Binomial Distribution

- Denote the binomial distribution with parameters $n$ and $p$ by
  
  $$b(j; n, p) \triangleq \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n-j)!} p^j (1 - p)^{n-j}.$$ 

- $n! = 1 \times 2 \times \cdots \times n$.
- Convention: $0! = 1$.

- Suppose you flip a coin $n$ times with $p$ being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting $j$ heads.
The Binomial Option Pricing Formula

- The stock prices at time $n$ are
  \[ Su^n, Su^{n-1}d, \ldots, Sd^n. \]

- Let $a$ be the minimum number of upward price moves for the call to finish in the money.

- So $a$ is the smallest nonnegative integer $j$ such that
  \[ Su^j d^{n-j} \geq X, \]
  or, equivalently,
  \[ a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \]
The Binomial Option Pricing Formula (concluded)

- Hence,

\[
C = \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( S^j d^{n-j} - X \right) \frac{R^n}{R^n} \tag{36}
\]

\[
= S \sum_{j=a}^{n} \binom{n}{j} (pu)^j \left[ (1 - p) d \right]^{n-j} \frac{R^n}{R^n} \]

\[- \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \]

\[
= S \sum_{j=a}^{n} b(j; n, pu/R) - X e^{-\hat{r}n} \sum_{j=a}^{n} b(j; n, p). \tag{37}
\]
Numerical Examples

• A non-dividend-paying stock is selling for $160.
• $u = 1.5$ and $d = 0.5$.
• $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
• Consider a European call on this stock with $X = 150$ and $n = 3$.
• The call value is $85.069$ by backward induction.
• Or, the PV of the expected payoff at expiration:

\[
\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
\]
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90 and invest $85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

\[ 90 - 85.069 = 4.931 \text{ dollars,} \]

is the arbitrage profit as we will see.
Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to $240.
- The new delta is 0.90625.
- Buy

\[ 0.90625 - 0.82031 = 0.08594 \]

more shares at the cost of \( 0.08594 \times 240 = 20.6256 \) dollars financed by borrowing.

- Debt now totals \( 20.6256 + 46.1806 \times 1.2 = 76.04232 \) dollars.
Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

\[ 0.90625 \times 240 - 76.04232 = 141.45768. \]

- It matches the corresponding call value!
Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to 
  \[ 76.04232 \times 1.2 - 78.75 = 12.5 \]
  dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

• The stock price moves to $180.

• The call we wrote finishes in the money.

• For a loss of $180 − 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.

• Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.

• It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

\[ 0.25 \times 60 = 15 \]

dollars.
- Use it to repay the debt of \( 12.5 \times 1.2 = 15 \) dollars.
Applications besides Exploiting Arbitrage Opportunities\textsuperscript{a}

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with $85.069.

- Hedge the options we issued.
  - Use $85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.\textsuperscript{b}

- \ldots

- Without hedge, one may end up forking out $390 in the worst case!\textsuperscript{c}

\textsuperscript{a}Thanks to a lively class discussion on March 16, 2011.
\textsuperscript{b}Hedging and replication are mirror images.
\textsuperscript{c}Thanks to a lively class discussion on March 16, 2016.
Binomial Tree Algorithms for European Options

• The BOPM implies the binomial tree algorithm that applies backward induction.

• The total running time is \( O(n^2) \) because there are \( \sim n^2/2 \) nodes.

• The memory requirement is \( O(n^2) \).
  
  – Can be easily reduced to \( O(n) \) by reusing space.\(^a\)

• To price European puts, simply replace the payoff.

\(^a\)But watch out for the proper updating of array entries.
Further Time Improvement for Calls

All zeros

0

X

©2019 Prof. Yuh-Dauh Lyuu, National Taiwan University
Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p)^j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[j]$:
- It runs in $O(n)$ steps.

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a}$;
2: for $j = a + 1, a + 2, \ldots, n$ do
3: $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j)$;
4: end for
Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (36) on p. 261 is trivial to compute.

- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.

- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.

- The above technique cannot be applied to American options because of early exercise.

- So binomial tree algorithms for American options usually run in $O(n^2)$ time.
Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As $n$ increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.\(^a\)
- Need to calibrate the BOPM’s parameters $u$, $d$, and $R$ to make it converge to the continuous-time model.
- We now skim through the proof.

\(^a\)Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!
Toward the Black-Scholes Formula (continued)

• Let $\tau$ denote the time to expiration of the option measured in years.

• Let $r$ be the continuously compounded annual rate.

• With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.

• Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, $\hat{r} = r\tau/n$.
  – Each period is $\tau/n$ years long.
  – The period gross return $R = e^{\hat{r}}$.

• Let
  \[\hat{\mu} \triangleq \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right]\]
  denote the expected value of the continuously compounded rate of return per period of the BOPM.

• Let
  \[\hat{\sigma}^2 \triangleq \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]\]
  denote the variance of that return.
Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that

\[
\hat{\mu} = q \ln(u/d) + \ln d,
\]
\[
\hat{\sigma}^2 = q(1 - q) \ln^2(u/d).
\]

- Assume the stock’s true continuously compounded rate of return over \( \tau \) years has mean \( \mu \tau \) and variance \( \sigma^2 \tau \).

- Call \( \sigma \) the stock’s (annualized) volatility.

\(^a\)The Bernoulli distribution.
Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

\[ n \hat{\mu} = n \left[ q \ln(u/d) + \ln d \right] \to \mu \tau, \quad \text{(38)} \]
\[ n \hat{\sigma}^2 = n q (1 - q) \ln^2(u/d) \to \sigma^2 \tau. \quad \text{(39)} \]

- We need one more condition to have a solution for \( u, d, q \).
Toward the Black-Scholes Formula (continued)

• Impose

\[ ud = 1. \]

  – It makes nodes at the same horizontal level of the tree have identical price (review p. 273).
  – Other choices are possible (see text).

• Exact solutions for \( u, d, q \) are feasible if Eqs. (38)–(39) are replaced by equations: 3 equations for 3 variables.\(^a\)

\(^a\)Chance (2008).
Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\frac{\tau}{n}}}, \quad d = e^{-\sigma \sqrt{\frac{\tau}{n}}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{\tau}{n}}. \quad (40) \]

- With Eqs. (40), it can be checked that

\[
\begin{align*}
n \hat{\mu} & = \mu \tau, \\
n \hat{\sigma}^2 & = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \frac{\tau}{n} \right] \sigma^2 \tau \rightarrow \sigma^2 \tau.
\end{align*}
\]
Toward the Black-Scholes Formula (continued)

- The choices (40) result in the CRR binomial model.\textsuperscript{a}
- With the above choice, even if $u$ and $d$ are not calibrated, the mean is still matched!\textsuperscript{b}

\textsuperscript{a}Cox, Ross, & Rubinstein (1979).
\textsuperscript{b}Recall Eq. (33) on p. 245. So $u$ and $d$ are related to volatility.
Toward the Black-Scholes Formula (continued)

• The no-arbitrage inequalities \( d < R < u \) may not hold under Eqs. (40) on p. 284 or Eq. (32) on p. 244.
  – If this happens, the probabilities lie outside \([0, 1]\).\(^a\)

• The problem disappears when \( n \) satisfies

\[
e^{\sigma \sqrt{\tau/n}} > e^{r \tau/n},
\]

i.e., when \( n > r^2 \tau / \sigma^2 \) (check it).
  – So it goes away if \( n \) is large enough.
  – Other solutions will be presented later.

\(^a\)Many papers and programs forget to check this condition!
Toward the Black-Scholes Formula (continued)

- The central limit theorem says \( \ln(S_\tau/S) \) converges to \( N(\mu\tau, \sigma^2\tau) \).\(^a\)

- So \( \ln S_\tau \) approaches \( N(\mu\tau + \ln S, \sigma^2\tau) \).

- Conclusion: \( S_\tau \) has a lognormal distribution in the limit.

\(^a\)The normal distribution with mean \( \mu\tau \) and variance \( \sigma^2\tau \).
Toward the Black-Scholes Formula (continued)

Lemma 9 The continuously compounded rate of return
\( \ln(S_\tau/S) \) approaches the normal distribution with mean
\( (r - \sigma^2/2) \tau \) and variance \( \sigma^2 \tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[
  p \overset{\Delta}{=} (e^{r\tau/n} - d)/(u - d).
  \]

- Let \( n \to \infty \).\(^a\)

- Then \( \mu = r - \sigma^2/2 \).

\(^a\)See Lemma 9.3.3 of the textbook.
Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is

\[ Se^{r\tau}. \]

- The stock’s expected annual rate of return is thus the riskless rate \( r \).

\( ^a \)By Lemma 9 (p. 289) and Eq. (28) on p. 175.

\( ^b \)In the sense of \( (1/\tau) \ln E[S_{\tau}/S] \) (arithmetic average rate of return) not \( (1/\tau)E[\ln(S_{\tau}/S)] \) (geometric average rate of return). In the latter case, it would be \( r - \sigma^2/2 \) by Lemma 9.
Theorem 10 (The Black-Scholes Formula)

\[
C = SN(x) - X e^{-r\tau} N(x - \sigma \sqrt{\tau}), \\
P = X e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - SN(-x),
\]

where

\[
x = \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
\]

---

\(^a\)On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!
Toward the Black-Scholes Formula (concluded)

• See Eq. (37) on p. 261 for the meaning of $x$.

• See Exercise 13.2.12 of the textbook for an interpretation of the probability measure associated with $N(x)$ and $N(-x)$. 
BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.
- Binomial tree algorithms take 6 inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.
- The connections are

\[
\begin{align*}
  u & = e^{\sigma \sqrt{\tau/n}}, \\
  d & = e^{-\sigma \sqrt{\tau/n}}, \\
  \hat{r} & = r\tau/n.
\end{align*}
\]
- $S = 100$, $X = 100$ (left), and $X = 95$ (right).
BOPM and Black-Scholes Model (concluded)

• The binomial tree algorithms converge reasonably fast.
• The error is $O(1/n)$.
• Oscillations are inherent, however.
• Oscillations can be dealt with by the judicious choices of $u$ and $d$.

\[\text{aL. Chang & Palmer (2007).}\]
\[\text{bSee Exercise 9.3.8 of the textbook.}\]
Implied Volatility

• Volatility is the sole parameter not directly observable.

• The Black-Scholes formula can be used to compute the market’s opinion of the volatility.\(^{a}\)
  
  – Solve for \(\sigma\) given the option price, \(S\), \(X\), \(\tau\), and \(r\) with numerical methods.
  
  – How about American options?

\(^{a}\)Implied volatility is hard to compute when \(\tau\) is small (why?).
Implied Volatility (concluded)

- Implied volatility is the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\(^a\)

- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

\(^a\)Rebonato (2004).
Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.

- A typical pattern is a “smile” in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

- Other patterns have also been observed.
The underlying Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) closed at 8132. Plot supplied by Mr. Lok, U Hou (D99922028) on December 6, 2017.
Solutions to the Smile

• To address this issue, volatilities are often combined to produce a composite implied volatility.

• This practice is not sound theoretically.

• The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

• So?
Binomial Tree Algorithms for American Puts

• Early exercise has to be considered.

• The binomial tree algorithm starts with the terminal payoffs

\[ \max(0, X - Su^j d^{m-j}) \]

and applies backward induction.

• At each intermediate node, it compares the payoff if exercised and the continuation value.

• It keeps the larger one.
Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.
Time-Dependent Instantaneous Volatility

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of $\sigma$.

- In the limit, the variance of $\ln(S_\tau/S)$ is
  \[ \int_0^\tau \sigma^2(t) \, dt \]
  rather than $\sigma^2 \tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be
  \[
  \sqrt{\frac{\int_0^\tau \sigma^2(t) \, dt}{\tau}}.
  \]

---

\textsuperscript{a}Merton (1973).
Time-Dependent Instantaneous Volatility (concluded)

- There is no guarantee that the implied volatility is constant.

- For the binomial model, $u$ and $d$ depend on time:

\[
\begin{align*}
    u &= e^{\sigma(t) \sqrt{\tau/n}}, \\
    d &= e^{-\sigma(t) \sqrt{\tau/n}}.
\end{align*}
\]

- How to make the binomial tree combine?\(^a\)

\(^a\)Amin (1991); C. I. Chen (R98922127) (2011).
Volatility (1990–2016)\textsuperscript{a}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{CBOE S&P 500 Volatility Index}
\end{figure}

\textsuperscript{a}Supplied by Mr. Lok, U Hou (D99922028) on July 17, 2017.
Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.

- The riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.

- In other words,

$$ r = \frac{\sum_{i=0}^{n-1} r_i \tau}{\tau}, $$

where $r_i$ is the continuously compounded short rate measured in periods for period $i$.a

- Will the binomial tree fail to combine?

---

*aThat is, one-period forward rate.
Trading Days and Calendar Days

- Interest accrues based on the calendar day.

- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.$^a$

- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?$^b$

---

$^a$Fama (1965); K. French (1980); K. French & Roll (1986).

$^b$Recall p. 158 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the annualized volatility of stock price *one year from now.*

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with\(^a\)

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$\(^a\)

\(^a\)D. French (1984).
Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?\(^a\)

\(^a\)Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.
Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.
Known Dividends

- Constant dividends introduce complications.
- Use $D$ to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $S_u - D$ and $S_d - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(S_u - D)_u$, $(S_u - D)_d$, $(S_d - D)_u$, $(S_d - D)_d$.
  - The binomial tree no longer combines.
\[(Su - D)u\]
\[Su - D\]
\[Su - D \downarrow \]
\[S\]
\[(Su - D)d\]
\[Sd - D\]
\[(Sd - D)u\]
\[(Sd - D)d\]
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.\(^a\)

- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.

- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, \(\sigma\) is the volatility of the process followed by the risky component.

- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

\(^a\)Roll (1977).
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 312 (Step 1)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[S - D/R\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 312 (Step 2)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R) + D/R = S\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 312

- The trees are different.
- The stock prices at maturity are also different.
  - \((Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d\) (p. 312).
  - \((S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2\) (ad hoc).
- Note that, as \(d < R < u\),
  \[
  (Su - D)u > (S - D/R)u^2, \\
  (Sd - D)d < (S - D/R)d^2,
  \]

\(^{a}\text{Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.}\)
The Ad-Hoc Approximation vs. P. 312 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach\(^a\)

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 766ff).
- Other approaches include adjusting \(\sigma\) and approximating the known dividend with a dividend yield.\(^b\)


\(^b\)Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).
Continuous Dividend Yields

• Dividends are paid continuously.
  – Approximates a broad-based stock market portfolio.

• The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  – A stock that grows from $S$ to $S_\tau$ with a continuous dividend yield of $q$ would grow from $S$ to $S_\tau e^{q\tau}$ without the dividends.

• A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, only the distribution of $S_\tau$ matters.
Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$.

\[
C = Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}),
\]

\[
P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x),
\]

where

\[
x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]

- Formulas (41) and (41′) remain valid as long as the dividend yield is predictable.

\[\text{aMerton (1973).}\]
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace $u$ with $ue^{-q\Delta t}$ and $d$ with $de^{-q\Delta t}$, where $\Delta t = \Delta \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, $p$ should use the original $u$ and $d!$\footnote{Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.}
Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$e^{(r-q)\Delta t} - d$$

$$u - d$$

(42)

where $\Delta t \triangleq \tau/n$.

- The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The $u$ and $d$ remain unchanged.

- Other than the change in Eq. (42), binomial tree algorithms stay the same as if there were no dividends.
Exercise Boundaries of American Options (in the Continuous-Time Model)\(^a\)

- The exercise boundary is a nondecreasing function of \( t \) for American puts (see the plot next page).

- The exercise boundary is a nonincreasing function of \( t \) for American calls.

\(^a\)See Section 9.7 of the textbook for the tree analog.
Risk Reversals\textsuperscript{a}

- From formulas (41) and (41\textsuperscript{′}) on p. 321, one can verify that $C = P$ when

$$X = S e^{(r-q)\tau}.$$  

- A risk reversal consists of a short out-of-the-money put and a long out-of-the-money call with the same maturity.\textsuperscript{b}

- Furthermore, the portfolio has zero value.

- A short risk reversal position is also called a collar.\textsuperscript{c}

\textsuperscript{a}Neftci (2008).

\textsuperscript{b}Thus their strike prices must be distinct.

\textsuperscript{c}Bennett (2014).
Sensitivity Analysis of Options
Cleopatra’s nose, had it been shorter, the whole face of the world would have been changed.
— Blaise Pascal (1623–1662)
Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.

- Let $x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}$ (recall p. 291).

- Recall that
  
  $$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

  the density function of standard normal distribution.
Delta

• Defined as

\[ \Delta \triangleq \frac{\partial f}{\partial S}. \]

  – \( f \) is the price of the derivative.
  – \( S \) is the price of the underlying asset.

• The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.\(^a\)

• The delta used in the BOPM (p. 238) is the discrete analog.

• The delta of a long stock is apparently 1.

\(^a\)Elementary calculus.
Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals

\[
\frac{\partial C}{\partial S} = N(x) > 0.
\]

- The delta of a European put equals

\[
\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.
\]

- So the deltas of a call and an otherwise identical put cancel each other when \( N(x) = 1/2 \), i.e., when\(^a\)

\[
X = S e^{(r+\sigma^2/2) \tau}.
\]  \hspace{1cm} (43)

\(^a\)The straddle (p. 206) \( C + P \) then has zero delta!
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curves: out-of-the-money calls or in-the-money puts.
Delta (continued)

• Suppose the stock pays a continuous dividend yield of $q$.

• Let

$$ x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} $$

(recall p. 321).

• Then

$$ \frac{\partial C}{\partial S} = e^{-q\tau} N(x) > 0, $$

$$ \frac{\partial P}{\partial S} = -e^{-q\tau} N(-x) < 0. $$
Delta (continued)

- Consider an $X_1$-strike call and an $X_2$-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let
  \[ x_i \equiv \frac{\ln(S/X_i) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \]  
  \[ (45) \]
- Then their deltas sum to zero when $x_1 = -x_2$.\(^a\)
- That implies
  \[ \frac{S}{X_1} = \frac{X_2}{S} e^{-(2r-2q+\sigma^2) \tau}. \]  
  \[ (46) \]

\(^a\)The strangle (p. 208) $C + P$ then has zero delta!
Delta (concluded)

• Suppose we demand \( X_1 = X_2 = X \) and have a straddle.

• Then

\[
X = S e^{(r-q+\sigma^2/2) \tau}
\]

leads to a straddle with zero delta.

– This generalizes Eq. (43) on p. 332.

• When \( C(X_1) \)'s delta and \( P(X_2) \)'s delta sum to zero, does the portfolio \( C(X_1) - P(X_2) \) have zero value?

• In general, no.
Delta Neutrality

• A position with a total delta equal to 0 is delta-neutral.
  – A delta-neutral portfolio is immune to *small* price changes in the underlying asset.

• Creating one serves for hedging purposes.
  – A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  – Short $\Delta$ shares of stock to hedge a long call.
  – Long $\Delta$ shares of stock to hedge a short call.

• In general, hedge a position in a security with delta $\Delta_1$ by shorting $\Delta_1/\Delta_2$ units of a security with delta $\Delta_2$. 
Theta (Time Decay)

- Defined as the rate of change of a security’s value with respect to time, or $\Theta \triangleq -\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial t}$.

- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$  

  - The call loses value with the passage of time.

- For a European put,

$$\Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$  

  - Can be negative or positive.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curve: out-of-the-money call or in-the-money put.
Theta (concluded)

- Suppose the stock pays a continuous dividend yield of $q$.
- Define $x$ as in Eq. (44) on p. 334.
- For a European call, add an extra term to the earlier formula for the theta:
  \[
  \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} - r X e^{-r\tau} N(x - \sigma \sqrt{\tau}) + q S e^{-q\tau} N(x).
  \]
- For a European put, add an extra term to the earlier formula for the theta:
  \[
  \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} + r X e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - q S e^{-q\tau} N(-x).
  \]
Gamma

• Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \triangleq \frac{\partial^2 \Pi}{\partial S^2}$.

• Measures how sensitive delta is to changes in the price of the underlying asset.

• In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.

• Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.

• The gamma of a European call or put on a non-dividend-paying stock is

$$\frac{N'(x)}{(S\sigma\sqrt{\tau})} > 0.$$
Dotted lines: in-the-money call or out-of-the-money put.
Solid lines: at-the-money option.
Dashed lines: out-of-the-money call or in-the-money put.
Vega\(a\) (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security’s value with respect to the volatility of the underlying asset

\[ \Lambda \triangleq \frac{\partial f}{\partial \sigma}. \]

- Volatility often changes over time.

- A security with a high vega is very sensitive to small changes or estimation error in volatility.

- The vega of a European call or put on a non-dividend-paying stock is \(S\sqrt{\tau} N'(x) > 0.\)
  - So higher volatility always increases the option value.

\(a\)Vega is not Greek.
Vega (continued)

• Note that

\[ \Lambda = \tau \sigma S^2 \Gamma. \]

• If the stock pays a continuous dividend yield of \( q \), then

\[ \Lambda = Se^{-q\tau} \sqrt{\tau} N'(x), \]

where \( x \) is defined in Eq. (44) on p. 334.

• Vega is maximized when \( x = 0 \), i.e., when

\[ S = X e^{-(r-q+\sigma^2/2) \tau}. \]

• Vega declines very fast as \( S \) moves away from that peak.

\(^{\text{a}}\text{Reiss & Wystup (2001).}\)
Vega (continued)

• Now consider a portfolio consisting of an $X_1$-strike call $C$ and a short $X_2$-strike put $P$, $X_1 \geq X_2$.

• The options’ vegas cancel out when

$$x_1 = -x_2,$$

where $x_i$ are defined in Eq. (45) on p. 335.

• This leads to Eq. (46) on p. 335.
  
  – Recall the same condition led to zero delta for the strangle $C + P$ (p. 335).
Vega (concluded)

• Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 296 for the Black-Scholes model).

• The Black-Scholes formula (p. 291) implies

$$C \to S,$$

$$P \to X e^{-r\tau},$$

as $\sigma \to \infty$.

• These boundary conditions may be handy for certain numerical methods.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curve: out-of-the-money call or in-the-money put.
Variance Vega\textsuperscript{a}

- Defined as the rate of change of a security’s value with respect to the variance (square of volatility) of the underlying asset

\[ \text{variance vega} = \frac{\Delta}{\partial\sigma^2}. \]

- Note that it is not \( \frac{\partial^2 f}{\partial\sigma^2} \)!

- It is easy to verify that

\[ \text{variance vega} = \frac{\Lambda}{2\sigma}. \]

\textsuperscript{a}Demeterfi, Derman, Kamal, & Zou (1999).
Volga (Vomma, Volatility Gamma, Vega Convexity)

- Defined as the rate of change of a security’s vega with respect to the volatility of the underlying asset

\[ \text{volga} \triangleq \frac{\partial \Lambda}{\partial \sigma} = \frac{\partial^2 f}{\partial \sigma^2}. \]

- It can be shown that

\[
\text{volga} = \Lambda \frac{x(x - \sigma \sqrt{\tau})}{\sigma} = \Lambda \left[ \frac{\ln^2(S/X)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4} \right],
\]

where \(x\) is defined in Eq. (44) on p. 334.\(^a\)

\(^a\)Derman & M. B. Miller (2016).
Volga (concluded)

- Volga is zero when $S = X e^{\pm \sigma^2 \tau / 2}$.

- For typical values of $\sigma$ and $\tau$, volga is positive except where $S \approx X$.

- Volga can be used to measure the 4th moment of the underlying asset and the smile of implied volatility at the same maturity.$^a$

---

$^a$Bennett (2014).
Rho

• Defined as the rate of change in its value with respect to interest rates

\[ \rho \triangleq \frac{\partial f}{\partial r}. \]

• The rho of a European call on a non-dividend-paying stock is

\[ X\tau e^{-r\tau}N(x - \sigma\sqrt{\tau}) > 0. \]

• The rho of a European put on a non-dividend-paying stock is

\[ -X\tau e^{-r\tau}N(-x + \sigma\sqrt{\tau}) < 0. \]
Dotted curves: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curves: out-of-the-money call or in-the-money put.