Example

• A hedger is *short* 10,000 European calls.
• $S = 50$, $\sigma = 30\%$, and $r = 6\%$.
• This call’s expiration is four weeks away, its strike price is $50$, and each call has a current value of $f = 1.76791$.
• As an option covers 100 shares of stock, $N = 1,000,000$.
• The trader adjusts the portfolio weekly.
• The calls are replicated well if the cumulative cost of trading *stock* is close to the call premium’s FV.\(^a\)

\(^a\)This example takes the replication viewpoint: One starts with zero dollar.
Example (continued)

• As \( \Delta = 0.538560 \)

\[
N \times \Delta = 538,560
\]

shares are purchased for a total cost of

\[
538,560 \times 50 = 26,928,000
\]
dollars to make the portfolio delta-neutral.

• The trader finances the purchase by borrowing

\[
B = N \times \Delta \times S - N \times f = 25,160,090
\]
dollars net.a

aThis takes the hedging viewpoint: One starts with the option premium. See Exercise 16.3.2 of the text.
Example (continued)

• At 3 weeks to expiration, the stock price rises to $51.

• The new call value is $f' = 2.10580$.

• So before rebalancing, the portfolio is worth

$$- N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622.$$  

(91)
Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.

- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

- In fact, the tracking error over one rebalancing act is positive about 68% of the time, but its expected value is essentially zero.\(^a\)

- The tracking error at maturity is proportional to vega.\(^b\)

\(^a\)Boyle & Emanuel (1980).
\(^b\)Kamal & Derman (1999).
Example (continued)

• In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.

• With a higher delta $\Delta' = 0.640355$, the trader buys

$$N \times (\Delta' - \Delta) = 101,795$$

shares for $5,191,545$.

• The number of shares is increased to $N \times \Delta' = 640,355$. 
Example (continued)

- The cumulative cost is\(^a\)

\[
26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634.
\]

- The portfolio is again delta-neutral.

\(^a\)We take the replication viewpoint here.
<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$S$</th>
<th>$f$</th>
<th>$\Delta$</th>
<th>$N \times (5)$</th>
<th>$(1) \times (6)$</th>
<th>FV(8′) + (7)</th>
<th>Cumulative cost</th>
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<tr>
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<td>160,175</td>
<td>8,649,450</td>
<td>51,524,853</td>
</tr>
</tbody>
</table>

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).
Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of
  \[51,524,853 - 50,000,000 = 1,524,853,\]
  which represents the replication cost.
- Compared with the FV of the call premium,
  \[1,767,910 \times e^{0.06 \times 4/52} = 1,776,088,\]
  the net gain is \[1,776,088 - 1,524,853 = 251,235.\]
Tracking Error Revisited

• Define the dollar gamma as $S^2 \Gamma$.

• The change in value of a delta-hedged long option position after a duration of $\Delta t$ is proportional to the dollar gamma.

• It is about

$$\frac{1}{2} S^2 \Gamma \left[ \left( \frac{\Delta S}{S} \right)^2 - \sigma^2 \Delta t \right].$$

$- (\Delta S/S)^2$ is called the daily realized variance.
Tracking Error Revisited (continued)

- In our particular case,
  \[ S = 50, \Gamma = 0.0957074, \Delta S = 1, \sigma = 0.3, \Delta t = 1/52. \]

- The estimated tracking error is
  \[ -\left( \frac{1}{2} \right) \times 50^2 \times 0.0957074 \times \left[ \left( \frac{1}{50} \right)^2 - \left( \frac{0.09}{52} \right) \right] = 159,205. \]

- It is very close to our earlier number of 171,622.\(^a\)

- Delta hedge is also called gamma scalping.\(^b\)

---

\(^a\)Recall Eq. (91) on p. 673.
\(^b\)Bennett (2014).
Tracking Error Revisited (continued)

• Let the rebalancing times be \( t_1, t_2, \ldots, t_n \).
• Let \( \Delta S_i = S_{i+1} - S_i \).
• The total tracking error at expiration is about
  \[
  \sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[ \left( \frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].
  \]
  
  • The tracking error is path dependent.
  • It is also known that\(^a\)
  \[
  \sum_{i=0}^{n-1} \left( \frac{\Delta S_i}{S_i} \right)^2 \rightarrow \sigma^2 T.
  \]

\(^a\)Protter (2005).
Tracking Error Revisited (concluded)a

- The tracking errorb $\epsilon_n$ over $n$ rebalancing acts has about the same probability of being positive as being negative.

- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\epsilon_n^2]}$ is $O(1/\sqrt{n})$.c

- The root-mean-square tracking error increases with $\sigma$ at first and then decreases.

---

aBertsimas, Kogan, & Lo (2000).
bSuch as 251,235 on p. 678.
Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, $\Delta f$, due to changes in the stock price, $\Delta S$.

- When $\Delta S$ is not small, the second-order term, gamma $\Gamma \equiv \frac{\Delta}{\Delta S} \frac{\partial^2 f}{\partial S^2}$, helps (theoretically).

- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.

- To meet this extra condition, one more security needs to be brought in.
Delta-Gamma Hedge (concluded)

• Suppose we want to hedge short calls as before.

• A hedging call $f_2$ is brought in.

• To set up a delta-gamma hedge, we solve

\[-N \times f + n_1 \times S + n_2 \times f_2 - B = 0\] (self-financing),
\[-N \times \Delta + n_1 \times \Delta_2 - 0 = 0\] (delta neutrality),
\[-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 = 0\] (gamma neutrality),

for $n_1, n_2$, and $B$.

- The gammas of the stock and bond are 0.

• See the numerical example on pp. 231–232 of the text.
Other Hedges

- If volatility changes, delta-gamma hedge may not work well.

- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.

- To accomplish this, one more security has to be brought into the process.

- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 275.
- We will now apply it to price barrier options.
The Reflection Principle\textsuperscript{a}

- Imagine a particle at position \((0, -a)\) on the integral lattice that is to reach \((n, -b)\).
- Without loss of generality, assume \(a > 0\) and \(b \geq 0\).
- This particle’s movement:
  
  \[
  (i, j) \quad \xrightarrow{(i + 1, j + 1)} \quad \text{up move } S \rightarrow Su
  
  (i, j) \quad \xleftarrow{(i + 1, j - 1)} \quad \text{down move } S \rightarrow Sd
  
  \]
- How many paths touch the \(x\) axis?

\textsuperscript{a}André (1887).
The Reflection Principle (continued)

• For a path from \((0, -a)\) to \((n, -b)\) that touches the \(x\) axis, let \(J\) denote the first point this happens.

• Reflect the portion of the path from \((0, -a)\) to \(J\).

• A path from \((0, a)\) to \((n, -b)\) is constructed.

• It also hits the \(x\) axis at \(J\) for the first time.

• The one-to-one mapping shows the number of paths from \((0, -a)\) to \((n, -b)\) that touch the \(x\) axis equals the number of paths from \((0, a)\) to \((n, -b)\).
The Reflection Principle (concluded)

- A path of this kind has \((n + b + a)/2\) down moves and \((n - b - a)/2\) up moves.\(^a\)

- Hence there are

\[
\binom{n}{\frac{n+a+b}{2}} = \binom{n}{\frac{n-a-b}{2}} \tag{92}
\]

such paths for even \(n + a + b\).

- Convention: \(\binom{n}{k} = 0\) for \(k < 0\) or \(k > n\).\(^a\)

\(^a\)Verify it!
Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define
  
  \[ a \triangleq \left[ \frac{\ln (X / (S d^n))}{\ln (u/d)} \right] = \left[ \frac{\ln(X/S)}{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right], \]
  \[ h \triangleq \left[ \frac{\ln (H / (S d^n))}{\ln (u/d)} \right] = \left[ \frac{\ln(H/S)}{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right]. \]

- $a$ is such that $\tilde{X} \triangleq Su^a d^{n-a}$ is the terminal price that is closest to $X$ from above.
- $h$ is such that $\tilde{H} \triangleq Su^h d^{n-h}$ is the terminal price that is closest to $H$ from below.\(^a\)

\(^a\)So we underestimate the price.
Pricing Barrier Options (continued)

• The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.

• A process with $n$ moves hence ends up in the money if and only if the number of up moves is at least $a$.

• The price $S u^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price $S d^n$ on the binomial tree.

\[
S u^k d^{n-k} = S d^{-k} d^{n-k} = S d^{n-2k}. \tag{93}
\]
\[ \tilde{X} = Su^a d^{n-a} \]
\[ \tilde{H} = Su^h d^{n-h} \]
Pricing Barrier Options (continued)

- A path from $S$ to the terminal price $S u^j d^{m-j}$ has probability $p^j (1 - p)^{n-j}$ of being taken.

- With reference to p. 695, the reflection principle (p. 690) can be applied with
  \[ a = n - 2h, \]
  \[ b = 2j - 2h, \]
  in Eq. (92) on p. 692 by treating the $\tilde{H}$ line as the $x$ axis.
Pricing Barrier Options (continued)

• Therefore,

\[
\left( \frac{n + (n-2h)+(2j-2h)}{2} \right) = \left( \frac{n}{n-2h+j} \right)
\]

paths hit \( \tilde{H} \) in the process for \( h \leq n/2 \).

• The terminal price \( S_{u^j d^{n-j}} \) is reached by a path that hits the effective barrier with probability

\[
\left( \frac{n}{n-2h+j} \right) p^j (1-p)^{n-j}, \quad j \leq 2h.
\]
Pricing Barrier Options (concluded)

• The option value equals

\[
\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1 - p)^{n-j} \frac{(S_u)^j (d^{n-j} - X)}{R^n}.
\]  

\[ (94) \]

\[ R \equiv e^{r\tau/n} \] is the riskless return per period.

• It yields a linear-time algorithm.\(^a\)

\(^a\)Lyuu (1998).
Convergence of BOPM

- Equation (94) results in the same sawtooth-like convergence shown on p. 395 (repeated on next page).
- The reasons are not hard to see.
- The true barrier $H$ most likely does not equal the effective barrier $\tilde{H}$. 
Convergence of BOPM (continued)

Down-and-in call value
Convergence of BOPM (continued)

- Convergence is actually good if we limit $n$ to certain values—191, for example.

- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx S d^j = S e^{-j \sigma \sqrt{\tau/n}}$$

for some integer $j$.

- The preferred $n$’s are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j \sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$
Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.
- However, the effective barrier above, $Sd^j$, corresponds to a terminal stock price only when $n - j$ is even.\(^a\)
- To close this gap, we decrement $n$ by one, if necessary, to make $n - j$ an even number.

\(^a\)This is because $j = n - 2k$ for some $k$ by Eq. (93) on p. 694. Of course we could have adopted the form $Sd^j \ (-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.
Convergence of BOPM (concluded)

- The preferred $n$’s are now

$$n = \begin{cases} 
\ell, & \text{if } \ell - j \text{ is even}, \\
\ell - 1, & \text{otherwise},
\end{cases}$$

$j = 1, 2, 3, \ldots$, where

$$\ell \triangleq \left\lfloor \frac{\tau}{\left(\ln(S/H)/(j\sigma)\right)^2} \right\rfloor.$$

- Evaluate pricing formula (94) on p. 698 only with the $n$’s above.
Practical Implications

• This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.\textsuperscript{a}

• Now that barrier options can be efficiently priced, we can afford to pick very large $n$’s (p. 706).

• This has profound consequences.\textsuperscript{b}

\textsuperscript{a}J. Lin (R95221010) (2008); J. Lin (R95221010) & Palmer (2010).
\textsuperscript{b}See pp. 720ff.
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Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because

$$n \sim 1/\ln^2(S/H).$$

- This is called the barrier-too-close problem.

- This observation is indeed true of standard quadratic-time binomial tree algorithms.

- But it no longer applies to linear-time algorithms (see p. 708).
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<th>Time</th>
<th>n</th>
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(All times in milliseconds.)
Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion\(^a\)

\[
\frac{dS}{S} = r \, dt + \sigma \, dW.
\]

- The three stock prices at time \(\Delta t\) are \(S\), \(Su\), and \(Sd\), where \(ud = 1\).

- Let the mean and variance of the stock price be \(SM\) and \(S^2V\), respectively.

\(^a\)Boyle (1988).
Trinomial Tree (continued)

• By Eqs. (29) on p. 176,

\[ M \triangleq e^{r\Delta t}, \]
\[ V \triangleq M^2(e^{\sigma^2\Delta t} - 1). \]

• Impose the matching of mean and that of variance:

\[ \begin{align*}
1 &= p_u + p_m + p_d, \\
SM &= (p_u u + p_m + (p_d/u))S, \\
S^2V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{align*} \]
Trinomial Tree (continued)

• Use linear algebra to verify that

\[ p_u = \frac{u(V + M^2 - M) - (M - 1)}{(u - 1)(u^2 - 1)}, \]

\[ p_d = \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}. \]

– We must also make sure the probabilities lie between 0 and 1.
Trinomial Tree (concluded)

- There are countless variations.
- But all converge to the Black-Scholes option pricing model.\(^a\)
- The trinomial model has a linear-time algorithm for European options.\(^b\)

\(^a\)Madan, Milne, & Shefrin (1989).
\(^b\)T. Chen (R94922003) (2007).
A Trinomial Tree

• Use $u = e^{\lambda \sigma \sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.

• Then

$$p_u \rightarrow \frac{1}{2\lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2\lambda \sigma},$$

$$p_d \rightarrow \frac{1}{2\lambda^2} - \frac{(r - 2\sigma^2) \sqrt{\Delta t}}{2\lambda \sigma}.$$

• A nice choice for $\lambda$ is $\sqrt{\pi/2}$.\(^a\)

\(^a\)Omberg (1988).
Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting $\lambda$ so that the barrier is hit exactly.\(^a\)
- When

$$Se^{-h\lambda\sigma\sqrt{\Delta t}} = H,$$

it takes $h$ down moves to go from $S$ to $H$, if $h$ is an integer.
- Then

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}.$$  

\(^a\)Ritchken (1995).
Barrier Options Revisited (continued)

• This is easy to achieve by adjusting $\lambda$.

• Typically, we find the smallest $\lambda \geq 1$ such that $h$ is an integer.\(^a\)
  
  − Such a $\lambda$ may not exist for very small $n$’s.
  
  − This is not hard to check.

• Toward that end, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ to be our $h$.

• Then let

\[
\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.
\]

\(^a\)Why must $\lambda \geq 1$?
Barrier Options Revisited (continued)

- Alternatively, we can pick

\[ h = \left\lfloor \frac{\ln(S/H)}{\sigma \sqrt{\Delta t}} \right\rfloor. \]

- Make sure \( h \geq 1 \).

- Then let

\[ \lambda = \frac{\ln(S/H)}{h \sigma \sqrt{\Delta t}}. \]
Barrier Options Revisited (concluded)

• This done, one of the layers of the trinomial tree coincides with the barrier.

• The following probabilities may be used,

\[ p_u = \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda \sigma}, \]

\[ p_m = 1 - \frac{1}{\lambda^2}, \]

\[ p_d = \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda \sigma}. \]

\[ \mu' \triangleq r - (\sigma^2/2). \]
Down-and-in call value

#Periods

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Algorithms Comparison\textsuperscript{a}

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the \( n \) value at which they converge.
  - The one with the smallest \( n \) wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not \( n \).\textsuperscript{b}

\textsuperscript{a}Lyuu (1998).
\textsuperscript{b}Patterson & Hennessy (1994).
Algorithms Comparison (continued)

• Pages 700 and 719 seem to show the trinomial model converges at a smaller $n$ than BOPM.

• It is in this sense when people say trinomial models converge faster than binomial ones.

• But does it make the trinomial model better then?
Algorithms Comparison (concluded)

• The linear-time binomial tree algorithm actually performs better than the trinomial one.

• See the next page, expanded from p. 706.

• The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.\textsuperscript{a}
  
  – See pp. 733ff for an alternative solution.

\textsuperscript{a} Lyuu (1998).
<table>
<thead>
<tr>
<th>$n$</th>
<th>Combinatorial method</th>
<th>Trinomial tree algorithm</th>
</tr>
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<td>Time</td>
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</table>

(All times in milliseconds.)
Double-Barrier Options

- Double-barrier options are barrier options with two barriers \( L < H \).
  - They make up “less than 5% of the light exotic market.”\(^a\)

- Assume \( L < S < H \).

- The binomial model produces oscillating option values (see plot on next page).\(^b\)

---

\(^a\) Bennett (2014).

\(^b\) Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2005).
Double-Barrier Options (concluded)

• The combinatorial method yields a linear-time algorithm.\textsuperscript{a}

• This binomial model is $O(1/\sqrt{n})$ convergent in general.\textsuperscript{b}

• If the barriers $L$ and $H$ depend on time, we have moving-barrier options.\textsuperscript{c}

---

\textsuperscript{a}See p. 241 of the textbook.
\textsuperscript{b}Gobet (1999).
\textsuperscript{c}Rogers & Zane (1998).
Double-Barrier Knock-Out Options

• We knew how to pick the $\lambda$ so that one of the layers of the trinomial tree coincides with one barrier, say $H$.

• This choice, however, does not guarantee that the other barrier, $L$, is also hit.

• One way to handle this problem is to lower the layer of the tree just above $L$ to coincide with $L$.\textsuperscript{a}
  
  – More general ways to make the trinomial model hit both barriers are available.\textsuperscript{b}

\textsuperscript{a}Ritchken (1995); Hull (1999).

\textsuperscript{b}Hsu (R7526001, D89922012) & Lyuu (2006). Dai (B82506025, R86526008, D8852600) & Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 733ff).
Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above $L$ must be adjusted.

- Let $\ell$ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^\ell.$$ 

- Hence the layer of the tree just above $L$ has price $Sd^\ell$.\(^a\)

\(^a\)You probably cannot do the same thing for binomial models (why?).
Thanks to a lively discussion on April 25, 2012.
Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying
  \[ L = S d^{\ell-1} e^{-\gamma \lambda \sigma \sqrt{\Delta t}}. \]

  - The prices between the barriers are (from low to high)
    \[ L, S d^{\ell-1}, \ldots, S d^2, S d, S, S u, S u^2, \ldots, S u^{h-1}, S u^h = H. \]

- The probabilities for the nodes with price equal to $S d^{\ell-1}$ are
  \[ p_u' = \frac{b + a \gamma}{1 + \gamma}, \quad p_d' = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p_m' = 1 - p_u' - p_d', \]

  where $a \triangleq \mu' \sqrt{\Delta t}/(\lambda \sigma)$ and $b \triangleq 1/\lambda^2$. 
Convergence: Binomial vs. Trinomial

Option value

Binomial

Trinomial
Ideas for Binomial Trees To Handle Two Barriers

\[
\begin{align*}
\ln(H) \\
\ln(H/L) \\
2\sigma \sqrt{\Delta t} \\
\ln(L)
\end{align*}
\]

\[
\begin{align*}
\ln(S') \\
\Delta t & \quad \Delta t & \quad \Delta t & \quad \Delta t \quad T
\end{align*}
\]
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.\(^a\)

- The resulting tree is called the binomial-trinomial tree.\(^b\)

- Suppose a binomial tree will be built with \(\Delta t\) as the duration of one period.

- Node X at time \(t\) needs to pick three nodes on the binomial tree at time \(t + \Delta t'\) as its successor nodes.

\[\Delta t \leq \Delta t' < 2\Delta t.\]

\(^a\)Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).

\(^b\)The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.
The Binomial-Trinomial Tree (continued)

\[ \Delta t' \]

\[ X \]

\[ A \]

\[ B \]

\[ C \]

\[ p_u \]

\[ p_m \]

\[ p_d \]

\[ \alpha \]

\[ \beta \]

\[ \gamma \]

\[ \hat{\mu} + 2\sigma \sqrt{\Delta t} \]

\[ \hat{\mu} \]

\[ \mu \]

\[ 0 \]

\[ \hat{\mu} - 2\sigma \sqrt{\Delta t} \]

\[ \Delta t \]

\[ \Delta t \]
The Binomial-Trinomial Tree (continued)

• These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.

• Let $S$ be the stock price at node $X$.

• Use $s(z)$ to denote the stock price at node $z$. 
The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return 
  \[ \ln(S_{t+\Delta t'/S}) \] at time \( t + \Delta t' \) equals\(^a\)

  \[ \mu \overset{\Delta}{=} \left( r - \frac{\sigma^2}{2} \right) \Delta t'. \] (95)

- Its variance equals

  \[ \text{Var} \overset{\Delta}{=} \sigma^2 \Delta t'. \] (96)

- Let node B be the node whose logarithmic return 
  \[ \hat{\mu} \overset{\Delta}{=} \ln(s(B)/S) \] is closest to \( \mu \) among all the nodes at 
  time \( t + \Delta t' \).

\(^{a}\text{See p. 290.}\)
The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at \(2\sigma\sqrt{\Delta t}\) apart.
- Review the figure on p. 734 for illustration.
The Binomial-Trinomial Tree (continued)

• The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.

• Recall that

$$\hat{\mu} \triangleq \ln (s(B)/S)$$

is the logarithmic return of the middle node B.

• Let $\alpha$, $\beta$, and $\gamma$ be the differences between $\mu$ and the logarithmic returns

$$\ln(s(Z)/S), \quad Z = A, B, C,$$

in that order.
The Binomial-Trinomial Tree (continued)

- In other words,

\[ \alpha \overset{\Delta}{=} \hat{\mu} + 2\sigma \sqrt{\Delta t} - \mu = \beta + 2\sigma \sqrt{\Delta t}, \quad (97) \]

\[ \beta \overset{\Delta}{=} \hat{\mu} - \mu, \quad (98) \]

\[ \gamma \overset{\Delta}{=} \hat{\mu} - 2\sigma \sqrt{\Delta t} - \mu = \beta - 2\sigma \sqrt{\Delta t}. \quad (99) \]

- The three branching probabilities \( p_u, p_m, p_d \) then satisfy

\[ p_u \alpha + p_m \beta + p_d \gamma = 0, \quad (100) \]

\[ p_u \alpha^2 + p_m \beta^2 + p_d \gamma^2 = \text{Var}, \quad (101) \]

\[ p_u + p_m + p_d = 1. \quad (102) \]
The Binomial-Trinomial Tree (concluded)

- Equation (100) matches the mean (95) of the logarithmic return \( \ln(S_{t+\Delta t'})/S \) on p. 736.
- Equation (101) matches its variance (96) on p. 736.
- The three probabilities can be proved to lie between 0 and 1 by Cramer’s rule.
Pricing Double-Barrier Options

• Consider a double-barrier option with two barriers $L$ and $H$, where $L < S < H$.

• We need to make each barrier coincide with a layer of the binomial tree for better convergence.

• The idea is to choose a $\Delta t$ such that

$$\frac{\ln(H/L)}{2\sigma \sqrt{\Delta t}}$$ (103)

is a positive integer.

  – The distance between two adjacent nodes such as nodes $Y$ and $Z$ in the figure on p. 742 is $2\sigma \sqrt{\Delta t}$.
Pricing Double-Barrier Options (continued)

\[
\begin{align*}
\ln(H/L) & \quad \ln(H/S) \\
2\sigma \sqrt{\Delta t} & \quad \ln(L/S) + 4\sigma \sqrt{\Delta t} \\
0 & \quad \ln(L/S) + 2\sigma \sqrt{\Delta t} \\
\end{align*}
\]
Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta \tau \overset{\Delta}{=} T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma \sqrt{\Delta \tau}}$ is an integer.
- So we pick a $\Delta t$ that is close to, but does not exceed, $\Delta \tau$ and makes $\frac{\ln(H/L)}{2\sigma \sqrt{\Delta t}}$ some integer $\kappa$.
- Specifically, we select

$$
\Delta t = \left( \frac{\ln(H/L)}{2\kappa \sigma} \right)^2,
$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma \sqrt{\Delta \tau}} \right\rceil$. 

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Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier $L$ upward.
- Automatically, a layer coincides with the high barrier $H$.
- It is unlikely that $\Delta t$ divides $T$, however.
- So the position at time 0 and with logarithmic return $\ln(S/S) = 0$ is not occupied by a binomial node to serve as the root node (recall p. 742).
Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.

- Between time 0 and time $T$, the binomial tree spans $\lceil T/\Delta t \rceil$ periods.

- Keep only the last $\lceil T/\Delta t \rceil - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left( \lceil \frac{T}{\Delta t} \rceil - 1 \right) \Delta t.$$

- Then these $\lceil T/\Delta t \rceil$ periods span $T$ years.

- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$. 

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return \( \ln(S/S') = 0 \).

- Find the three nodes on the binomial tree at time \( \Delta t' \) as described earlier.

- Calculate the three branching probabilities to them.

- Grow the binomial tree from these three nodes until time \( T \) to obtain a binomial-trinomial tree with \( \lfloor T/\Delta t \rfloor \) periods.

- See p. 742 for illustration.
Pricing Double-Barrier Options (continued)

• Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.

• That takes quadratic time.

• But a linear-time algorithm exists for double-barrier options on the \textit{binomial} tree.\footnote{See p. 241 of the textbook; Chao (R86526053) (1999); Dai (B82506025, R86526008, D8852600) & Lyuu (2008).}

• Apply that algorithm to price the double-barrier option’s prices at the three nodes at time $\Delta t'$.
  – That is, nodes A, B, and C on p. 742.

• Then calculate their expected discounted value for the root node.
Pricing Double-Barrier Options (continued)

• The overall running time is only linear!

• Binomial trees have troubles pricing barrier options.\(^a\)

• Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 749 and p. 750).

• In fact, the binomial-trinomial tree has an error of \(O(1/n)\) for single-barrier options.\(^b\)

• It has an error of \(O(1/n^{1-a})\) for any \(0 < a < 1\) for double-barrier options.\(^c\)

\(^a\)See p. 395, p. 725, and p. 731.
\(^b\)Lyuu & Palmer (2010).
\(^c\)Appolloni, Gaudenziy, & Zanette (2014).
Pricing Double-Barrier Options\textsuperscript{a} (continued)

\textsuperscript{a}Generated by Mr. Lin, Ying-Hung (R01723029) on June 6, 2014.
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).
The Barrier-Too-Close Problem (p. 707) Revisited

- Our idea solves it even if one barrier is very close to \( S \).
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 715ff).
  - Unlike an earlier solution using combinatorics (p. 698), now the choice of \( n \) is not that restricted.

- So it combines the strengths of binomial and trinomial trees.

- This holds for single-barrier options too.

- Here is how.
The Barrier-Too-Close Problem Revisited (continued)

- We can build the tree treating $S$ as if it were a second barrier.
- So both $H$ and $S$ are matched.
- Alternatively, we can pick $\Delta \tau \overset{\Delta}{=} T/m$ as our length of a period $\Delta t$ without any adjustment.
- Then build the tree from the price $H$ down.
- So $H$ is matched.
- As before, the initial price $S$ will be matched by the *trinomial* structure.
The Barrier-Too-Close Problem Revisited (concluded)

- The earlier trinomial tree is impractical as it needs a very large $n$ when the barrier $H$ is very close to $S$.\(^a\)
  - It needs at least one up move to connect $S$ to $H$ as its middle branch is flat.
  - But when $S \approx H$, that up move must take a very small step, necessitating a small $\Delta t$.

- Our trinomial structure’s middle branch is not necessarily flat.

- So $S$ can be connected to $H$ via the middle branch, and the need of a very large $n$ no longer exists!

\(^a\)Recall the table on p. 708.