Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \stackrel{\Delta}{=} \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 292).
- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as

$$\Delta \triangleq \frac{\partial f}{\partial S}.$$

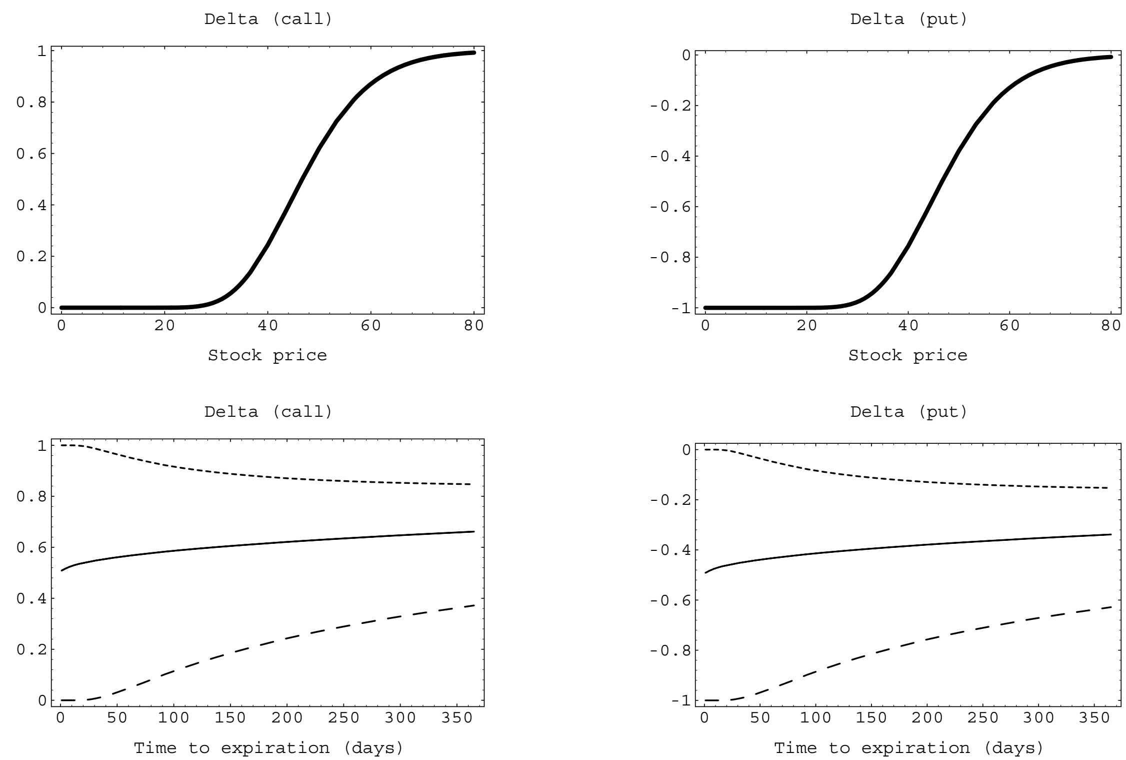
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM (p. 239) is the discrete analog.
- The delta of a long stock is apparently 1.

[a]Elementary calculus.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.$$
- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]
$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{44}$$

[a]The straddle (p. 207) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{45}$$

(recall p. 322).

- Then

$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{46}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r-2q+\sigma^2)\,\tau}. \tag{47}$$

[a]The strangle (p. 209) $C + P$ then has zero delta!

Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then

$$X = S e^{(r-q+\sigma^2/2)\,\tau}$$

  leads to a straddle with zero delta.
  - This generalizes Eq. (44) on p. 332.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

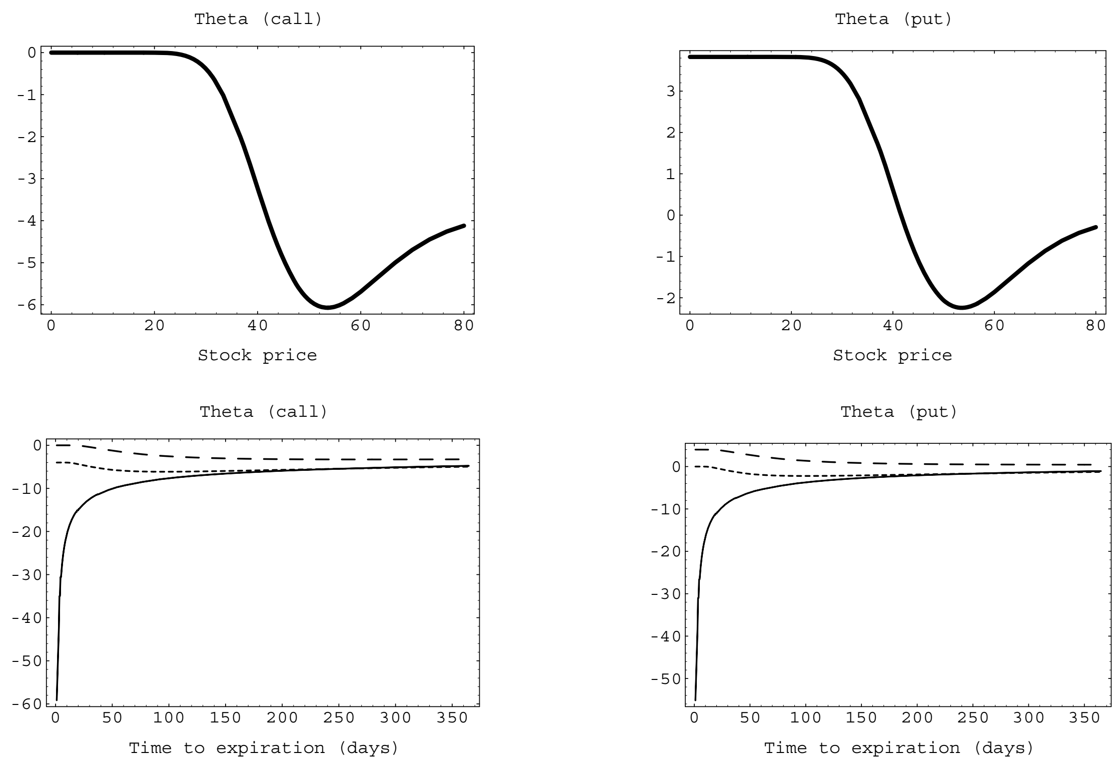
- Defined as the rate of change of a security's value with respect to time, or $\Theta \stackrel{\Delta}{=} -\partial f/\partial\tau = \partial f/\partial t$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$

  - The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$

  - Can be negative or positive.



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q.
- Define x as in Eq. (45) on p. 334.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x-\sigma\sqrt{\tau}) + qSe^{-q\tau}\,N(x).$$

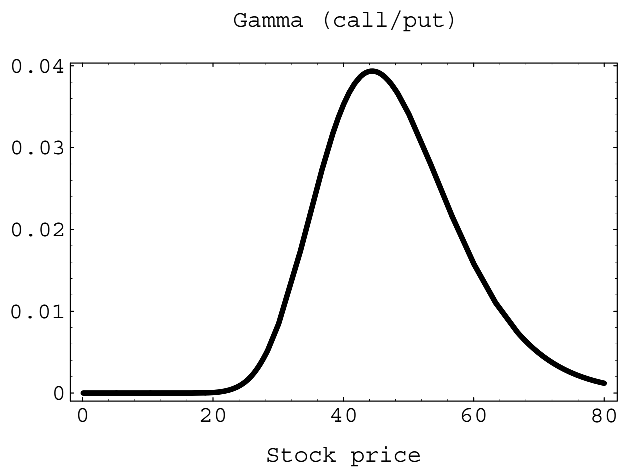
- For a European put, add an extra term to the earlier formula for the theta:

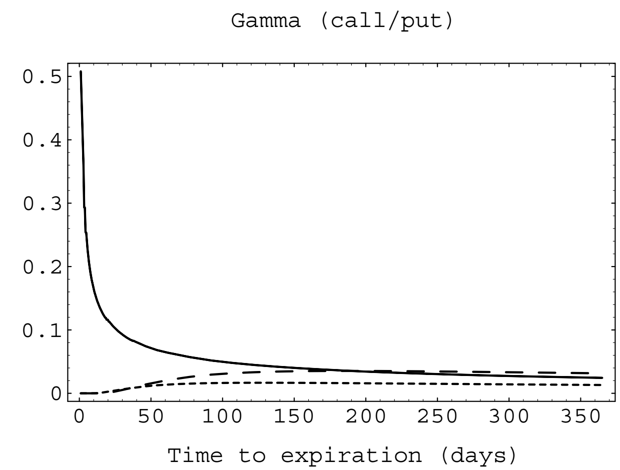
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x+\sigma\sqrt{\tau}) - qSe^{-q\tau}\,N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \stackrel{\Delta}{=} \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$





Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \stackrel{\Delta}{=} \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - So higher volatility always increases the option value.

[a]Vega is not Greek.

Vega (continued)

- Note that[a]

$$\Lambda = \tau \sigma S^2 \Gamma.$$

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = S e^{-q\tau} \sqrt{\tau}\, N'(x),$$

where x is defined in Eq. (45) on p. 334.

- Vega is maximized when $x = 0$, i.e., when

$$S = X e^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

[a]Reiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

where x_i are defined in Eq. (46) on p. 335.

- This also leads to Eq. (47) on p. 335.
  - Recall the same condition led to zero delta for the strangle $C + P$ (p. 335).

Vega (concluded)

- Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

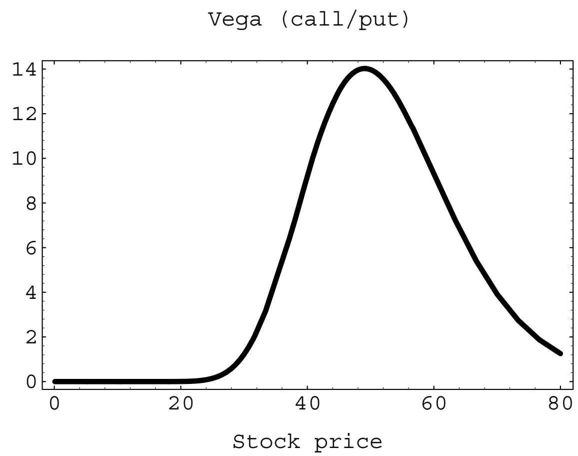
(which answers the question on p. 297 for the Black-Scholes model).

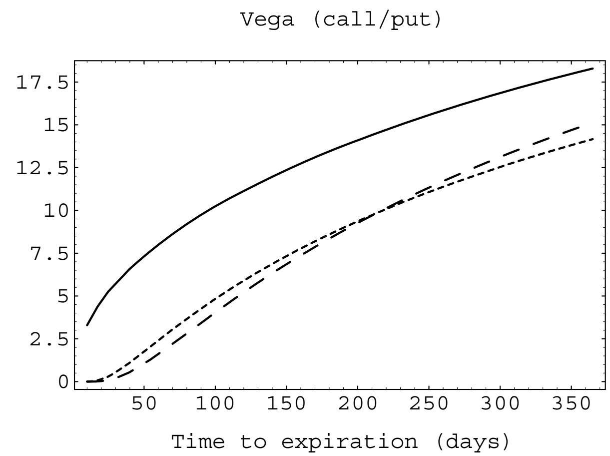
- The Black-Scholes formula (p. 292) implies

$$\begin{aligned} C &\to S, \\ P &\to Xe^{-r\tau}, \end{aligned}$$

as $\sigma \to \infty$.

- These boundary conditions may be handy for certain numerical methods.





Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Variance Vega[a]

- Defined as the rate of change of a security's value with respect to the variance (square of volatility) of the underlying asset

$$\text{variance vega} \triangleq \frac{\partial f}{\partial \sigma^2}.$$

  - Note that it is not defined as $\partial^2 f / \partial \sigma^2$!

- It is easy to verify that

$$\text{variance vega} = \frac{\Lambda}{2\sigma}.$$

[a]Demeterfi, Derman, Kamal, & Zou (1999).

Volga (Vomma, Volatility Gamma, Vega Convexity)

- Defined as the rate of change of a security's vega with respect to the volatility of the underlying asset

$$\text{volga} \triangleq \frac{\partial \Lambda}{\partial \sigma} = \frac{\partial^2 f}{\partial \sigma^2}.$$

- It can be shown that

$$\begin{aligned} \text{volga} &= \Lambda \frac{x(x - \sigma\sqrt{\tau})}{\sigma} \\ &= \frac{\Lambda}{\sigma} \left[\frac{\ln^2(S/X)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4} \right], \end{aligned}$$

where x is defined in Eq. (45) on p. 334.[a]

[a] Derman & M. B. Miller (2016).

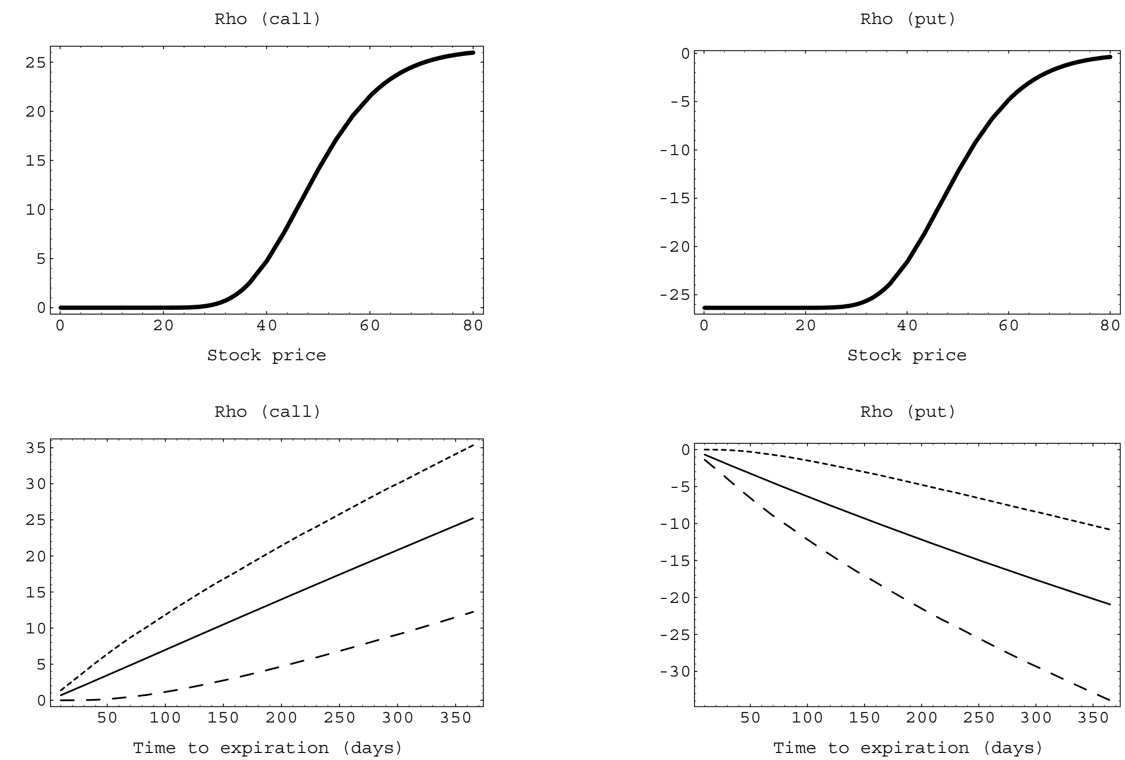
Volga (concluded)

- Volga is zero when $S = Xe^{\pm\sigma^2\tau/2}$.
- For typical values of σ and τ, volga is positive except where $S \approx X$.
- Volga can be used to measure the 4th moment of the underlying asset and the smile of implied volatility at the same maturity.[a]

[a] Bennett (2014).

Rho

- Defined as the rate of change in its value with respect to interest rates
$$\rho \stackrel{\Delta}{=} \frac{\partial f}{\partial r}.$$
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curves: out-of-the-money call or in-the-money put.

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$
- The computation time roughly doubles that for evaluating the derivative security itself.

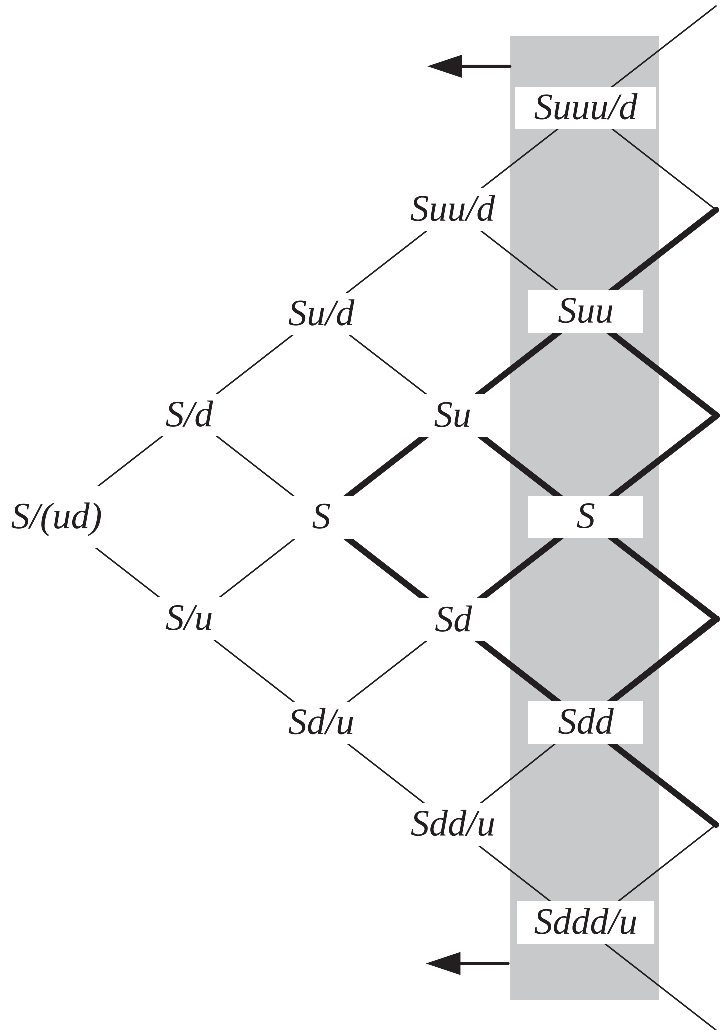
An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}. \tag{48}$$

- Almost zero extra computational effort.

[a]Pelsser & Vorst (1994).


Suuu/d
Suu/d
Su/d
Suu
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}. \tag{49}$$

- Alternative formulas exist (p. 659).

Alternative Numerical Delta and Gamma

- Let $\epsilon \equiv \ln u$.
- Think in terms of $\ln S$.
- Then

$$\left(\frac{f_u - f_d}{2\epsilon} \right) \frac{1}{S}$$

approximates the numerical delta.

- And

$$\left(\frac{f_{uu} - 2f_{ud} + f_{dd}}{\epsilon^2} - \frac{f_{uu} - f_{dd}}{2\epsilon} \right) \frac{1}{S^2}$$

approximates the numerical gamma.

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$

- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.
- But why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  – In fact, the theta of a European option can be derived from delta and gamma (p. 658).

- The vega of a European option can be derived from gamma (p. 344).

- For rho, there seems no alternative but to run the binomial tree algorithm twice.[a]

---

[a] But see p. 840 and pp. 1030ff.

Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to price corporate securities.
  - The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a] Black & Scholes (1973); Merton (1974).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call[a] on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V.

[a] See p. 198.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned} nS &= C, \\ B &= V - C. \end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 10 (p. 292) and the put-call parity,[a]

$$nS = VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \qquad (50)$$

$$B = VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \qquad (51)$$

  - Above,

$$x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

[a]Merton (1974).

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned} &\frac{\ln(X/B)}{\tau} - r \\ = \quad & -\frac{1}{\tau} \ln \left(N(-z) + \frac{1}{\omega} N(z - \sigma\sqrt{\tau}) \right). \end{aligned}$$

  - $\omega \stackrel{\Delta}{=} Xe^{-r\tau}/V$.
  - $z \stackrel{\Delta}{=} (\ln \omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of the remaining firm value.
- Then Eqs. (50)–(51) on p. 367 become, respectively,

$$\begin{aligned} nS &= Ve^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ B &= (1-\alpha) Ve^{-q\tau} N(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  – Above,

$$x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

  dollars.
  - Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.[a]
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

[a] See p. 221.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility, dividend, and strike price are under partial control of the stockholders or their boards.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 374 says the total market value of the bonds should be $42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

  dollars.
- The remaining stock is worth $1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$

  – This is called claim dilution.[a]

[a]Fama & M. H. Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a $14,833.3 cash dividend.
- The stockholders now have $14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning 2/3 of a call on n Merck shares with a strike price of $45,000.
- n such calls are worth $1,937.5 (p. 374).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

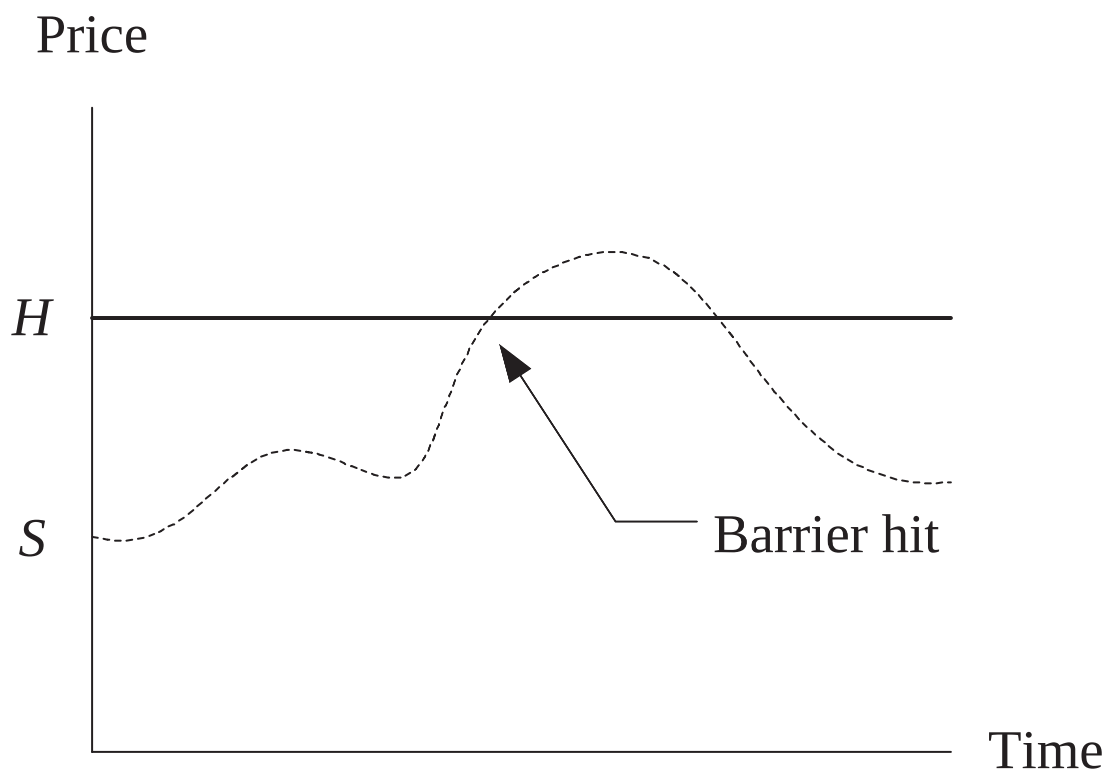
- Other Examples:
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
- Securities with a complex liability structure must be solved by trees.[a]

[a] Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H *throughout its life*.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

[a] A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.


Price
H
S
Barrier hit
Time

Barrier Options (continued)

- A knock-in (KI) option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all the possible barrier options mentioned above.[a]

[a] Haug (2006).

Barrier Options (concluded)

- Knock-in puts are the most popular barrier options.[a]
- Knock-out puts are the second most popular barrier options.[b]
- Knock-out calls are the most popular among barrier call options.[c]

---

[a] Bennett (2014).
[b] Bennett (2014).
[c] Bennett (2014).

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{52}$$

  - $x \stackrel{\Delta}{=} \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \stackrel{\Delta}{=} (r - q + \sigma^2/2)/\sigma^2$.
- A European down-and-out call can be priced via the in-out parity (see text).

[a]Merton (1973). See Exercise 17.1.6 of the textbook for a proof.

A Formula for Up-and-In Puts[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- Again, a European up-and-out put can be priced via the in-out parity.

---

[a]Merton (1973).

Are American Options Barrier Options?[a]

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

---

[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on March 25, 2009.

Interesting Observations

- Assume $H < X$.
- Replace S in the Merton pricing formula Eq. (42) on p. 322 for the call with H^2/S.
- Equation (52) on p. 386 for the down-and-in call becomes Eq. (42) when $r - q = \sigma^2/2$.
- Equation (52) becomes S/H times Eq. (42) when $r - q = 0$.

Interesting Observations (concluded)

- Replace S in the pricing formula for the down-and-in call, Eq. (52), with H^2/S.
- Equation (52) becomes Eq. (42) when $r - q = \sigma^2/2$.
- Equation (52) becomes H/S times Eq. (42) when $r - q = 0$.[a]
- Why?[b]

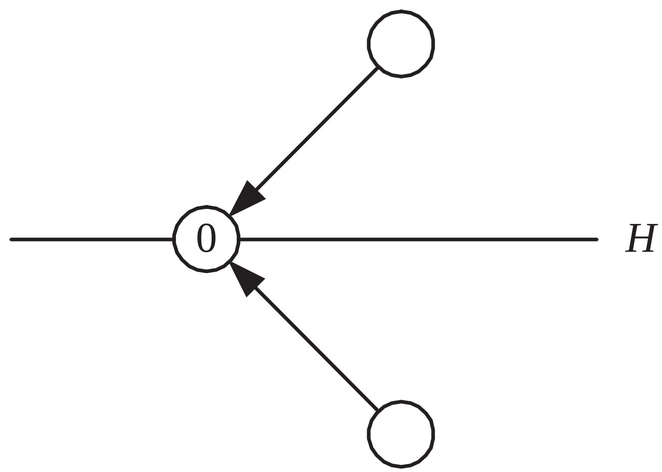
---

[a]Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.

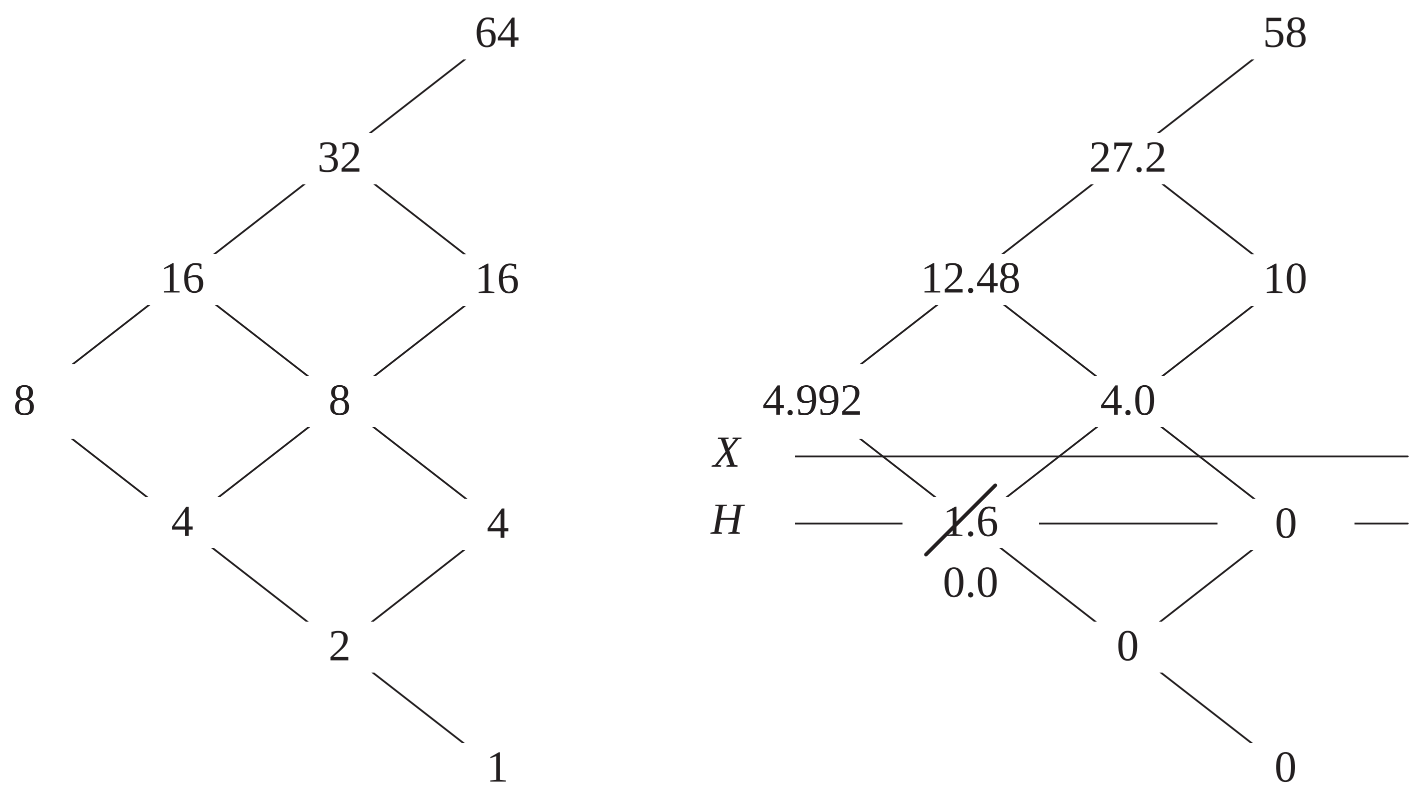
[b]Apply the reflection principle (p. 688), Eq. (41) on p. 285, and Lemma 9 (p. 290).

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.



- Pricing down-and-in options is subtler.



$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

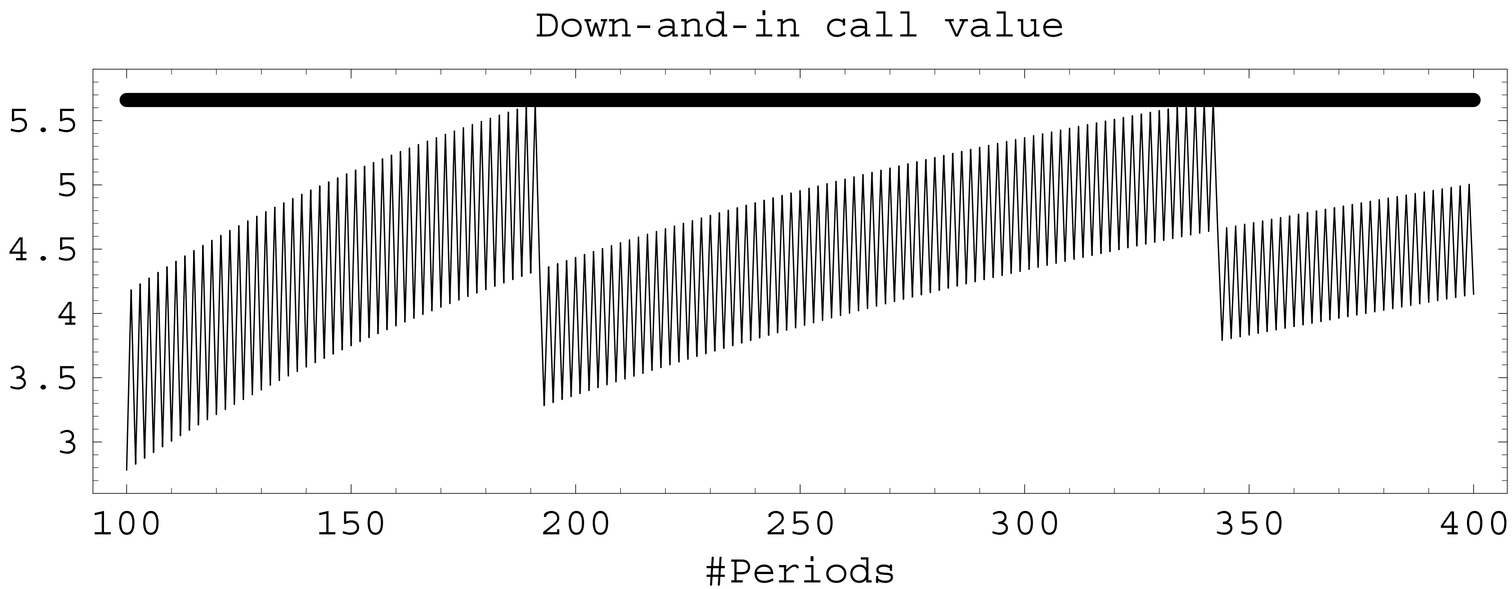
Binomial Tree Algorithms (continued)

- But convergence is erratic because H is not at a price level on the tree (see plot on next page).[a]
  - The barrier H is moved lower (or higher) to a close-by node price.
  - This "effective barrier" thus changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]
- Solutions will be presented later.

---

[a] Boyle & Lau (1994).
[b] J. Lin (R95221010) (2008).

Binomial Tree Algorithms (concluded)[a]



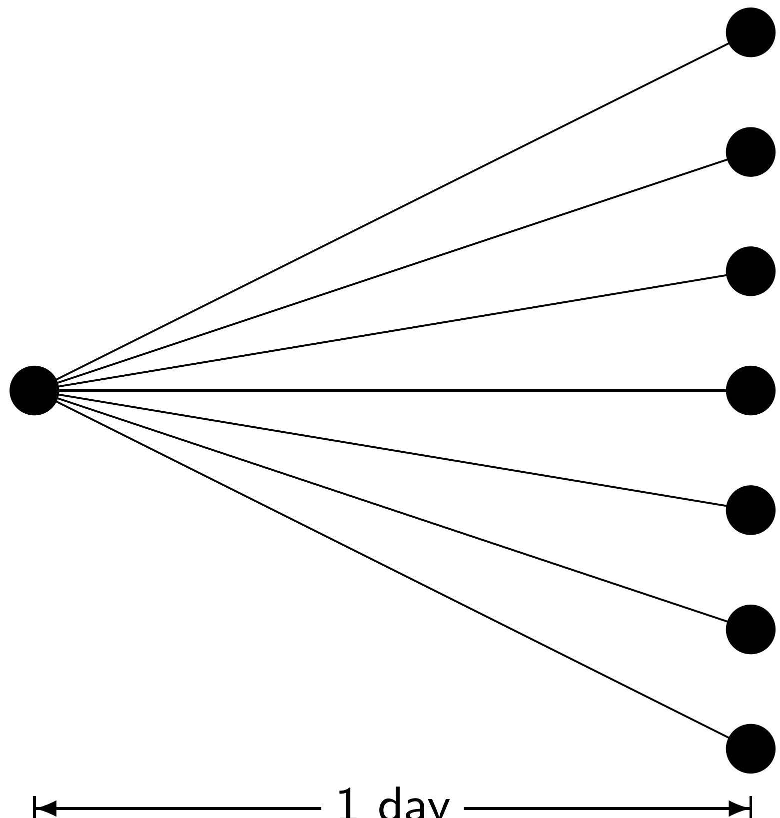
[a]Lyuu (1998).

Daily Monitoring

- Many barrier options monitor the barrier only for daily *closing prices.*
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into d periods.
- Does this save time or space?[a]

[a]Contributed by Ms. Chen, Tzu-Chun (`R94922003`) and others on April 12, 2006.

A Heptanomial Tree (6 Periods Per Day)



Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.[a]
- They are equally popular for indices.[b]

---

[a]Bennett (2014).
[b]Bennett (2014).

Data! data! data!
— Arthur Conan Doyle (1892),
The Adventures of Sherlock Holmes

Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
  - By that we mean the number of domestic currencies per unit of foreign currency.[a]
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.

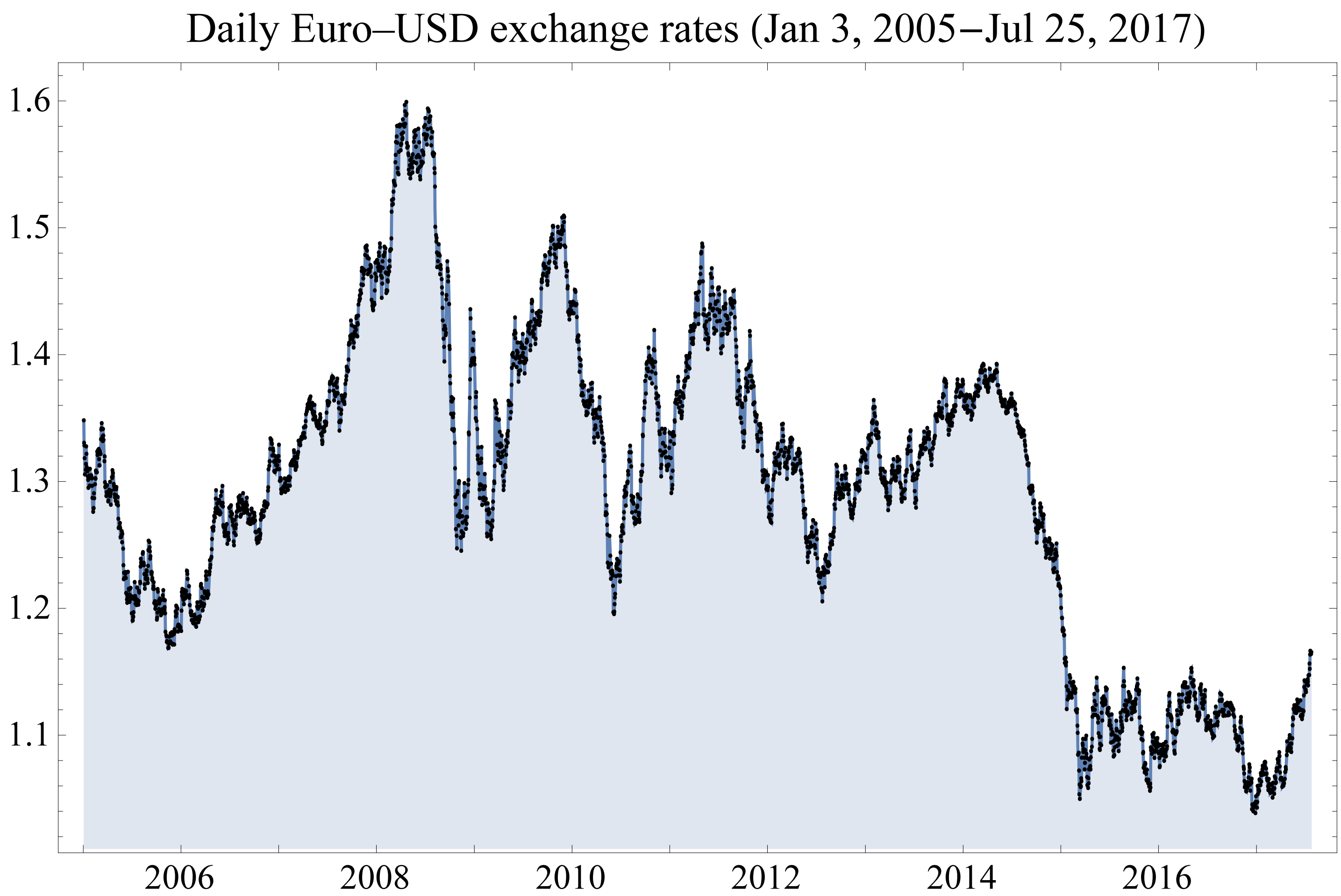
---

[a]The market convention is the opposite: A/B $= x$ means one unit of currency A (the reference currency or base currency) is equal to x units of currency B (the counter-value currency).

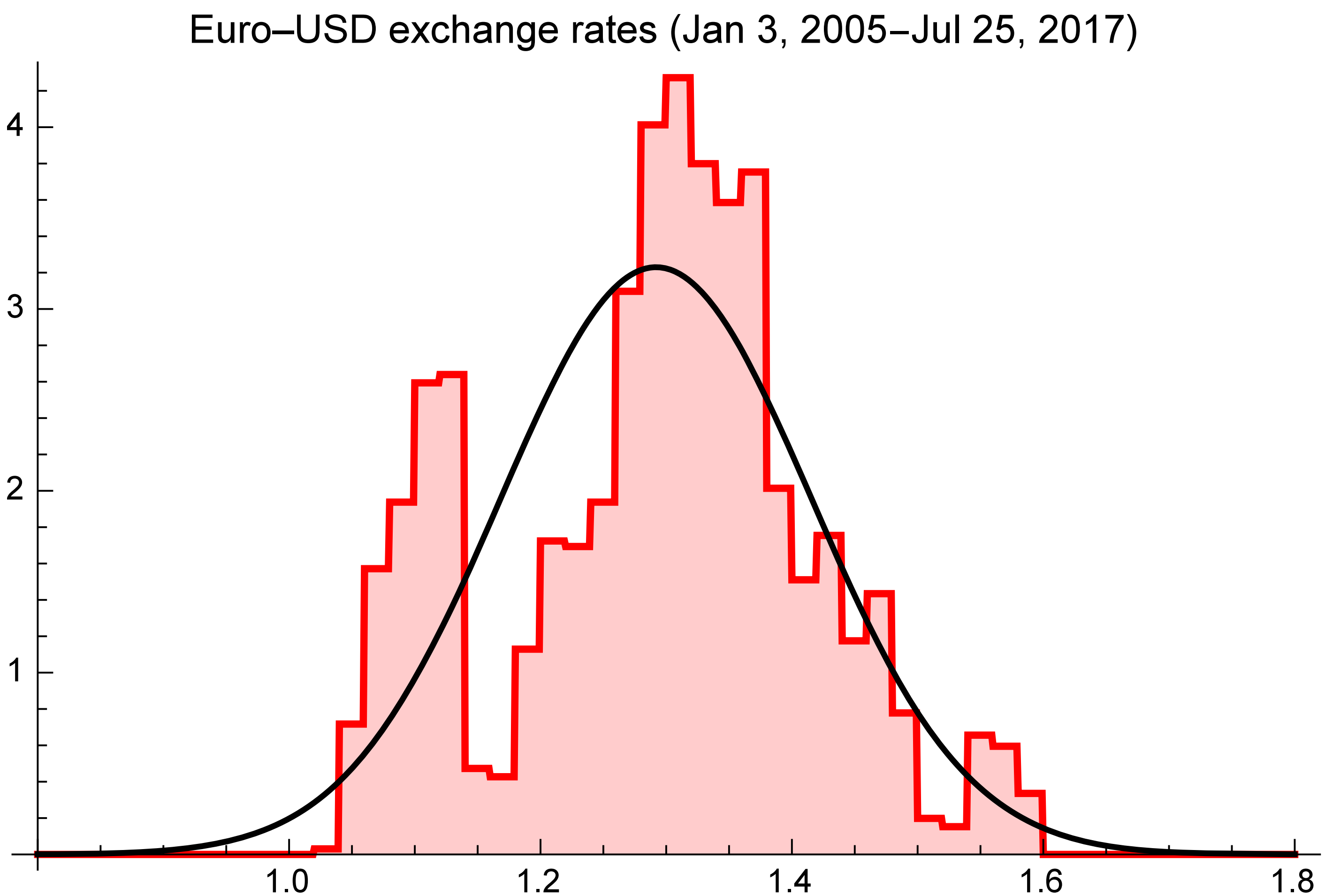
Foreign Currencies (concluded)

- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

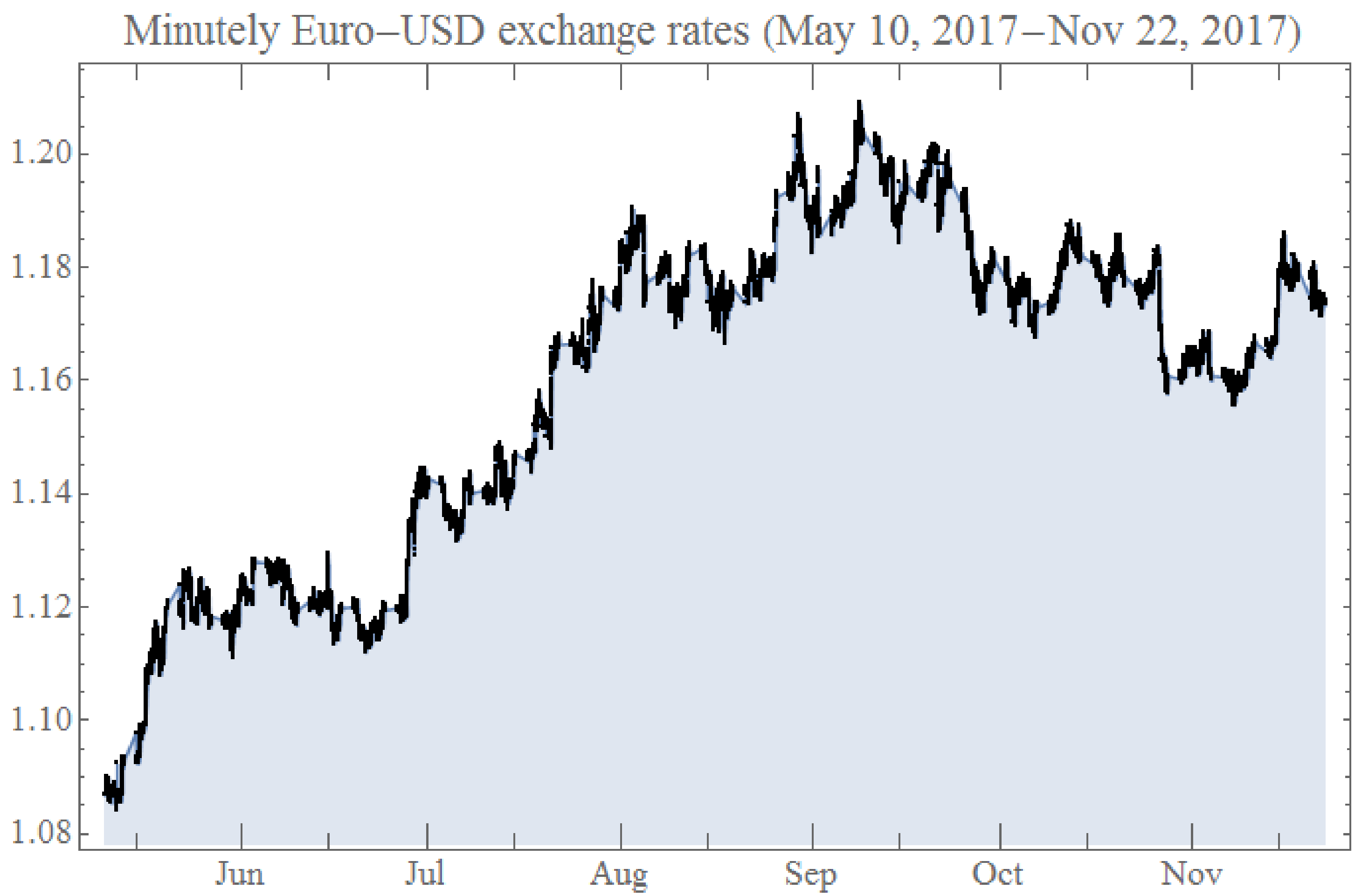
Time Series of the Daily Euro–USD Exchange Rate



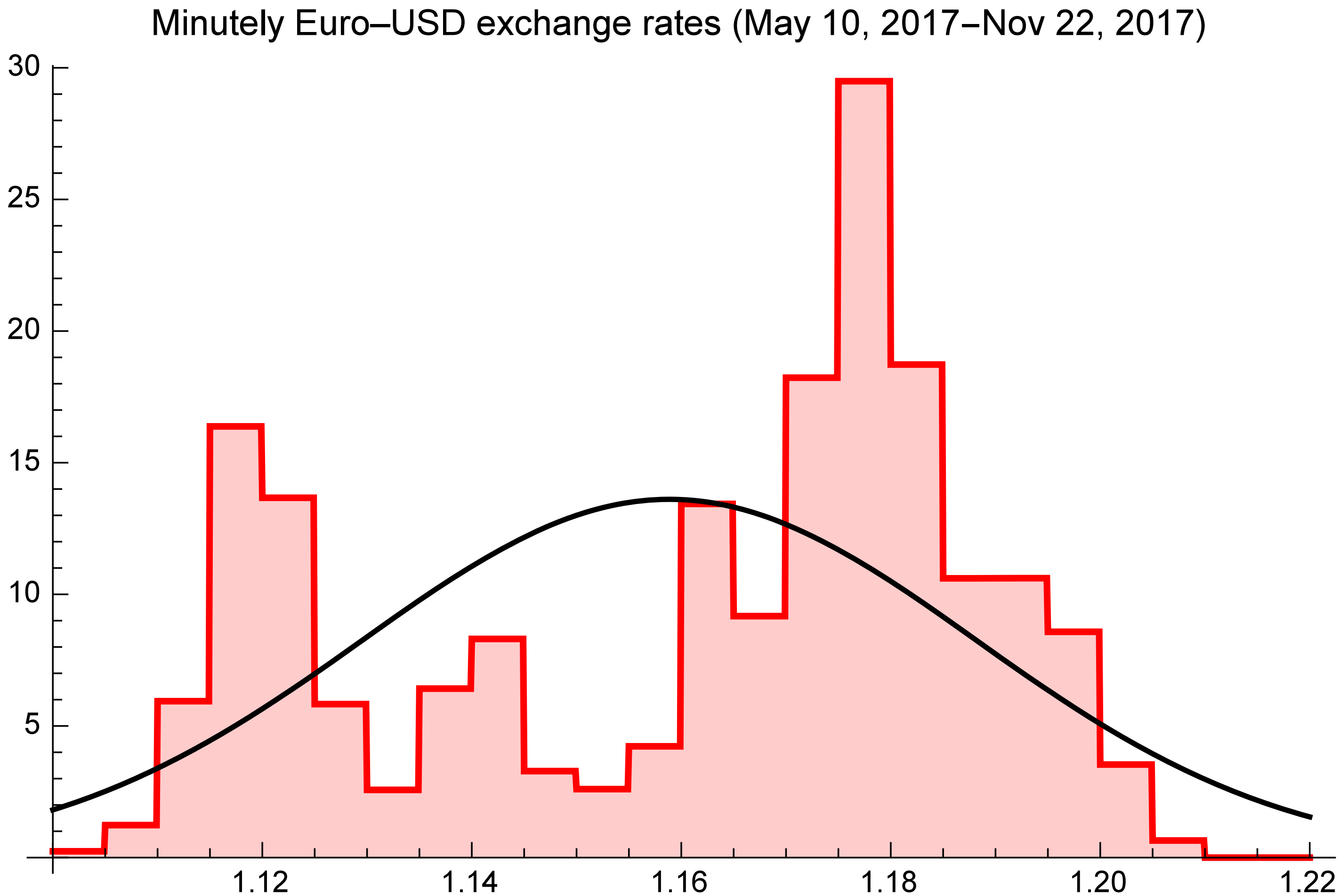
Distribution of the Daily Euro–USD Exchange Rate



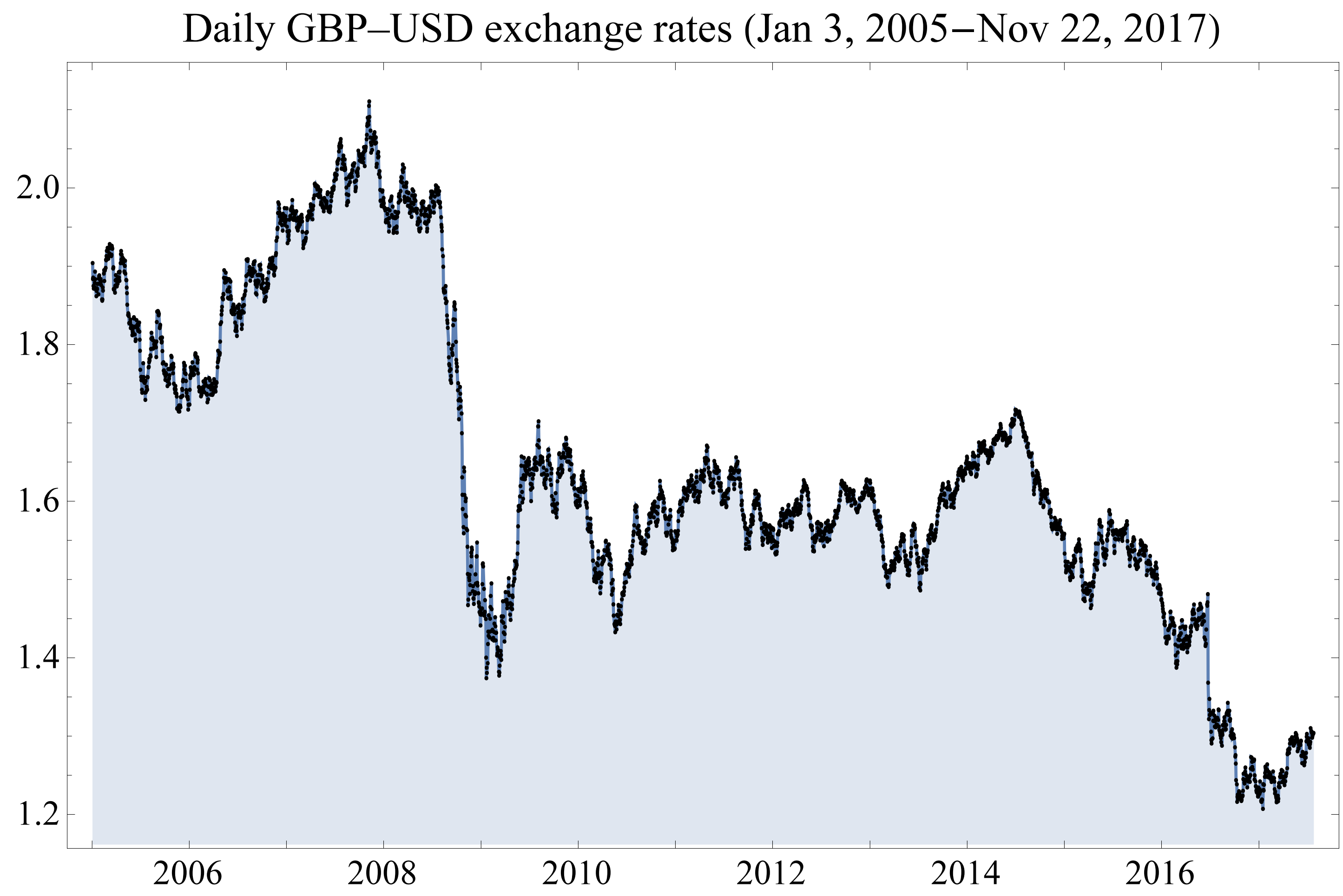
Time Series of the Minutely Euro–USD Exchange Rate



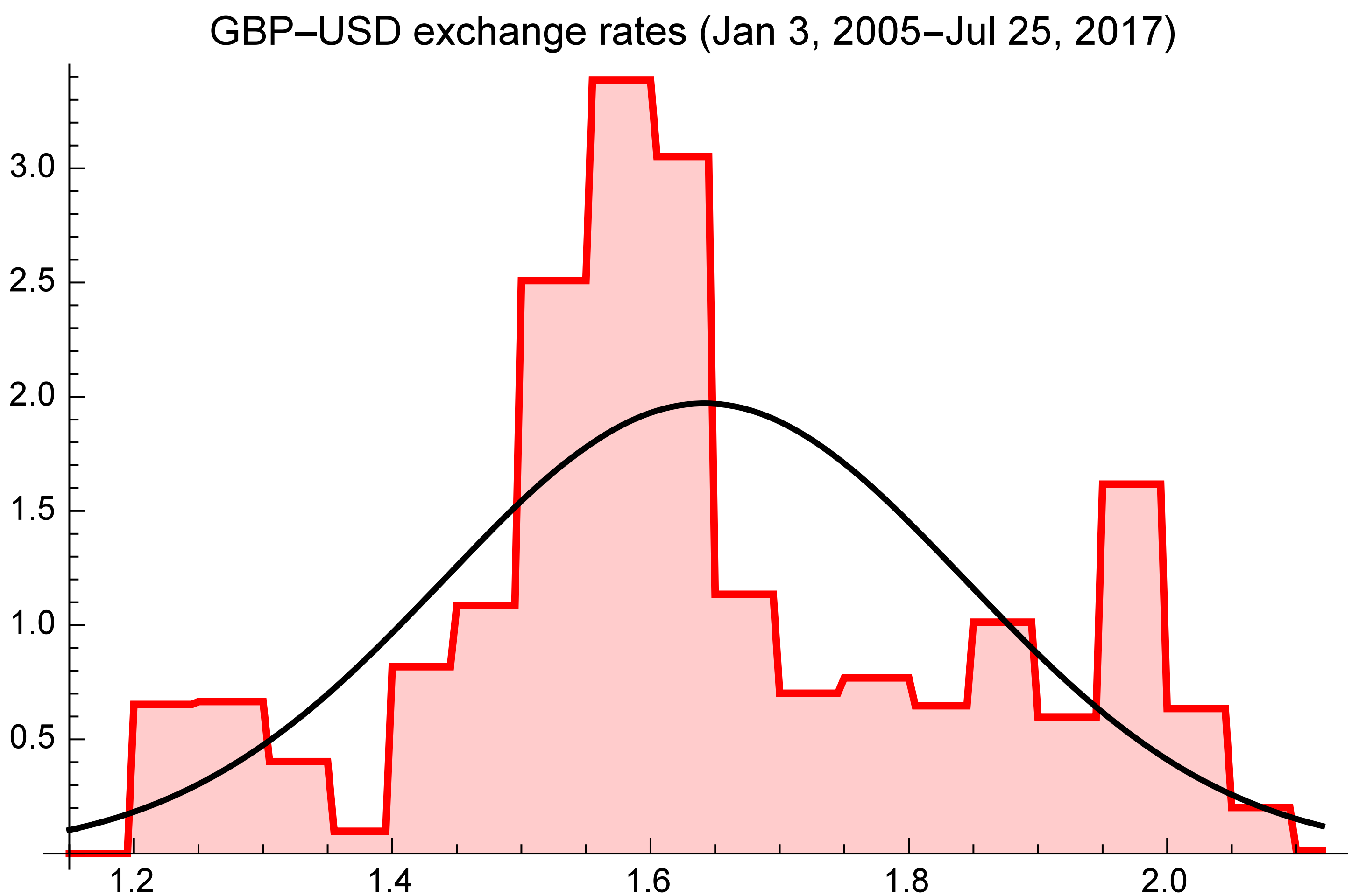
Distribution of the Minutely Euro–USD Exchange Rate



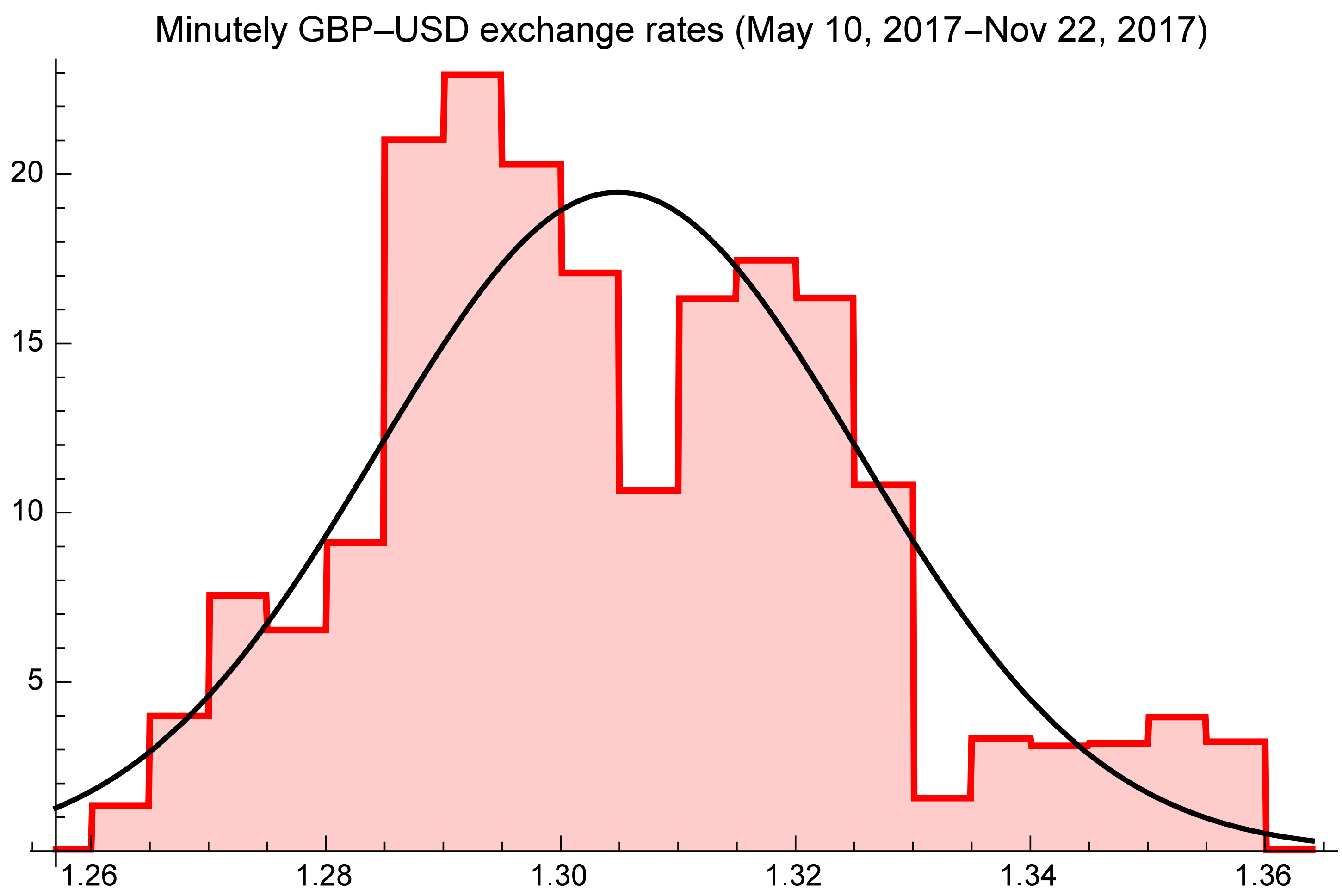
Time Series of the Daily GBP–USD Exchange Rate



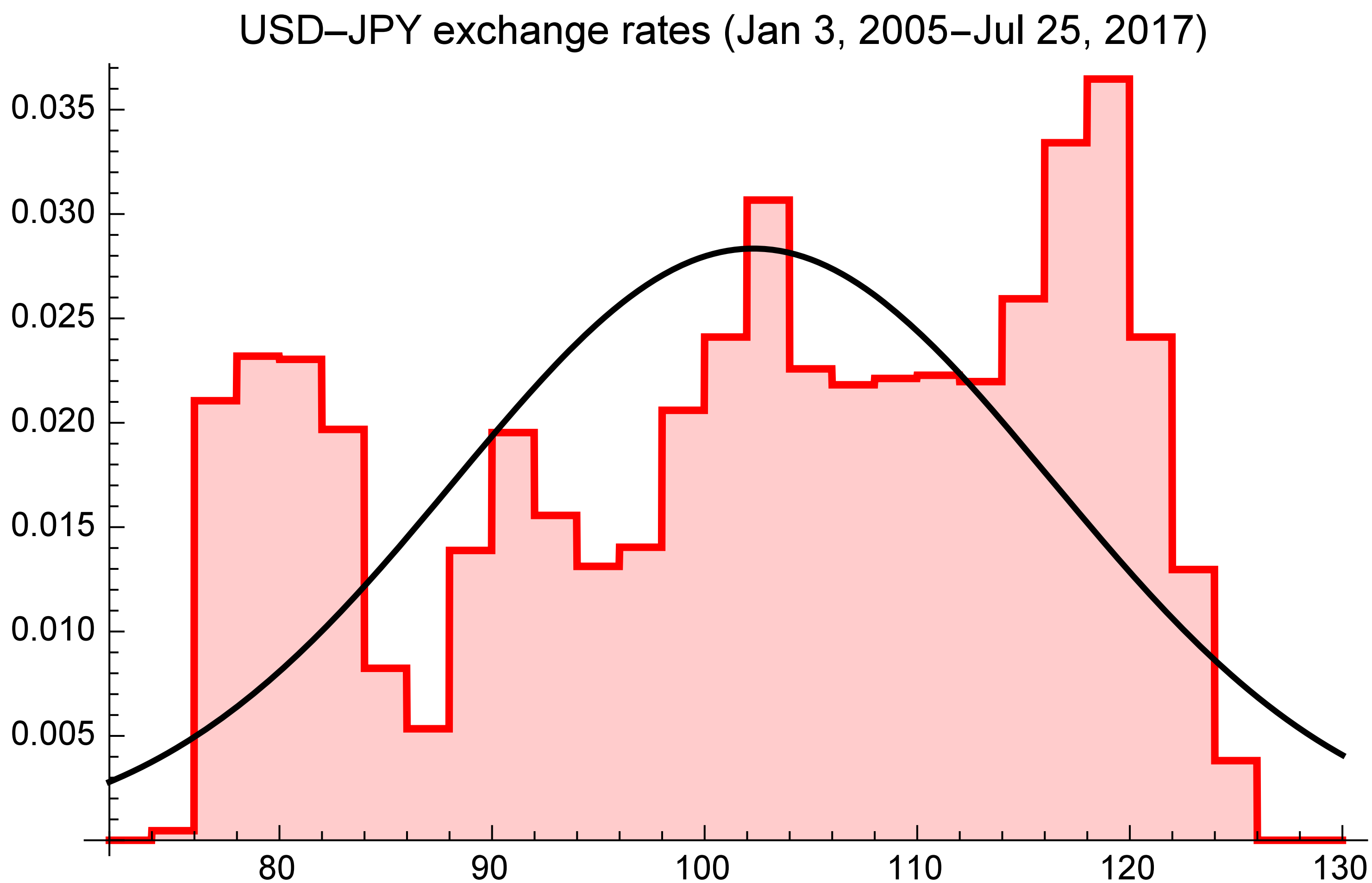
Distribution of the Daily GBP–USD Exchange Rate



Distribution of the Minutely GBP–USD Exchange Rate



Distribution of the Daily JPY–USD Exchange Rate



Foreign Exchange Options

- In 2000 the total notional volume of foreign exchange options was US$13 trillion.[a]
  - 38.5% were vanilla calls and puts with a maturity less than one month.
  - 52.5% were vanilla calls and puts with a maturity between one and 18 months.
  - 4% were barrier options.
  - 1.5% were vanilla calls and puts with a maturity more than 18 months.
  - 1% were digital options (see p. 839).
  - 0.7% were Asian options (see p. 420).

---

[a]Lipton (2002).

Foreign Exchange Options (continued)

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases

$$\frac{100{,}000{,}000}{6{,}250{,}000} = 16$$

puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.

- This gives the company the right to sell 100,000,000 Japanese yen for

$$100{,}000{,}000 \times .0088 = 880{,}000$$

U.S. dollars.

Foreign Exchange Options (concluded)

- Assume the exchange rate S is lognormally distributed.
- The formulas derived for stock index options in Eqs. (42) on p. 322 apply with the dividend yield equal to $\hat{r}$:

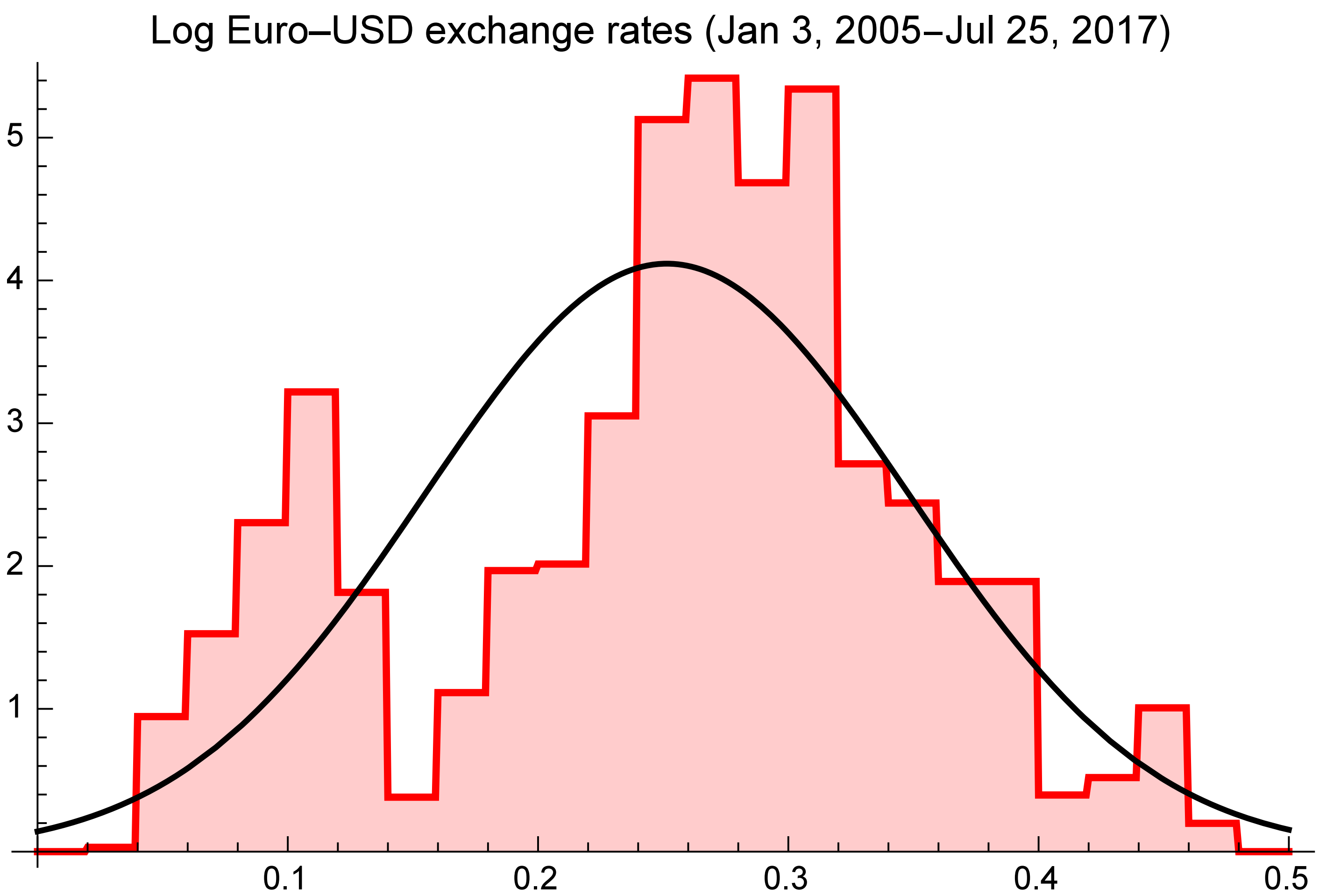
$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{53}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x). \tag{53'}$$

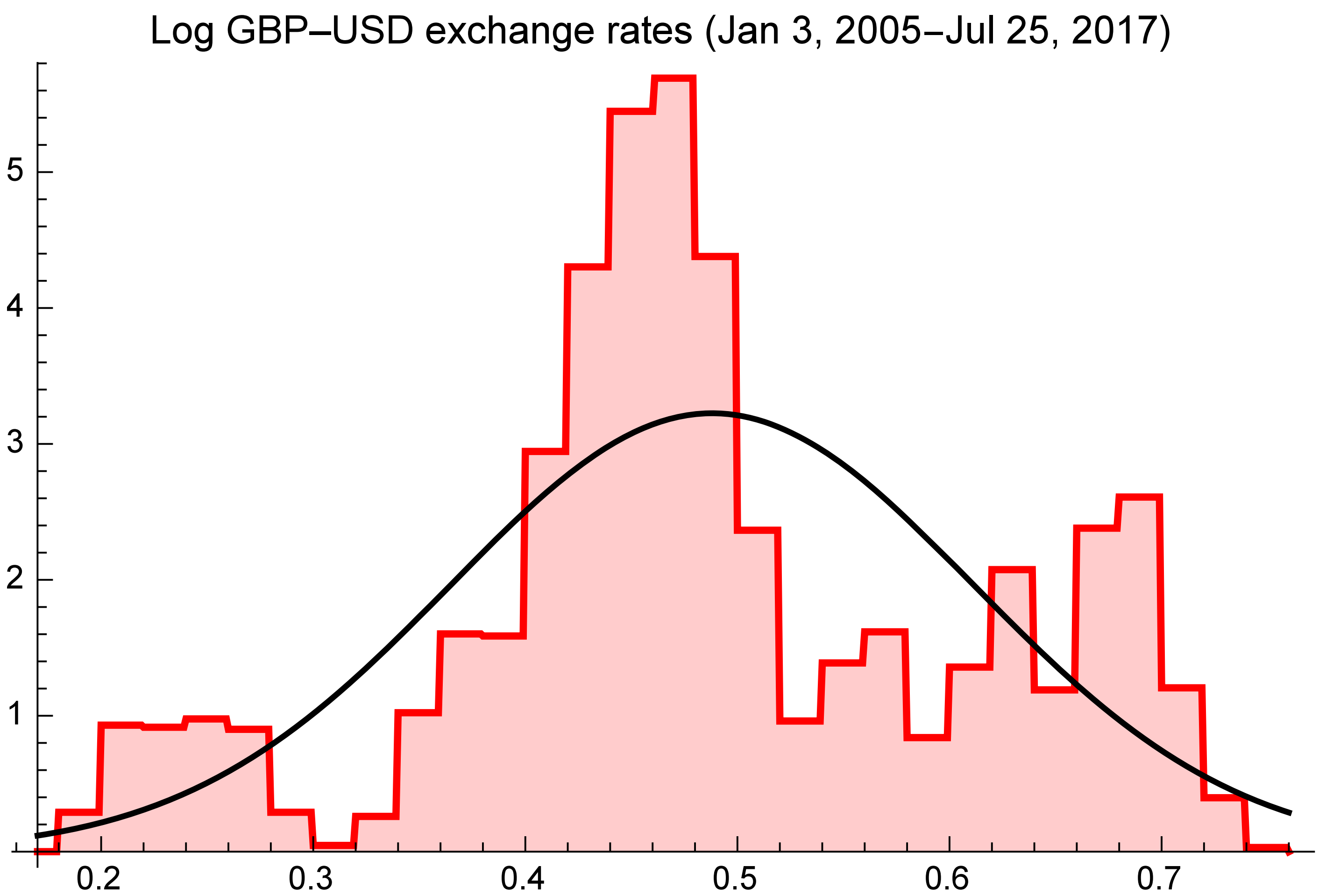
  – Above,

$$x \triangleq \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

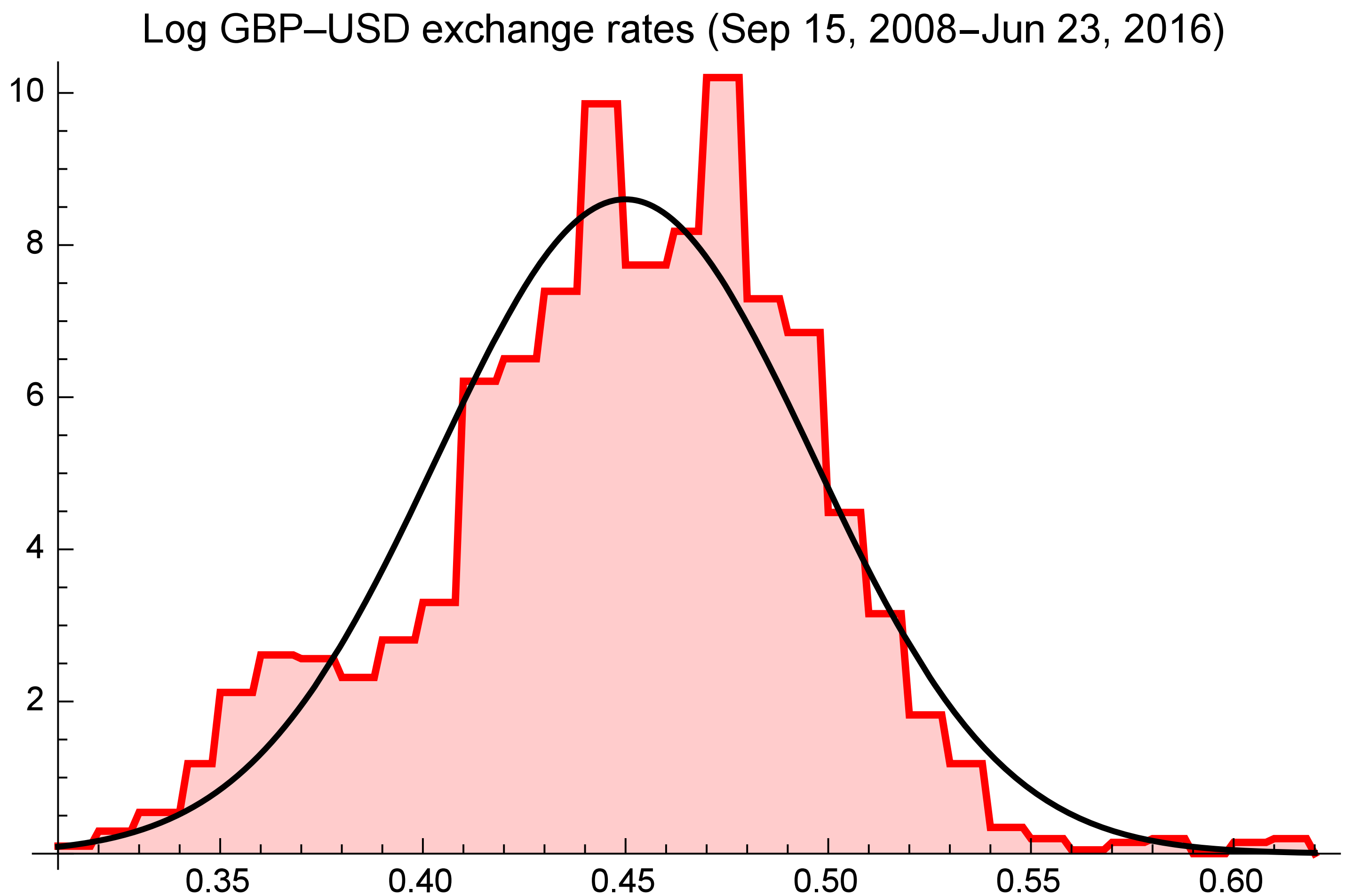
Distribution of the Logarithmic Euro–USD Exchange Rate



Distribution of the Logarithmic GBP–USD Exchange Rate



Distribution of the Logarithmic GBP–USD Exchange Rate (after the Collapse of Lehman Brothers and before Brexit)



Distribution of the Logarithmic JPY–USD Exchange Rate