function $P$ is a monotonically decreasing function of the short rate $r$.
  - Such as a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.

---

a Jamshidian (1989).
Options on Coupon Bonds (continued)

- The bond has cash flows \(c_1, c_2, \ldots, c_n\) at times \(t_1, t_2, \ldots, t_n\), where \(t_i > T\) for all \(i\).

- The payoff for the option is

\[
\max \left\{ \left[ \sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - X, 0 \right\}.
\]

- At time \(T\), there is a unique value \(r^*\) for \(r(T)\) that renders the coupon bond’s price equal the strike price \(X\).
Options on Coupon Bonds (continued)

• This $r^*$ can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for $r$.

• Let

$$X_i \triangleq P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value $\$1$ and maturing at time $t_i$ if $r(T) = r^*$.

• Note that $P(r, T, t_i) \geq X_i$ if and only if $r \leq r^*$. 
Options on Coupon Bonds (concluded)

• As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\max\left\{ \left[ \sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - \left[ \sum_{i=1}^{n} c_i X_i \right], 0 \right\}$$

$$= \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

• Thus the call is a package of $n$ options on the underlying zero-coupon bond.

• Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?

— Arthur Eddington (1882–1944)
Motivations

• Recall the difficulties facing equilibrium models mentioned earlier.
  – They usually require the estimation of the market price of risk.$^a$
  – They cannot fit the market term structure.
  – But consistency with the market is often mandatory in practice.

$^a$Recall p. 1068.
No-Arbitrage Models\textsuperscript{a}

- No-arbitrage models utilize the full information of the term structure.

- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.

- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.

- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

\textsuperscript{a}T. Ho & S. B. Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to Bloomberg on May 26, 2012.
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.
The Ho-Lee Model\textsuperscript{a}

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

\textsuperscript{a}T. Ho & S. B. Lee (1986).
\[
\begin{align*}
& r_1 \rightarrow r_2 \rightarrow r_3 \rightarrow \cdots \rightarrow r_3 + v_3 \\
& r_2 \rightarrow r_3 + v_3 \\
& r_2 + v_2 \rightarrow r_3 + 2v_3
\end{align*}
\]
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices \( P(t, t + 1), P(t, t + 2), \ldots \) at time \( t \) identified with the root of the tree.

- Let the discount factors in the next period be
  \[
P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots, \]
  if short rate moves down,
  \[
P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots, \]
  if short rate moves up.

- By backward induction, it is not hard to see that for \( n \geq 2 \), \(^a\)
  \[
P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{-(v_2 + \cdots + v_n)}. \] \(^{(152)}\)

\(^a\)See p. 376 of the textbook.
The Ho-Lee Model (continued)

- It is also not hard to check that the \( n \)-period zero-coupon bond has yields

\[
y_d(n) \triangleq -\frac{\ln P_d(t + 1, t + n)}{n - 1}
\]

\[
y_u(n) \triangleq -\frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}
\]

- The volatility of the yield to maturity for this bond is therefore

\[
\kappa_n \triangleq \sqrt{py_u(n)^2 + (1 - p)y_d(n)^2 - [py_u(n) + (1 - p)y_d(n)]^2}
\]

\[
= \sqrt{p(1 - p)} \left[ y_u(n) - y_d(n) \right]
\]

\[
= \sqrt{p(1 - p) \frac{v_2 + \cdots + v_n}{n - 1}}.
\]
The Ho-Lee Model (concluded)

• In particular, the short rate volatility is determined by taking \( n = 2 \):

\[
\sigma = \sqrt{p(1 - p)} \ v_2. \tag{153}
\]

• The volatility of the short rate therefore equals

\[
\sqrt{p(1 - p)} (r_u - r_d),
\]

where \( r_u \) and \( r_d \) are the two successor rates.\(^a\)

\(^a\)Contrast this with the lognormal model (130) on p. 990.
The volatility term structure is composed of

\[ \kappa_2, \kappa_3, \ldots . \]

- The volatility structure is supplied by the market.
- For the Ho-Lee model, it is independent of

\[ r_2, r_3, \ldots . \]

It is easy to compute the \( v_i \)s from the volatility structure, and vice versa.\(^a\)

The \( r_i \)s can be computed by forward induction.

\(^a\)Review p. 1126.
The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy

\[
P(t, t+n) = \left[p P_u(t+1, t+n) + (1-p) P_d(t+1, t+n)\right] P(t, t+1).
\]

• Combine the above with Eq. (152) on p. 1125 and assume \( p = 1/2 \) to obtain

\[
P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]},
\]

\[
P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}.
\]

\(^a\)Recall Eq. (144) on p. 1057.

\(^b\)In the limit, only the volatility matters; the first formula is similar to multiple logistic regression.
The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.\textsuperscript{a}

- Suppose all $v_i$ equal some constant $v$ and $\delta \overset{\Delta}{=} e^v > 0$.

- Then

$$P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}},$$

$$P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.$$

- Short rate volatility $\sigma = v/2$ by Eq. (153) on p. 1127.

- Price derivatives by taking expectations under the risk-neutral probability.

\textsuperscript{a}See Exercise 26.2.3 of the textbook.
Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in $O(n)$ time.\(^a\)
  - Derive the $v_i$’s in linear time.
  - Derive the $r_i$’s in linear time.

\(^a\)See Programming Assignment 26.2.6 of the textbook.
The Ho-Lee Model: Yields and Their Covariances

• The one-period rate of return of an $n$-period zero-coupon bond is

$$r(t, t + n) \triangleq \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).$$

• Its two possible value are

$$\ln \frac{P_d(t + 1, t + n)}{P(t, t + n)} \quad \text{and} \quad \ln \frac{P_u(t + 1, t + n)}{P(t, t + n)}.$$

• Thus the variance of return is\(^{a}\)

$$\text{Var}[r(t, t + n)] = p(1 - p) \left[ (n - 1) v \right]^2 = (n - 1)^2 \sigma^2.$$  

\(^{a}\)Recall that $\sigma$ is the short rate volatility by Eq. (153) on p. 1127.
The Ho-Lee Model: Yields and Their Covariances (concluded)

• The covariance between $r(t, t + n)$ and $r(t, t + m)$ is

$$(n - 1)(m - 1) \sigma^2.$$ 

• As a result, the correlation between any two one-period rates of return is one.

• Strong correlation between rates is inherent in all one-factor Markovian models.

\(^a\text{See Exercise 26.2.7 of the textbook.}\)
The Ho-Lee Model: Short Rate Process

• The continuous-time limit of the Ho-Lee model is\(^a\)

\[
    dr = \theta(t) \, dt + \sigma \, dW. \tag{154}
\]

• This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

• A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

\[
    dr = \theta(t) \, dt + \sigma(t) \, dW.
\]

• This corresponds to the discrete-time model in which \(v_i\) are not all identical.

\(^a\)See Exercise 26.2.10 of the textbook.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.\textsuperscript{a}

\textsuperscript{a}Recall pp. 111ff. See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.
Problems with No-Arbitrage Models in General

• Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.

• Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.

• But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  – A new model is thus born every day.
Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.