Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \triangleq \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you flip a coin n times with p being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting j heads.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

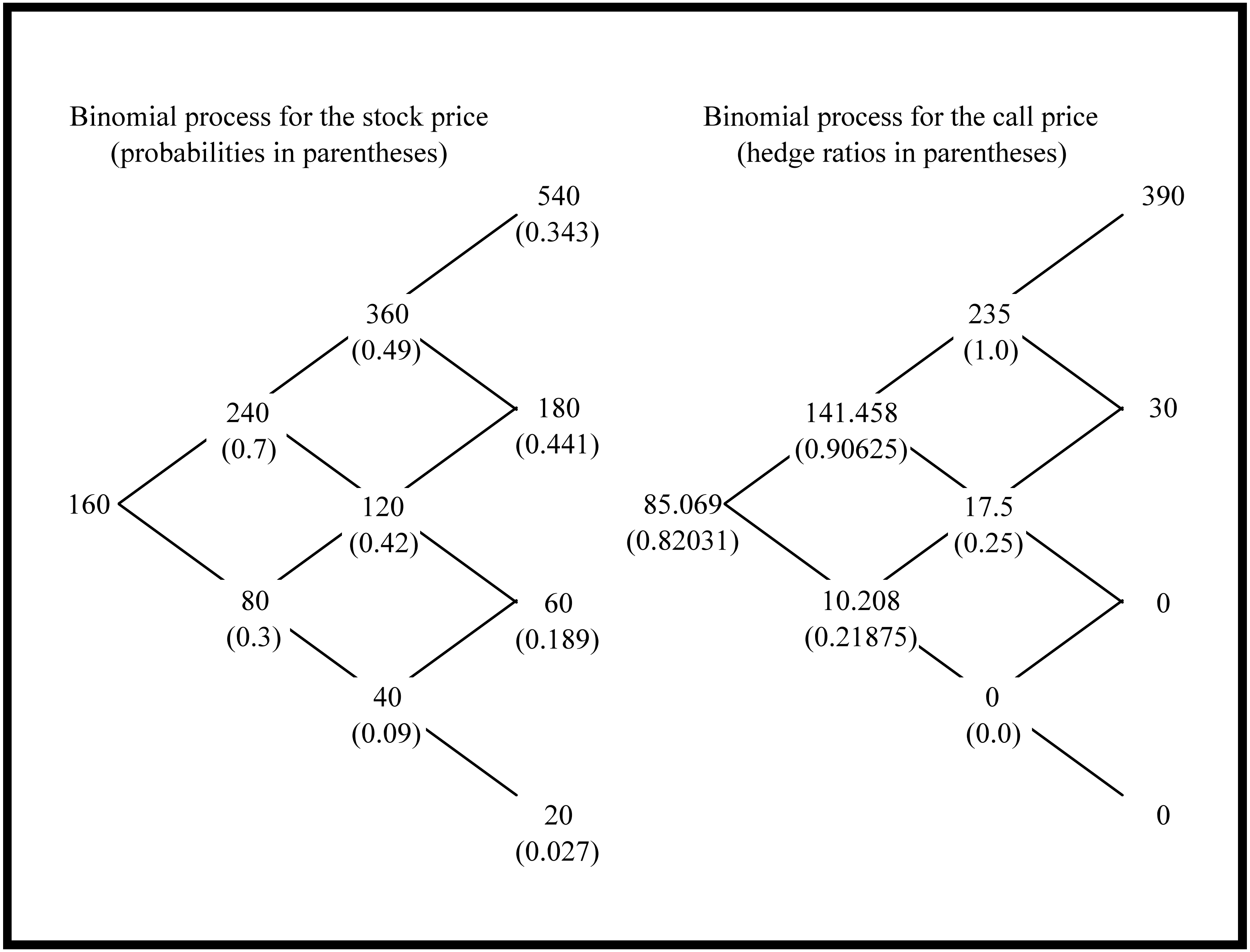
- Hence,

$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X \right)}{R^n} && (36) \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p). && (37)
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90.
- Invest \$85.069 in the *replicating* portfolio with 0.82031 shares of stock as required by the delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

is the arbitrage profit, as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy
$$0.90625 - 0.82031 = 0.08594$$
more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value!

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

  dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- Close out the call's short position by buying back the call or buying a share of stock for delivery.
- This results in a loss of $180 - 150 = 30$ dollars.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

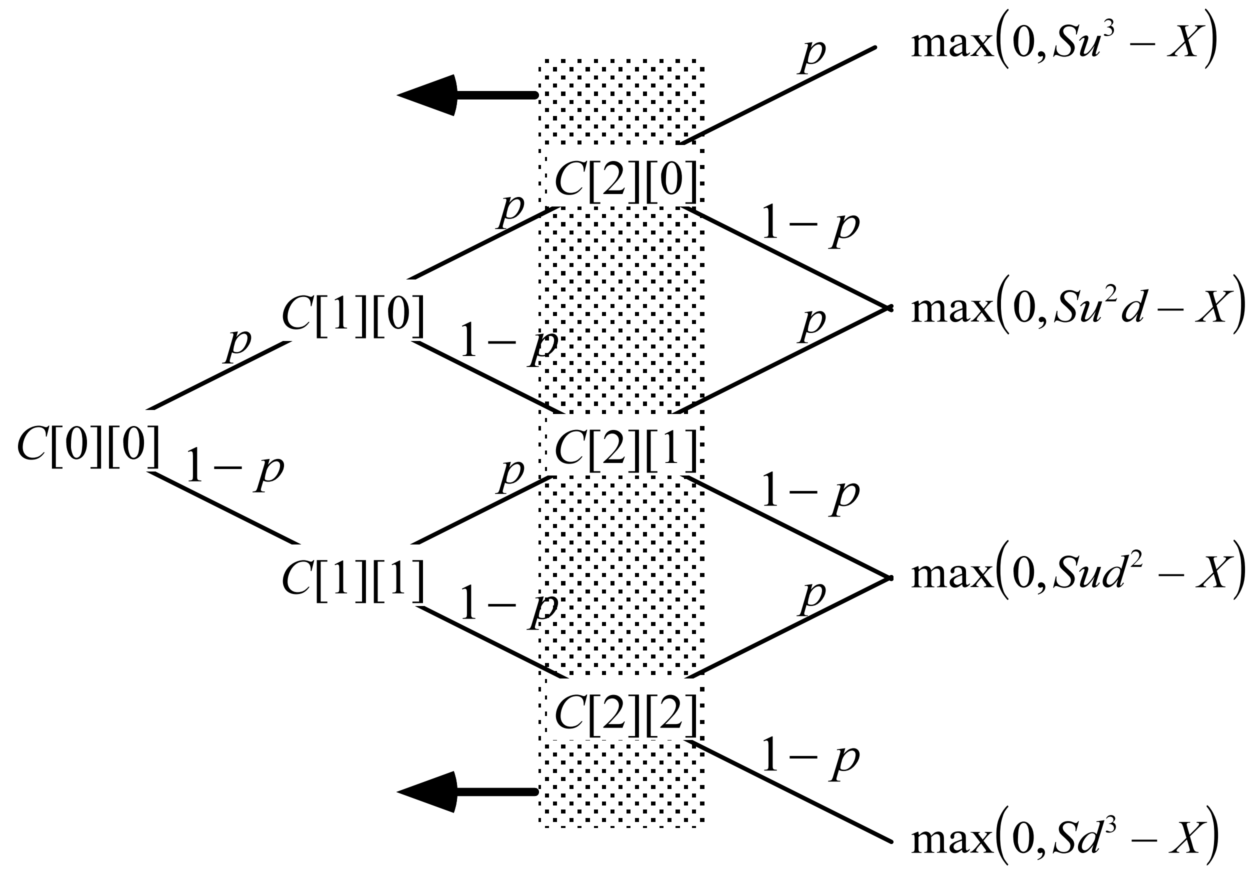
- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Use \$85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.[b]
- $\cdots$
- Without hedge, one may end up forking out \$390 in the worst case![c]

[a]Thanks to a lively class discussion on March 16, 2011.
[b]Hedging and replication are mirror images.
[c]Thanks to a lively class discussion on March 16, 2016.

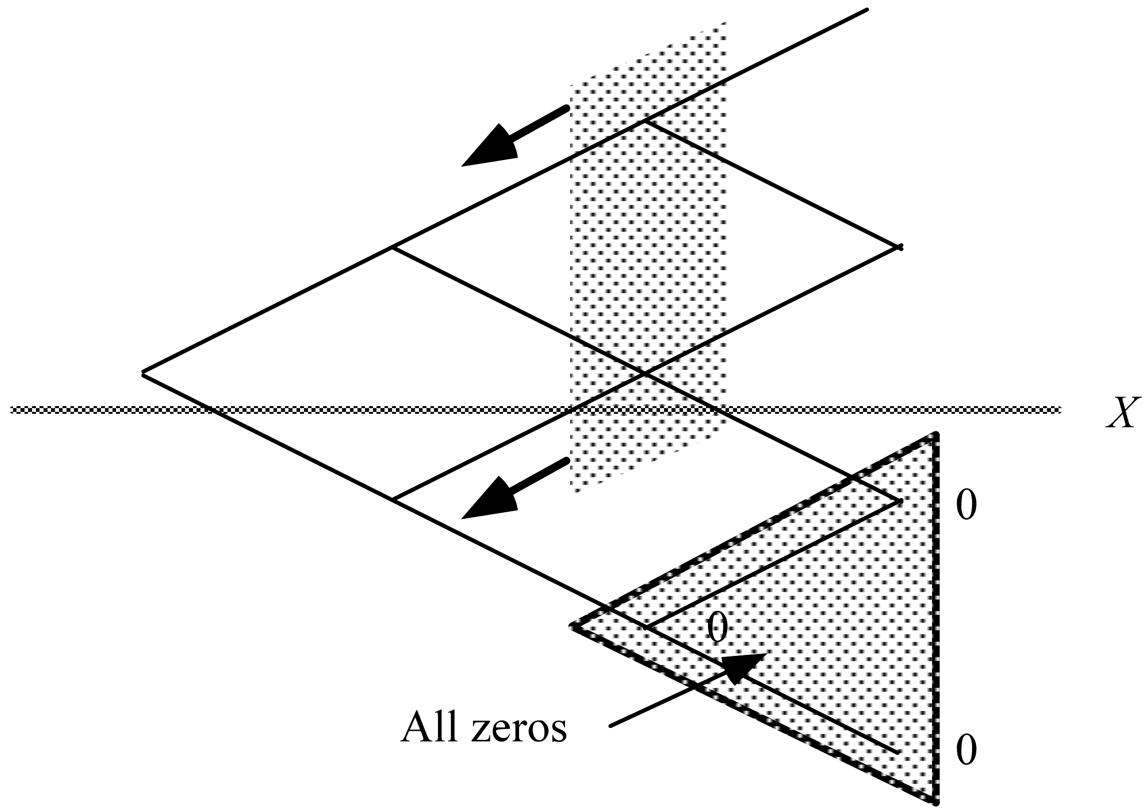
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To price European puts, simply replace the payoff.

[a] But watch out for the proper updating of array entries.


$\max(0, Su^3 - X)$
p
$C[2][0]$
p
$1-p$
$C[1][0]$
$\max(0, Su^2d - X)$
p
$1-p$
p
$C[0][0]$
$C[2][1]$
$1-p$
p
$1-p$
$C[1][1]$
$\max(0, Sud^2 - X)$
$1-p$
p
$C[2][2]$
$1-p$
$\max(0, Sd^3 - X)$

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j;n,p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1;n,p).$$

Optimal Algorithm (continued)

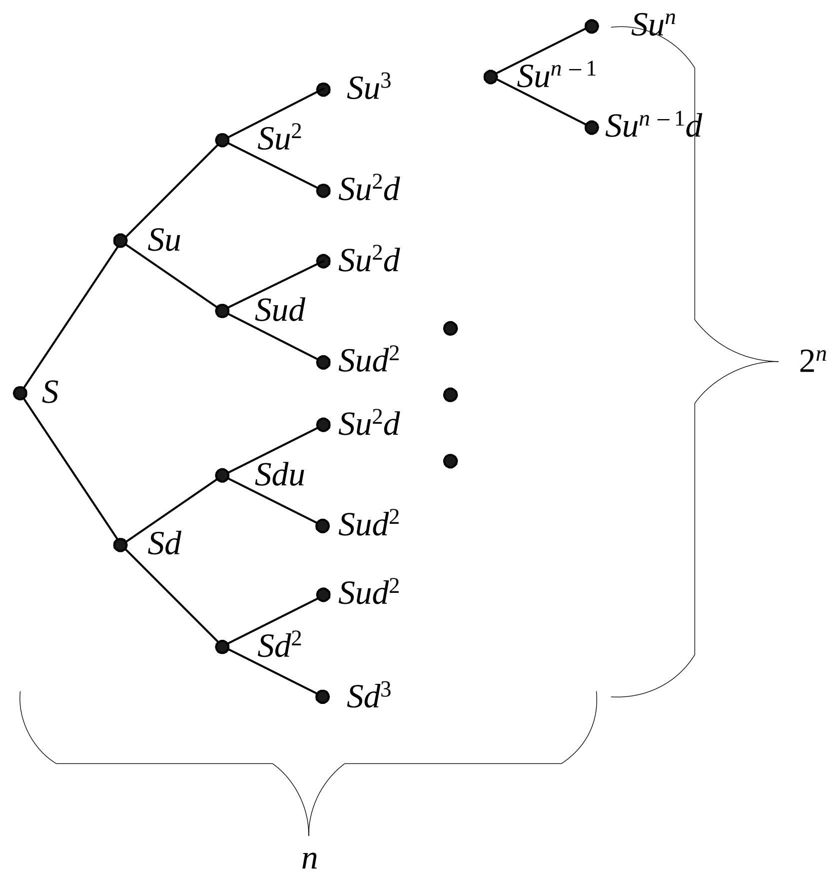
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a};$
for $j = a+1, a+2, \ldots, n$ **do**
    $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j);$
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (36) on p. 261 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.[a]
- Need to calibrate the BOPM's parameters u, d, and R to make it converge to the continuous-time model.
- We now skim through the proof.

[a] Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- First, $\hat{r} = r\tau/n$.
  - Each period is τ/n years long.
  - The period gross return $R = e^{\hat{r}}$.
- Let
$$\widehat{\mu} \stackrel{\Delta}{=} \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period of the BOPM.
- Let
$$\widehat{\sigma}^2 \stackrel{\Delta}{=} \frac{1}{n} \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that[a]

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

- Assume the stock's *true* continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

[a]Recall the Bernoulli distribution.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$n\widehat{\mu} = n[\,q\ln(u/d) + \ln d\,] \to \mu\tau, \tag{38}$$

$$n\widehat{\sigma}^2 = nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \tag{39}$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 273).
  - Other choices are possible (see text).

- Exact solutions for u, d, q are feasible if Eqs. (38)–(39) are replaced by equations: 3 equations for 3 variables.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \tag{40}$$

- With Eqs. (40), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

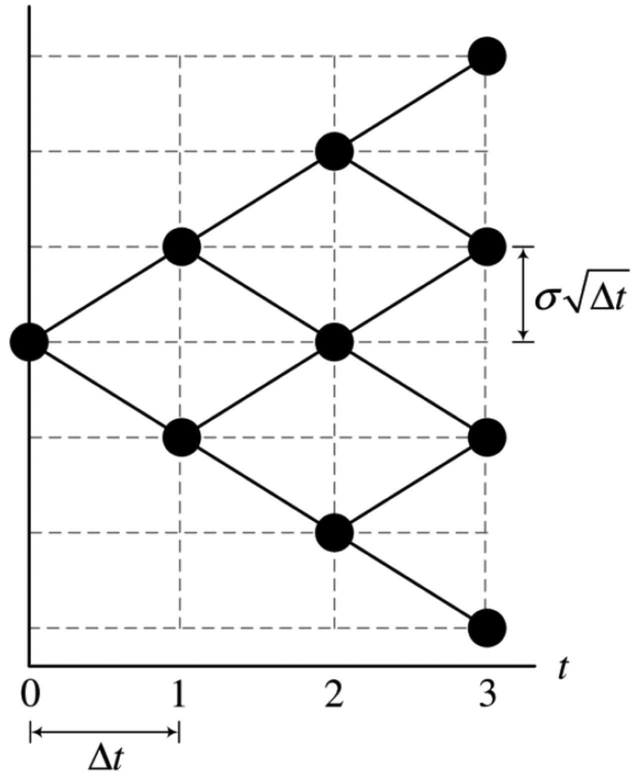
Toward the Black-Scholes Formula (continued)

- The choices (40) result in the CRR binomial model.[a]
- With the above choice, even if u and d are not calibrated, the mean is still matched![b]

---

[a] Cox, Ross, & Rubinstein (1979).

[b] Recall Eq. (33) on p. 245. So u and d are related to volatility exclusively. They do not depend on r.



$\sigma\sqrt{\Delta t}$
t
0
1
2
3
Δt

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (40) on p. 284 or Eq. (32) on p. 244.
  - If this happens, the probabilities lie outside $[0, 1]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

  i.e., when $n > r^2\tau/\sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions can be found in the textbook[b] or will be presented later.

---

[a] Many papers and programs forget to check this condition!

[b] See Exercise 9.3.1 of the textbook.

Toward the Black-Scholes Formula (continued)

- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

---

[a]The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.

Toward the Black-Scholes Formula (continued)

Lemma 9 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \stackrel{\Delta}{=} (e^{r\tau/n} - d)/(u-d).$$

- Let $n \to \infty$.[a]
- Then $\mu = r - \sigma^2/2$.

[a]See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return[b] is thus the riskless rate r.

---

[a]By Lemma 9 (p. 289) and Eq. (28) on p. 175.

[b]In the sense of $(1/\tau) \ln E[\, S_\tau / S\,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S)\,]$ (geometric average rate of return). In the latter case, it would be $r - \sigma^2/2$ by Lemma 9.

Toward the Black-Scholes Formula (continued)[a]

Theorem 10 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

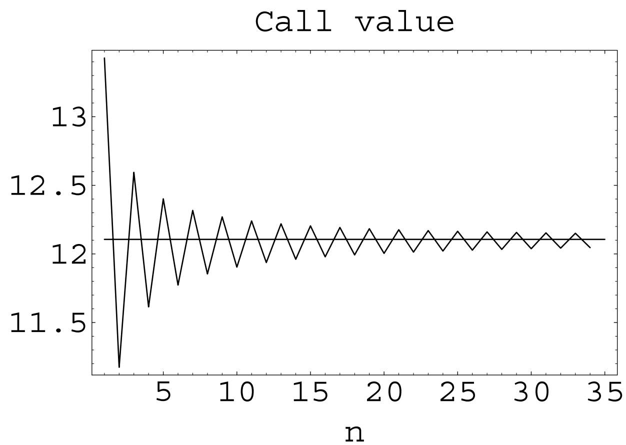
Toward the Black-Scholes Formula (concluded)

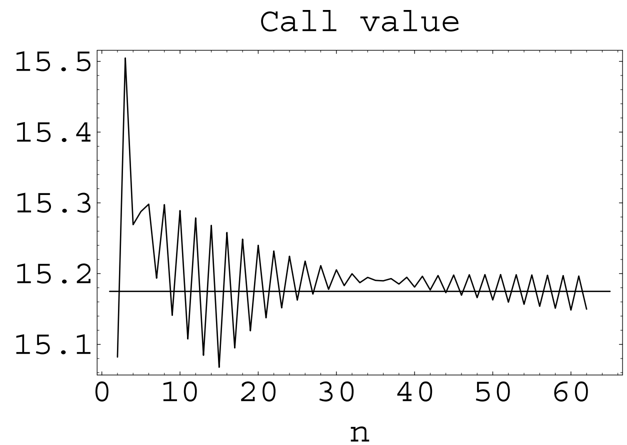
- See Eq. (37) on p. 261 for the meaning of x.
- See Exercise 13.2.12 of the textbook for an interpretation of the probability associated with $N(x)$ and $N(-x)$.

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\tau/n}}, \\
d &= e^{-\sigma\sqrt{\tau/n}}, \\
\hat{r} &= r\tau/n.
\end{aligned}
$$





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of u and d.[b]

[a] L. Chang & Palmer (2007).
[b] See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

[a]Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  > the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Just think of it as an alternative to quoting option prices.
- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

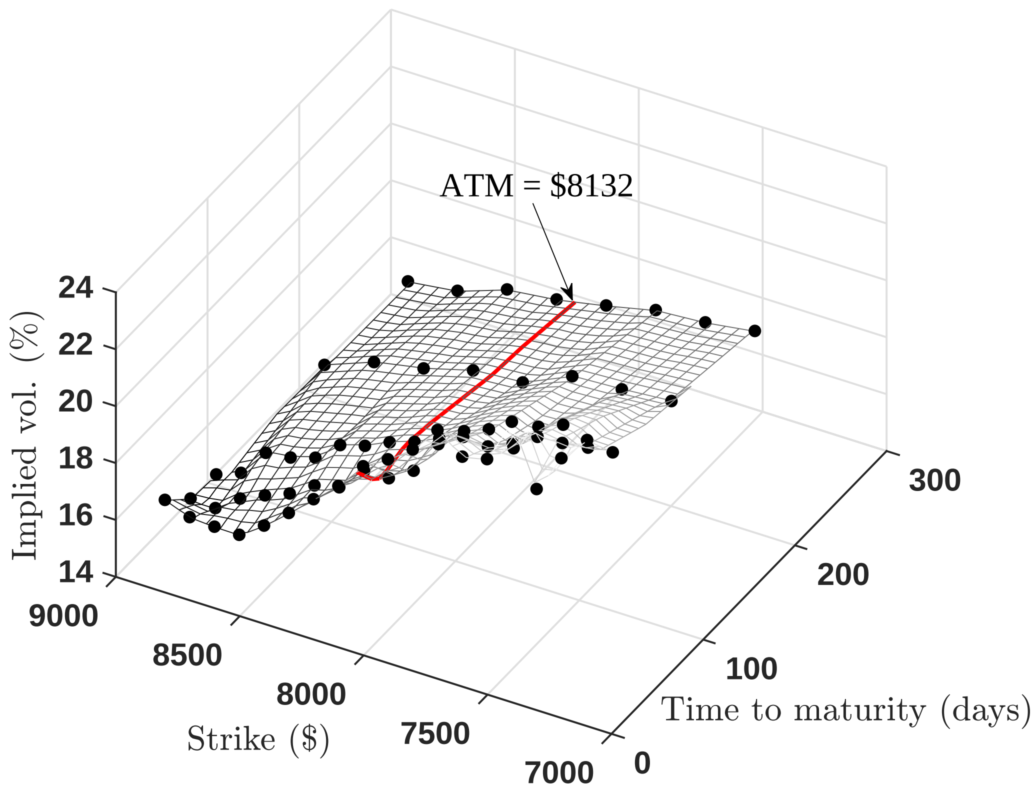
---

[a]Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

TXO Calls (September 25, 2015)[a]



[a]The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.

Solutions to the Smile

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the *continuation value*.
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Instantaneous Volatility[a]

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ.
- In the limit, the variance of $\ln(S_\tau/S)$ is
$$\int_0^\tau \sigma^2(t)\,dt$$
rather than $\sigma^2\tau$.
- The annualized volatility to be used in the Black-Scholes formula should now be
$$\sqrt{\frac{\int_0^\tau \sigma^2(t)\,dt}{\tau}}\,.$$

[a] Merton (1973).

Time-Dependent Instantaneous Volatility (concluded)

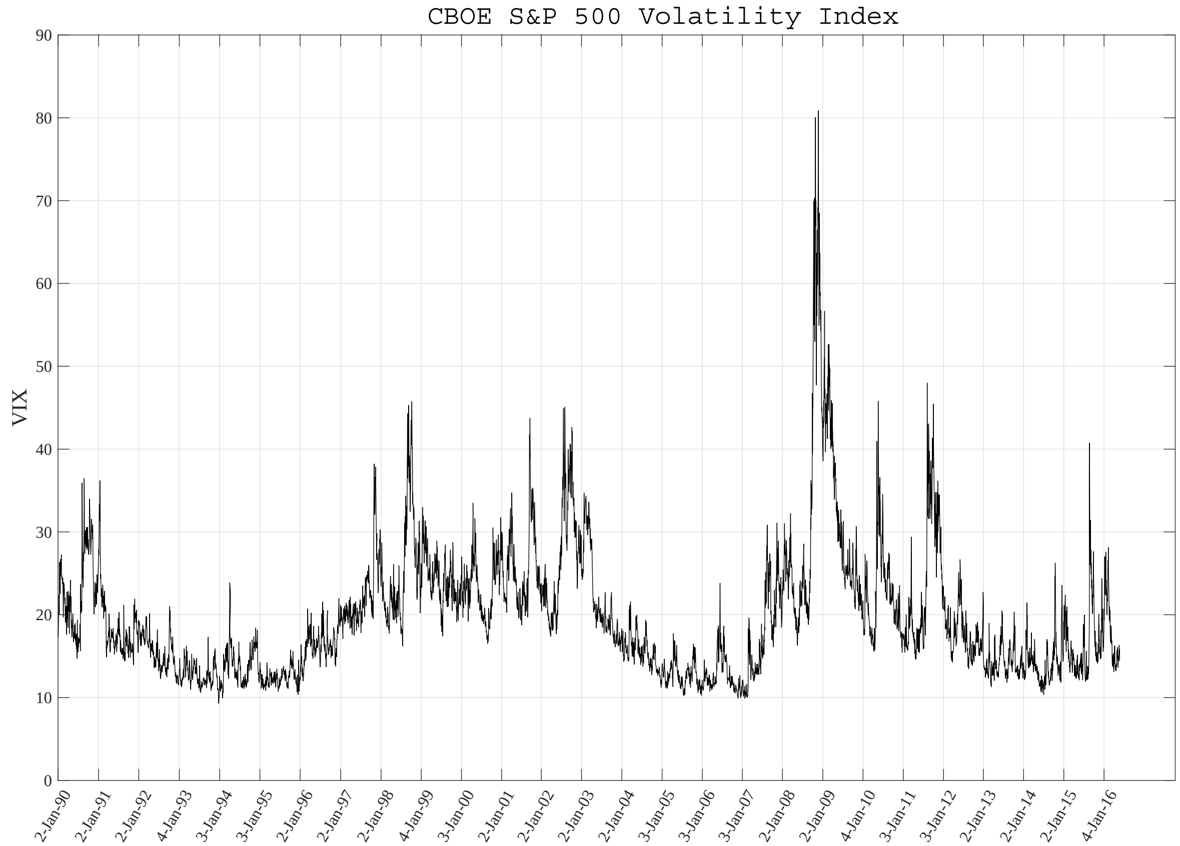
- For the binomial model, u and d depend on time:

$$\begin{aligned} u &= e^{\sigma(t)\sqrt{\tau/n}}, \\ d &= e^{-\sigma(t)\sqrt{\tau/n}}. \end{aligned}$$

- But how to make the binomial tree combine?[a]

[a] Amin (1991); C. I. Chen (R98922127) (2011).

Volatility (1990–2016)[a]



[a]Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.
- The riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,
$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$
where r_i is the continuously compounded short rate measured in periods for period i.[a]
- Will the binomial tree fail to combine?

[a] That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a] Fama (1965); K. French (1980); K. French & Roll (1986).

[b] Recall p. 158 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

[a]D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

[a]Contributed by Mr. Lu, Zheng-Liang (`D00922011`) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.

$$
\begin{array}{ccccc}
 & & & & (Su-D)\,u \\
 & & & \nearrow & \\
 & & Su-D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su-D)\,d \\
S & & & & \\
 & & & & (Sd-D)\,u \\
 & \searrow & & \nearrow & \\
 & & Sd-D & & \\
 & & & \searrow & \\
 & & & & (Sd-D)\,d
\end{array}
$$

An Ad-Hoc Approximation

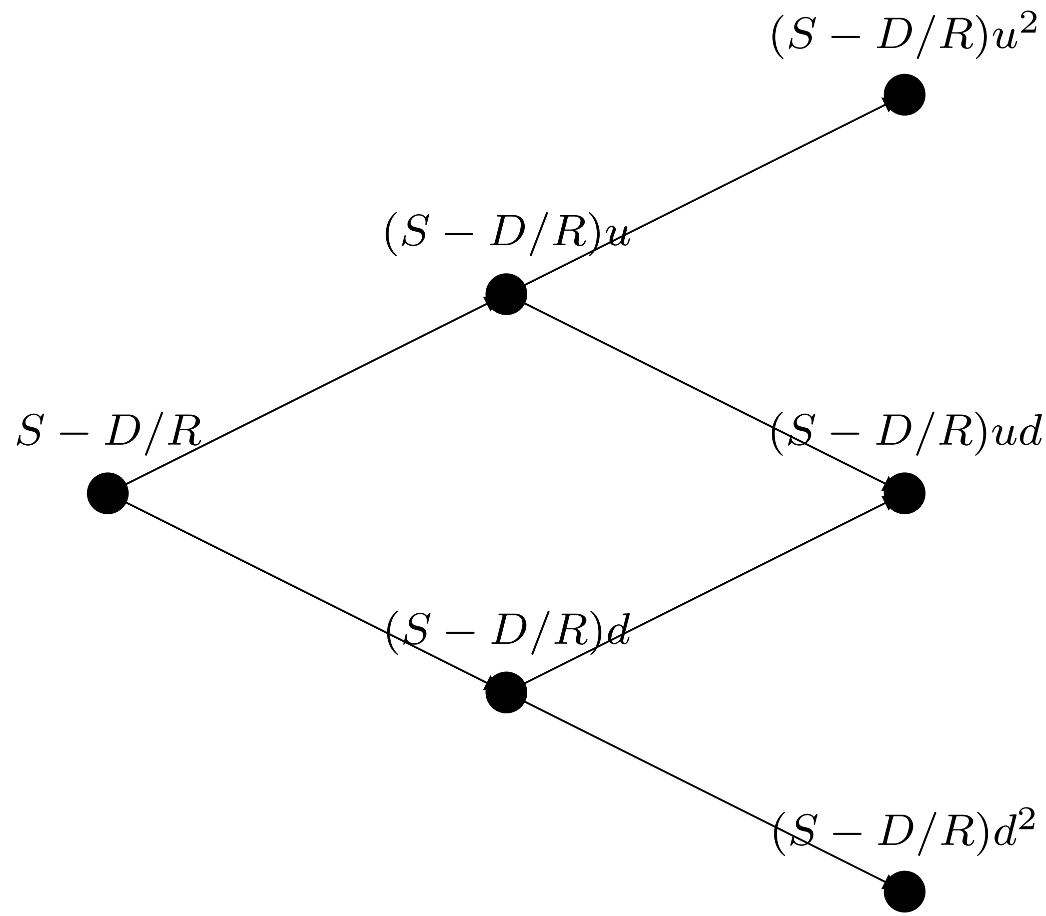
- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

[a]Roll (1977).

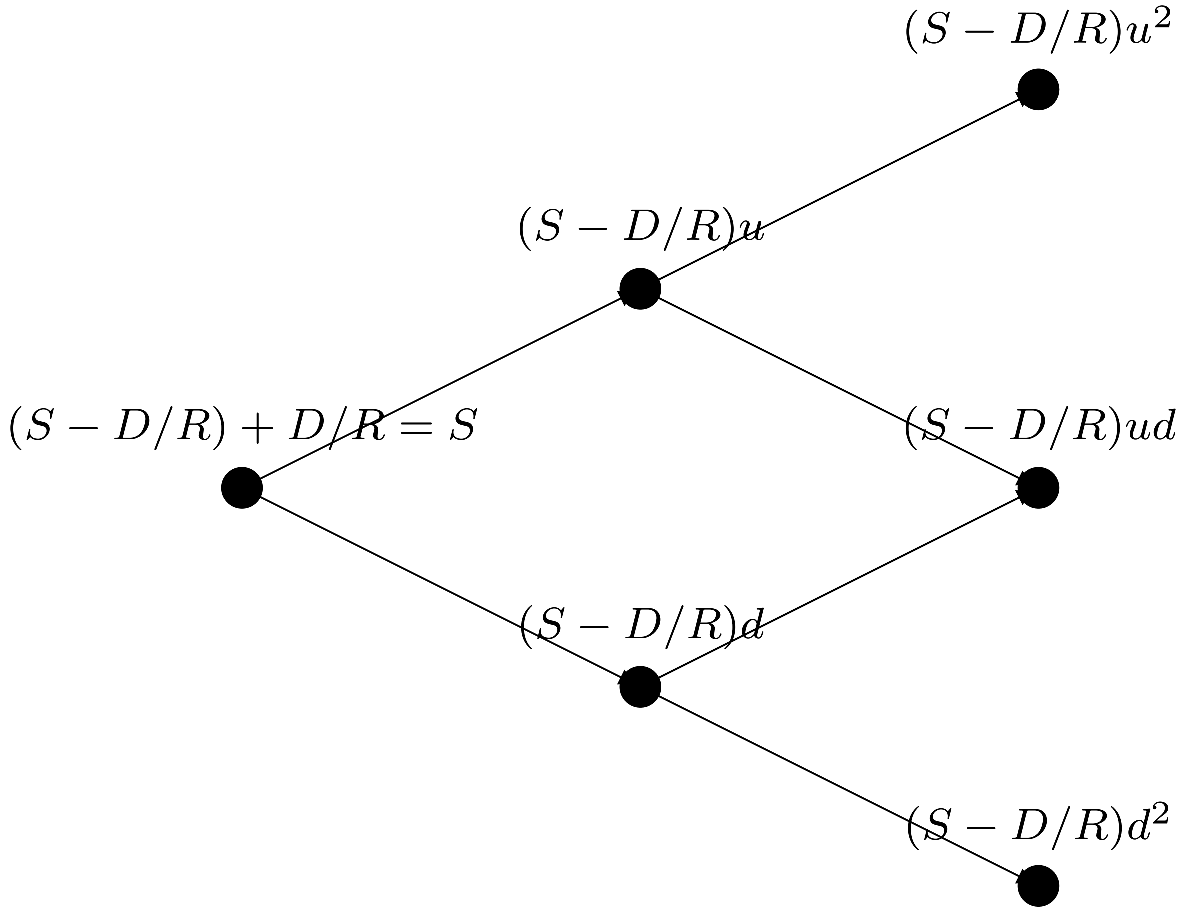
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 312 (Step 1)



The Ad-Hoc Approximation vs. P. 312 (Step 2)



The Ad-Hoc Approximation vs. P. 312[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\, u, (Su - D)\, d, (Sd - D)\, u, (Sd - D)\, d$ (p. 312).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$
\begin{aligned}
(Su - D)\, u &> (S - D/R)u^2, \\
(Sd - D)\, d &< (S - D/R)d^2,
\end{aligned}
$$

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 312 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually increased when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 765ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

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[a] Dai (`B82506025`, `R86526008`, `D8852600`) & Lyuu (2004). Also Arealy & Rodrigues (2013).

[b] Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{41}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{41'}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (41) and (41′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \stackrel{\Delta}{=} \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

[a]Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{42}$$

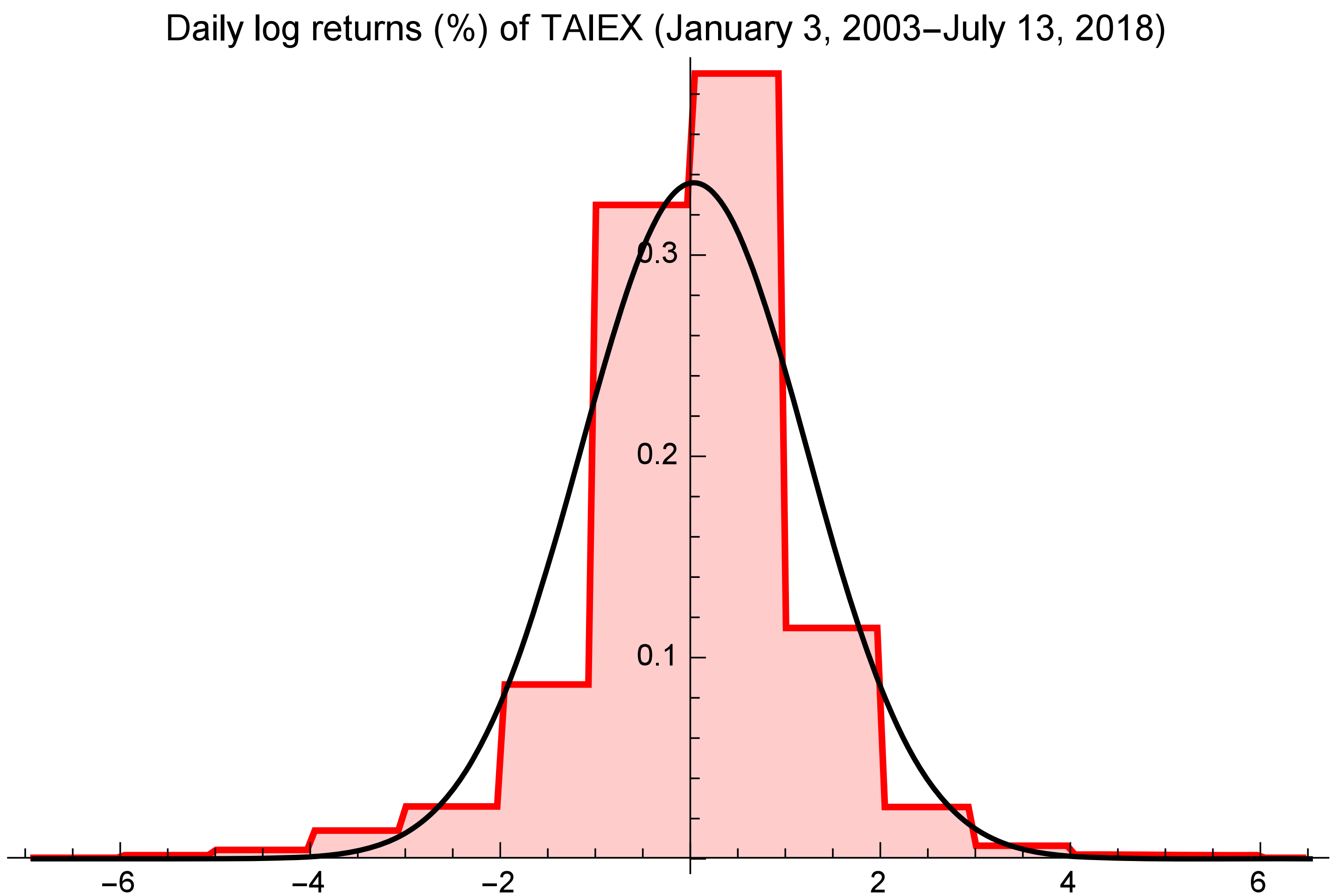
where $\Delta t \stackrel{\Delta}{=} \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.

- Other than the change in Eq. (42), binomial tree algorithms stay the same *as if there were no dividends*.

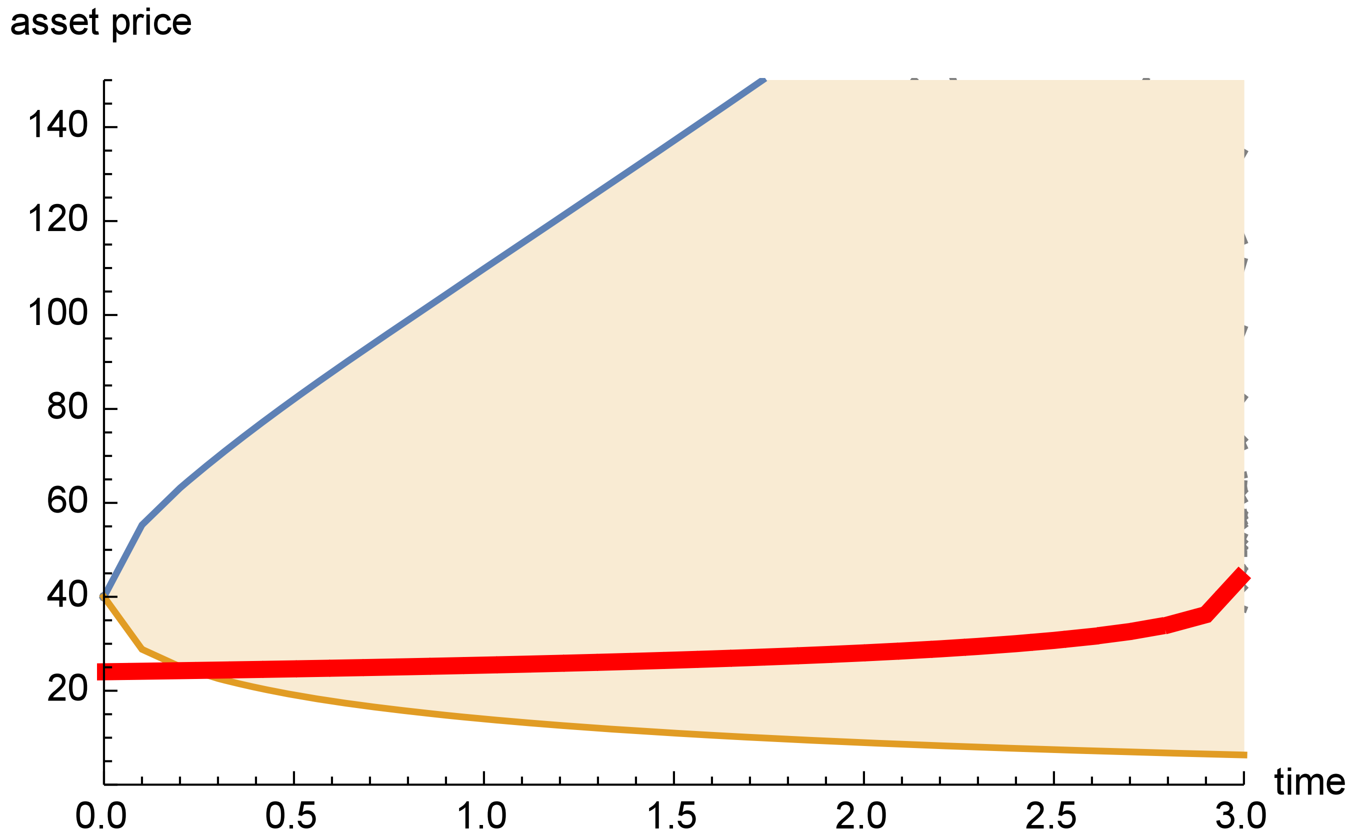
Distribution of Logarithmic Returns of TAIEX



Exercise Boundaries of American Options (in the Continuous-Time Model)[a]

- The exercise boundary is a nondecreasing function of t for American puts (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.

[a]See Section 9.7 of the textbook for the tree analog.



asset price
140
120
100
80
60
40
20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
time