Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.

- Let

\[ \omega \triangleq \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \]

(12)
Full Price (continued)

\[ C(1 - \omega) \]

coupon payment date

(1 - \omega)

\omega

coupon payment date
Full Price (concluded)

- The price is now calculated by

\[
PV = C \left(1 + \frac{r}{m}\right)\omega + C\left(1 + \frac{r}{m}\right)\omega + 1 + \cdots
\]

\[
= \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega + i}} + \frac{F}{(1 + \frac{r}{m})^{\omega + n-1}}. \tag{13}
\]
Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price plus the accrued interest (AI).
- The accrued interest equals

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega). \]
Accrued Interest (concluded)

- The yield to maturity is the $r$ satisfying Eq. (13) on p. 82 when PV is the invoice price:

$$\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.$$

©2019 Prof. Yuh-Dauh Lyuu, National Taiwan University
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.
Example ("30/360") (concluded)

• The yield to maturity is 3%.

• This can be verified by Eq. (13) on p. 82 with
  - \( \omega = 60/180 \),
  - \( n = 4 \),
  - \( m = 2 \),
  - \( F = 100 \),
  - \( C = 5 \),
  - \( PV = 111.2891 + 3.3333 \),
  - \( r = 0.03 \).
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.\textsuperscript{a}
  
  – The short reason: Exponential growth to $C$ is replaced by linear growth, hence “overpaying.”

\textsuperscript{a}See Exercise 3.5.6 of the textbook for proof.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[- \frac{\partial P}{\partial y} \cdot \frac{P}{P} . \]  

(14)
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[
- \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

– $F$ is the par value.
– $C$ is the coupon payment per period.
– Formula can be simplified a bit with $C = Fc/m$.

For the above bond,

\[
- \frac{\partial P}{\partial y} > 0.
\]
Macaulay Duration$^a$

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.
- The weights are the cash flows’ PVs divided by the asset’s price.
- Formally,

\[
\text{MD} \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} i.
\]

- The Macaulay duration, in periods, is equal to

\[
\text{MD} = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}.
\]  

\(^a\)Macaulay (1938).
MD of Bonds

- The MD of a level-coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right].
\]  

(16)

- It can be simplified to

\[
MD = \frac{c(1+y)}{cy} \left[ (1+y)^{n-1} + \frac{ny(y-c)}{(1+y)^n - 1} \right],
\]

where \(c\) is the period coupon rate.

- The MD of a zero-coupon bond equals \(n\), its term to maturity.

- The MD of a level-coupon bond is less than \(n\).
Remarks

• Equations (15) on p. 93 and (16) on p. 94 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility\(^a\) may decrease.

\(^a\)As originally defined in Eq. (14) on p. 91.
How Not To Think about MD

• The MD has its origin in measuring the length of time a bond investment is outstanding.

• But it should be seen mainly as measuring price volatility.

• Duration of a security can be longer than its maturity or negative!

• Neither makes sense under the maturity interpretation.

• Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

- For the MD to be year-based, modify Eq. (16) on p. 94 to
  \[
  \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \left(1 + \frac{y}{k}\right)^i + \frac{n}{k} \left(1 + \frac{y}{k}\right)^n \right],
  \]
  where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

- Equation (15) on p. 93 also becomes
  \[
  \text{MD} = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
  \]

- By definition, \( \text{MD} \) (in years) = \( \frac{\text{MD} \text{ (in periods)}}{k} \).
Modified Duration

• Modified duration is defined as

\[
\text{modified duration} \triangleq - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \tag{17}
\]

• By the Taylor expansion,

percent price change \approx -\text{modified duration} \times \text{yield change}.
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\%.\]
Modified Duration of a Portfolio

• The modified duration of a portfolio equals

\[ \sum \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[ \frac{P_\text{−} - P_\text{+}}{P_0(y_\text{+} - y_\text{−})}. \]

- \( P_\text{−} \) is the price if the yield is decreased by \( \Delta y \).
- \( P_\text{+} \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.

- An alternative is to use

\[ \frac{P_0 - P_+}{P_0 \Delta y}. \]

  - More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^a\)

• The percentage price change expressed in percentage terms is then approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\(^a\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

  \[ \text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}. \]

- The approximate dollar price change is

  \[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

- One can hedge a bond portfolio with a dollar duration \(D\) by bonds with a dollar duration \(-D\).
Convexity

- Convexity is defined as

\[ \text{convexity (in periods)} = \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

- The convexity of a level-coupon bond is positive (prove it!).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

- So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

- The approximation $\Delta P/P \approx - \text{duration} \times \text{yield change}$ works for small yield changes.

- For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$ 

- Recall the figure on p. 107.
The Practices

• Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

− Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

• $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

• The effective convexity is defined as

$$\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},$$

- $P_-$ is the price if the yield is decreased by $\Delta y$.
- $P_+$ is the price if the yield is increased by $\Delta y$.
- $P_0$ is the initial price, $y$ is the initial yield.
- $\Delta y$ is small.

• Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

• Numerically, choosing the right $\Delta y$ is a delicate matter.
Approximate \( \frac{d^2 f(x)^2}{dx^2} \) at \( x = 1 \), Where \( f(x) = x^2 \)

- The difference of \( ((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2 \) and 2:

\[
\text{Error}
\]

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 838ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots the yields to maturity of coupon bonds against maturity.
- A par yield curve is constructed from bonds trading near par.
Yield Curve as of July 24, 2015

Yield (%) vs. Year

©2019 Prof. Yuh-Dauh Lyuu, National Taiwan University
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.

- The PV of one dollar $i$ periods from now is by definition

$$[1 + S(i)]^{-i}.$$

  - It is the price of an $i$-period zero-coupon bond.$^a$

- The one-period spot rate is called the short rate.

- Spot rate curve: Plot of spot rates against maturity:

$$S(1), S(2), \ldots, S(n).$$

---

$^a$Recall Eq. (9) on p. 66.
Problems with the PV Formula

• In the bond price formula (3) on p. 39,

\[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]

every cash flow is discounted at the same yield \( y \).

• Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

\[ PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \]

\[ PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn’t they be discounted at the *same* rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

\[ C_1, C_2, \ldots, C_n \]
Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \]  \hspace{1cm} (18)

- This pricing method incorporates information from the term structure.

- It discounts each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow \( C_1, C_2, \ldots, C_n \) should have a market price of

\[
P = \sum_{i=1}^{n} C_i d(i).
\]

– Above, \( d(i) \triangleq [1 + S(i)]^{-i}, \ i = 1, 2, \ldots, n \), are called the discount factors.

– \( d(i) \) is the PV of one dollar \( i \) periods from now.

– This formula—now just a definition—will be justified on p. 215.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

- Then $S(n)$ can be computed from Eq. (18) on p. 124, repeated below,

$$ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. $$

- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\(^a\)

\(^a\)Often without economic justifications.
Which One?
Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.
Static Spread

• Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

• But as risk must be compensated, in reality \( P < P^* \).

• The static spread is the amount \( s \) by which the spot rate curve has to shift in parallel to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

• Unlike the yield spread, the static spread incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

- \( y_k \): yield to maturity for the \( k \)-period coupon bond.
- \( S(k) \geq y_k \) if \( y_1 < y_2 < \cdots \) (yield curve is normal).
- \( S(k) \leq y_k \) if \( y_1 > y_2 > \cdots \) (yield curve is inverted).
- \( S(k) \geq y_k \) if \( S(1) < S(2) < \cdots \) (spot rate curve is normal).
- \( S(k) \leq y_k \) if \( S(1) > S(2) > \cdots \) (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

• The yield curve contains information regarding future interest rates currently “expected” by the market.

• Invest $1 for \( j \) periods to end up with \( [1 + S(j)]^j \) dollars at time \( j \).
  – The maturity strategy.

• Invest $1 in bonds for \( i \) periods and at time \( i \) invest the proceeds in bonds for another \( j - i \) periods where \( j > i \).

• Will have \( [1 + S(i)]^i[1 + S(i, j)]^{j-i} \) dollars at time \( j \).
  – \( S(i, j) \): \((j - i)\)-period spot rate \( i \) periods from now.
  – The rollover strategy.
Forward Rates (concluded)

• When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$  \hspace{1cm} (19)

we will end up with $[1 + S(j)]^j$ dollars again.

• By definition, $f(0, j) = S(j)$.

• $f(i, j)$ is called the (implied) forward rates.
  – More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

- $f(0, 1)$
- $f(1, 2)$
- $f(2, 3)$
- $f(3, 4)$

Time 0

- $S(1)$
- $S(2)$
- $S(3)$
- $S(4)$
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, if realized, will equate the two investment strategies.

• $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

• When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i). \]

• When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i). \]
(a) forward rate curve  
spot rate curve  
yield curve

(b) yield curve  
spot rate curve  
forward rate curve
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n-1, n)]. \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

\[
S(n) = \left\{ [1 + S(1)][1 + f(1, 2)] \right\} \cdot \cdots \cdot [1 + f(n - 1, n)]^{1/n} - 1. \tag{20}
\]

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

- Other equivalencies can be derived similarly, such as

\[
f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \tag{21}
\]
Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ $m$-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time $n$ there will be a cash inflow of $1$.
- At time $m$ there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the interest rate between times $n$ and $m$ equals $f(n, m)$ by Eq. (19) on p. 135.

\[
(1 + S(m))^m / (1 + S(n))^n
\]
Forward Loans

• We had generated the cash flow of a type of forward contract called the forward loan.

• Agreed upon today, it enables one to
  – Borrow money at time $n$ in the future, and
  – Repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

• Can the spot rate curve be an arbitrary curve?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.
Synthetic Bonds

- We had seen that

\[
\text{forward loan} = n\text{-period zero} - \left[1 + f(n, m)\right]^{m-n} \times m\text{-period zero}.
\]

- Thus

\[
n\text{-period zero} = \text{forward loan} + \left[1 + f(n, m)\right]^{m-n} \times m\text{-period zero}.
\]

- We have created a *synthetic* zero-coupon bond with forward loans and other zero-coupon bonds.

- Very useful if the \(n\)-period zero is unavailable or illiquid.
Spot and Forward Rates under Continuous Compounding

- The pricing formula:

\[ P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}. \]

- The market discount function:

\[ d(n) = e^{-nS(n)}. \]

- The spot rate is an arithmetic average of forward rates,

\[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}. \]

\[ ^a \text{Compare it with Eq. (20) on p. 141.} \]
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

\[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]  \hspace{1cm} (22)

- Compare the above formula with Eq. (19) on p. 135.

- The one-period forward rate:\(^a\)

\[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]

\(^a\)Compare it with Eq. (21) on p. 141.
Spot and Forward Rates under Continuous Compounding (concluded)

• Now,

\[ f(T) \overset{\Delta}{=} \lim_{\Delta T \to 0} f(T, T + \Delta T) \]

\[ = S(T) + T \frac{\partial S}{\partial T}. \]

• So \( f(T) > S(T) \) if and only if \( \frac{\partial S}{\partial T} > 0 \) (i.e., a normal spot rate curve).

• If \( S(T) < -T(\frac{\partial S}{\partial T}) \), then \( f(T) < 0 \).\(^a\)

\(^a\)Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
Unbiased Expectations Theory

• Forward rate equals the average future spot rate,

\[ f(a, b) = E[S(a, b)]. \]  \hspace{1cm} (23)

• It does not imply that the forward rate is an accurate predictor for the future spot rate.

• It implies the maturity strategy and the rollover strategy produce the same result at the horizon on average.
Unbiased Expectations Theory and Spot Rate Curve

• It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  \[- f(j, j + 1) > S(j + 1) \text{ if and only if } S(j + 1) > S(j) \text{ from Eq. (19) on p. 135.} \]

  \[- \text{So } E[S(j, j + 1)] > S(j + 1) > \cdots > S(1) \text{ if and only if } S(j + 1) > \cdots > S(1). \]

• Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
A “Bad” Expectations Theory

- The expected returns\(^a\) on all possible riskless bond strategies are equal for all holding periods.

- So

\[
(1 + S(2))^2 = (1 + S(1)) \, E[1 + S(1, 2)] \tag{24}
\]

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

\[
\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]

\(^a\)More precisely, the one-plus returns.
A “Bad” Expectations Theory (continued)

• Now consider two one-period strategies.
  – Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  – The expected return is
    \[
    E[(1 + S(1, 2))^{-1}] / (1 + S(2))^{-2}.
    \]
    – Strategy two buys a one-period bond with a return of \(1 + S(1)\).
A “Bad” Expectations Theory (continued)

• The theory says the returns are equal:
\[
\frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
\]

• Combine this with Eq. (24) on p. 151 to obtain
\[
E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.
\]
A “Bad” Expectations Theory (concluded)

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that $E[g(X)] > g(E[X])$
    for any nondegenerate random variable $X$ and
    strictly convex function $g$ (i.e., $g''(x) > 0$).
  – Use
    \[ g(x) \triangleq (1 + x)^{-1} \]
    to prove our point.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1, n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \quad \text{for all} \ n > 1.
\]

- This theory is the basis of many interest rate models.
Duration in Practice

• To handle more general types of spot rate curve changes, define a vector \([c_1, c_2, \ldots, c_n]\) that characterizes the perceived type of change.
  
  – Parallel shift: \([1, 1, \ldots, 1]\).
  
  – Twist: \([1, 1, \ldots, 1, -1, \ldots, -1]\),
    \([1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]\), etc.
  
  – \ldots

• At least one \(c_i\) should be 1 as the reference point.
Duration in Practice (concluded)

• Let

\[ P(y) \overset{\Delta}{=} \sum_i C_i/(1 + S(i) + yc_i)^i \]

be the price associated with the cash flow \( C_1, C_2, \ldots \).

• Define duration as

\[ \left. -\frac{\partial P(y)/P(0)}{\partial y} \right|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y) - P(-\Delta y)}{2P(0)\Delta y}. \]

• Modified duration equals the above when

\[ [c_1, c_2, \ldots, c_n] = [1, 1, \ldots, 1], \quad S(1) = S(2) = \cdots = S(n). \]
Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days \((T + 2, \text{ etc.})\).
- Shall we treat a year as 1 year whether it has 365 or 366 days?
Fundamental Statistical Concepts
There are three kinds of lies: lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli (1804–1881)

If 50 million people believe a foolish thing, it’s still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy, but a million deaths are a statistic.
— Josef Stalin (1879–1953)
Moments

• The variance of a random variable $X$ is defined as

$$\operatorname{Var}[X] \triangleq E \left[ (X - E[X])^2 \right].$$

• The covariance between random variables $X$ and $Y$ is

$$\operatorname{Cov}[X,Y] \triangleq E \left[ (X - \mu_X)(Y - \mu_Y) \right],$$

where $\mu_X$ and $\mu_Y$ are the means of $X$ and $Y$, respectively.

• Random variables $X$ and $Y$ are uncorrelated if

$$\operatorname{Cov}[X,Y] = 0.$$
Correlation

• The standard deviation of $X$ is the square root of the variance,

$$\sigma_X \triangleq \sqrt{\text{Var}[X]}.$$

• The correlation (or correlation coefficient) between $X$ and $Y$ is

$$\rho_{X,Y} \triangleq \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.\(^a\)

\(^a\)Wilmott (2009), “the correlations between financial quantities are notoriously unstable.” It may even break down “at high-frequency time intervals” (Budish, Cramton, & Shim, 2015).
Variance of Sum

• Variance of a weighted sum of random variables equals

\[
\text{Var} \left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \text{Cov}[X_i, X_j].
\]

• It becomes

\[
\sum_{i=1}^{n} a_i^2 \text{Var}[X_i]
\]

when \( X_i \) are uncorrelated.
Conditional Expectation

• “$X | I$” denotes $X$ conditional on the information set $I$.
• The information set can be another random variable’s value or the past values of $X$, say.
• The conditional expectation

$$E[X | I]$$

is the expected value of $X$ conditional on $I$; it is a random variable.
• The law of iterated conditional expectations\(^a\) says

$$E[X] = E[E[X | I]].$$

\(^a\)Or the tower law.
Conditional Expectation (concluded)

• If $I_2$ contains at least as much information as $I_1$, then

\[ E[X \mid I_1] = E[E[X \mid I_2] \mid I_1]. \]  

\[ (25) \]

– $I_1$ contains price information up to time $t_1$, and $I_2$ contains price information up to a later time $t_2 > t_1$.

• In general,

\[ I_1 \subseteq I_2 \subseteq \cdots \]

means the players never forget past data so the information sets are increasing over time.\(^a\)

---

\(^a\)Hirsa & Neftci (2013). This idea is used in sigma fields and filtration.
The Normal Distribution

- A random variable $X$ has the normal distribution with mean $\mu$ and variance $\sigma^2$ if its probability density function is
  \[
  \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
  \]
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function
  \[
  \text{Prob}[X \leq z] = N(z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.
  \]