An Example

- Consider a two-tranche sequential-pay CMO backed by \$1,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The cash flow pattern for each tranche with zero prepayment and zero servicing fee is shown on p. 1216.
- The calculation can be carried out first for the Total columns, which make up the amortization schedule.
- Then the cash flow is allocated.
- Tranche A is retired after 4 months, and tranche B starts principal paydown at the end of month 4.

CMO Cash Flows without Prepayments

	Interest			Principal			Remaining principal		
Month	A	B	Total	A	B	Total	A	B	Tot
							500,000	500,000	1,000,
1	5,000	5,000	10,000	162,548	0	162,548	337,452	500,000	837,
2	3,375	5,000	8,375	164,173	0	164,173	173,279	500,000	673,
3	1,733	5,000	6,733	165,815	0	165,815	7,464	500,000	507,
4	75	5,000	5,075	7,464	160,009	167,473	0	339,991	339,
5	0	3,400	3,400	0	169,148	169,148	0	170,843	170,
6	0	1,708	1,708	0	170,843	170,843	0	0	
Total	10,183	25,108	35,291	500,000	500,000	1,000,000			

The total monthly payment is \$172,548. Month-i numbers reflect the ith monthly payment.

Another Example

- When prepayments are present, the calculation is only slightly more complex.
- Suppose the single monthly mortality (SMM) per month is 5%.
- This means the prepayment amount is 5% of the *remaining* principal.
- The remaining principal at month i *after* prepayment then equals the scheduled remaining principal as computed by Eq. (7) on p. 53 times $(0.95)^i$.
- This done for all the months, the interest payment at any month is the remaining principal of the previous month times 1%.

Another Example (continued)

- The prepayment amount equals the remaining principal times $0.05/0.95$.
  - The division by 0.95 yields the remaining principal *before* prepayment.
- Page 1219 tabulates the cash flows of the same two-tranche CMO under 5% SMM.

Another Example (continued)

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Total
							500,000	500,000	1,000,0
1	5,000	5,000	10,000	204,421	0	204,421	295,579	500,000	795,5
2	2,956	5,000	7,956	187,946	0	187,946	107,633	500,000	607,6
3	1,076	5,000	6,076	107,633	64,915	172,548	0	435,085	435,0
4	0	4,351	4,351	0	158,163	158,163	0	276,922	276,9
5	0	2,769	2,769	0	144,730	144,730	0	132,192	132,1
6	0	1,322	1,322	0	132,192	132,192	0	0	
Total	9,032	23,442	32,474	500,000	500,000	1,000,000			

Month-i numbers reflect the ith monthly payment.

Another Example (continued)

- For instance, the total principal payment at month one, \$204,421, can be verified as follows.
- The *scheduled* remaining principal is \$837,452 from p. 1216.
- The remaining principal is hence

$$837452 \times 0.95 = 795579.$$

- That makes the total principal payment

$$1000000 - 795579 = 204421.$$

Another Example (concluded)

- As tranche A's remaining principal is \$500,000, all 204,421 dollars go to tranche A.
  - Incidentally, the prepayment is

$$837452 \times 5\% = 41873.$$

- Tranche A is retired after 3 months, and tranche B starts principal paydown at the end of month 3.

Stripped Mortgage-Backed Securities (SMBSs)[a]

- The principal and interest are divided between the PO strip and the IO strip.
- In the scenarios on p. 1215 and p. 1217:
  - The IO strip receives all the interest payments under the Interest/Total column.
  - The PO strip receives all the principal payments under the Principal/Total column.

[a]They were created in February 1987 when Fannie Mae issued its Trust 1 stripped MBS.

Stripped Mortgage-Backed Securities (SMBSs) (concluded)

- These new instruments allow investors to better exploit anticipated changes in interest rates.[a]
- The collateral for an SMBS is a pass-through.
- CMOs and SMBSs are usually called derivative MBSs.

[a]See p. 357 of the textbook.

Prepayments

- The prepayment option sets MBSs apart from other fixed-income securities.
- The exercise of options on most securities is expected to be "rational."
- This kind of "rationality" is weakened when it comes to the homeowner's decision to prepay.
- For example, even when the prevailing mortgage rate exceeds the mortgage's loan rate, some loans are prepaid.

Prepayment Risk

- Prepayment risk is the uncertainty in the amount and timing of the principal prepayments in the pool of mortgages that collateralize the security.
- This risk can be divided into contraction risk and extension risk.
- Contraction risk is the risk of having to reinvest the prepayments at a rate lower than the coupon rate when interest rates decline.
- Extension risk is due to the slowdown of prepayments when interest rates climb, making the investor earn the security's lower coupon rate rather than the market's higher rate.

Prepayment Risk (concluded)

- Prepayments can be in whole or in part.
  - The former is called liquidation.
  - The latter is called curtailment.
- The holder of a pass-through security is exposed to the total prepayment risk associated with the underlying pool of mortgage loans.
- CMOs are designed to alter the distribution of that risk among investors.

Other Risks

- Investors in mortgages are exposed to at least three other risks.
  - Interest rate risk is inherent in any fixed-income security.
  - Credit risk is the risk of loss from default.
    - For privately insured mortgage, the risk is related to the credit rating of the company that insures the mortgage.
  - Liquidity risk is the risk of loss if the investment must be sold quickly.

Prepayment: Causes

Prepayments have at least five components.

Home sale ("housing turnover"). The sale of a home generally leads to the prepayment of mortgage because of the full payment of the remaining principal.

Refinancing. Mortgagors can refinance their home mortgage at a lower mortgage rate. This is the most volatile component of prepayment and constitutes the bulk of it when prepayments are extremely high.

Default. Caused by foreclosure and subsequent liquidation of a mortgage. Relatively minor in most cases.[a]

---

[a]Except the subprime mortgage crisis that started the financial crisis of 2007–2008 (Lewis, 2011; Zuckerman, 2010).

Prepayment: Causes (concluded)

Curtailment. As the extra payment above the scheduled payment, curtailment applies to the principal and shortens the maturity of fixed-rate loans. Its contribution to prepayments is minor.

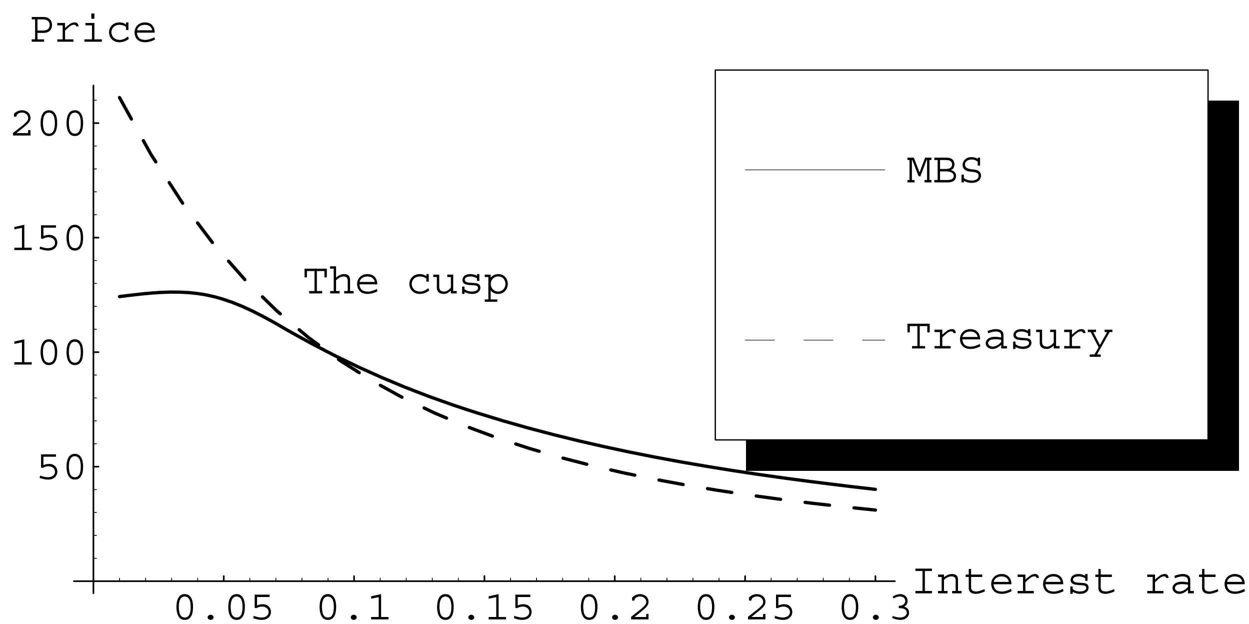
Full payoff (liquidation). There is evidence that many mortgagors pay off their mortgage completely when it is very seasoned and the remaining balance is small. Full payoff can also be due to natural disasters.

Prepayment: Characteristics

- Prepayments usually increase as the mortgage ages — first at an increasing rate and then at a decreasing rate.
- They are higher in the spring and summer and lower in the fall and winter.
- They vary by the geographic locations of the underlying properties.
- They increase when interest rates drop but with a time lag.

Prepayment: Characteristics (continued)

- If prepayments were higher for some time because of high refinancing rates, they tend to slow down.
  - Perhaps, homeowners who do not prepay when rates have been low for a prolonged time tend never to prepay.
- Plot on p. 1232 illustrates the typical price/yield curves of the Treasury and pass-through.



Price compression occurs as yields fall through a threshold. The cusp represents that point.

Prepayment: Characteristics (concluded)

- As yields fall and the pass-through's price moves above a certain price, it flattens and then follows a downward slope.
- This phenomenon is called the price compression of premium-priced MBSs.
- It demonstrates the negative convexity of such securities.

Analysis of Mortgage-Backed Securities

Oh, well, if you cannot measure,

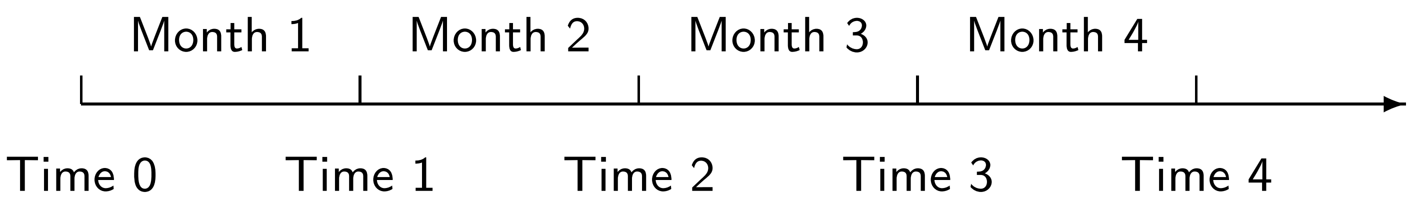
measure anyhow.

— Frank H. Knight (1885–1972)

Uniqueness of MBS

- Compared with other fixed-income securities, the MBS is unique in two respects.
- Its cash flow consists of principal and interest (P&I).
- The cash flow may vary because of prepayments in the underlying mortgages.

Time Line

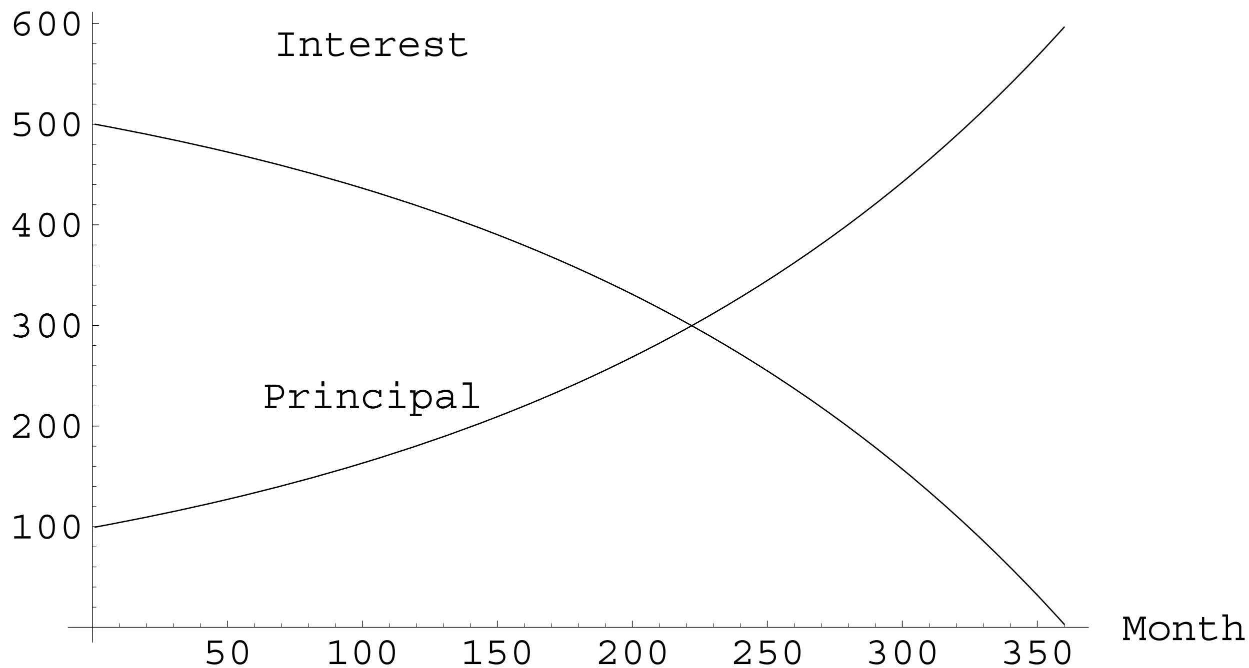


- Mortgage payments are paid in arrears.
- A payment for month i occurs at time i, that is, end of month i.
- The end of a month will be identified with the beginning of the coming month.

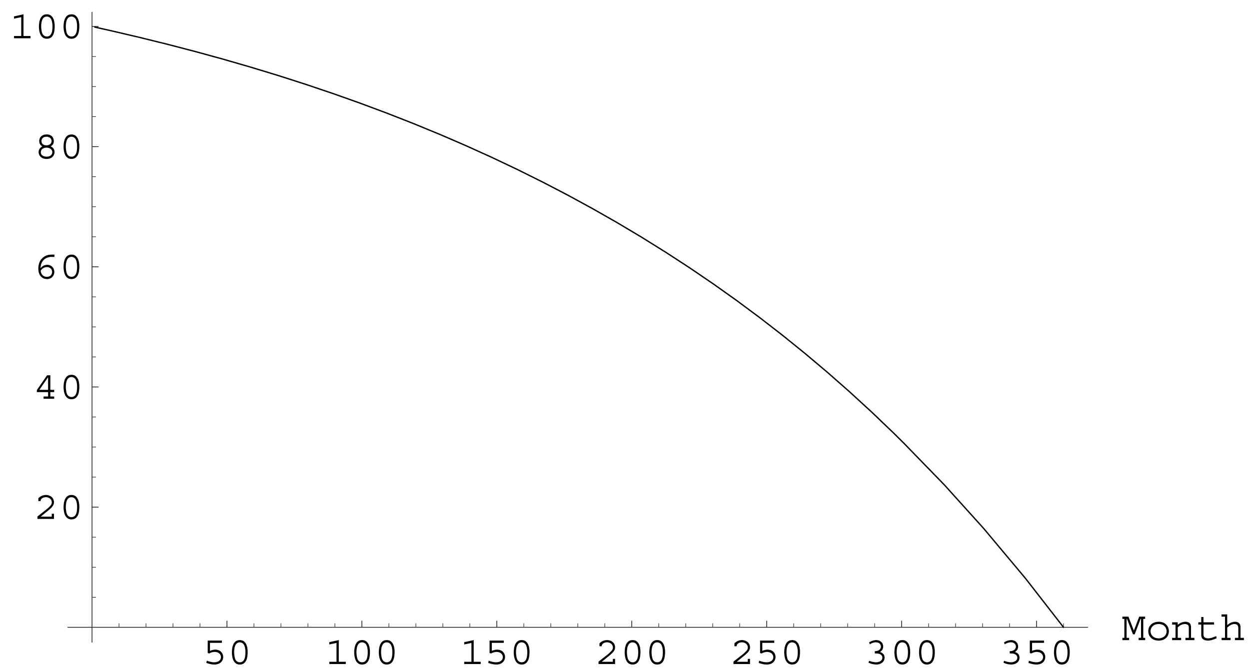
Cash Flow Analysis

- A traditional mortgage has a fixed term, a fixed interest rate, and a fixed monthly payment.
- Page 1239 illustrates the scheduled P&I for a 30-year, 6% mortgage with an initial balance of \$100,000.
- Page 1240 depicts how the remaining principal balance decreases over time.

Scheduled Principal and Interest Payments



Scheduled Remaining Principal Balances



Cash Flow Analysis (continued)

- In the early years, the P&I consists mostly of interest.
- Then it gradually shifts toward principal payment with the passage of time.
- However, the total P&I payment remains the same each month, hence the term *level* pay.
- In the absence of prepayments and servicing fees, identical characteristics hold for the pool's P&I payments.

Cash Flow Analysis (continued)

- From Eq. (7) on p. 53 the remaining principal balance after the kth payment is

$$C \, \frac{1-(1+r/m)^{-n+k}}{r/m}. \tag{165}$$

  - C is the scheduled P&I payment of an n-month mortgage making m payments per year.[a]
  - r is the annual mortgage rate.

- For mortgages, $m = 12$.

[a] See Eq. (6) on p. 45.

Cash Flow Analysis (continued)

- The scheduled remaining principal balance after k payments can be expressed as a portion of the original principal balance:

$$\begin{aligned}\mathrm{Bal}_k &\stackrel{\Delta}{=} 1 - \frac{(1+r/m)^k - 1}{(1+r/m)^n - 1} \\ &= \frac{(1+r/m)^n - (1+r/m)^k}{(1+r/m)^n - 1}. \end{aligned} \tag{166}$$

- This equation can be verified by dividing Eq. (165) (p. 1242) by the same equation with $k = 0$.

Cash Flow Analysis (continued)

- The remaining principal balance after k payments is

$$\mathrm{RB}_k \stackrel{\Delta}{=} \mathcal{O} \times \mathrm{Bal}_k,$$

  where $\mathcal{O}$ will denote the original principal balance.

- The term factor denotes the portion of the remaining principal balance to its original principal balance.
- So Bal_k is the monthly factor when there are no prepayments.
- It is also known as the amortization factor.

Cash Flow Analysis (concluded)

- The idea of factor can also be applied to a mortgage pool.
- Then it is called the paydown factor on the pool or simply the pool factor.

An Example

- The remaining balance of a 15-year mortgage with a 9% mortgage rate after 54 months is

$$
\begin{aligned}
& \mathcal{O} \times \frac{(1+(0.09/12))^{180}-(1+(0.09/12))^{54}}{(1+(0.09/12))^{180}-1} \\
= \quad & \mathcal{O} \times 0.824866.
\end{aligned}
$$

- In other words, roughly 82.49% of the original loan amount remains after 54 months.

P&I Analysis

- By the amortization principle, the tth interest payment equals

$$I_t \stackrel{\Delta}{=} \mathrm{RB}_{t-1} \times \frac{r}{m} = \mathcal{O} \times \frac{r}{m} \times \frac{(1+r/m)^n - (1+r/m)^{t-1}}{(1+r/m)^n - 1}.$$

- The principal part of the tth monthly payment is

$$\begin{aligned} P_t &\stackrel{\Delta}{=} \mathrm{RB}_{t-1} - \mathrm{RB}_t \\ &= \mathcal{O} \times \frac{(r/m)(1+r/m)^{t-1}}{(1+r/m)^n - 1}. \end{aligned} \tag{167}$$

P&I Analysis (concluded)

- The scheduled P&I payment at month t, or $P_t + I_t$, is

$$
\begin{aligned}
& (\mathrm{RB}_{t-1} - \mathrm{RB}_t) + \mathrm{RB}_{t-1} \times \frac{r}{m} \\
= \quad & \mathcal{O} \times \left[\frac{(r/m)(1 + r/m)^n}{(1 + r/m)^n - 1} \right], \qquad (168)
\end{aligned}
$$

  indeed a level pay independent of t.[a]

- The term within the brackets is called the payment factor or annuity factor.

- It is the monthly payment for each dollar of mortgage.

[a]This formula is identical to Eq. (6) on p. 45.

An Example

- The mortgage on pp. 47ff has a monthly payment of

$$250000 \times \frac{(0.08/12) \times (1 + (0.08/12))^{180}}{(1 + (0.08/12))^{180} - 1} = 2389.13$$

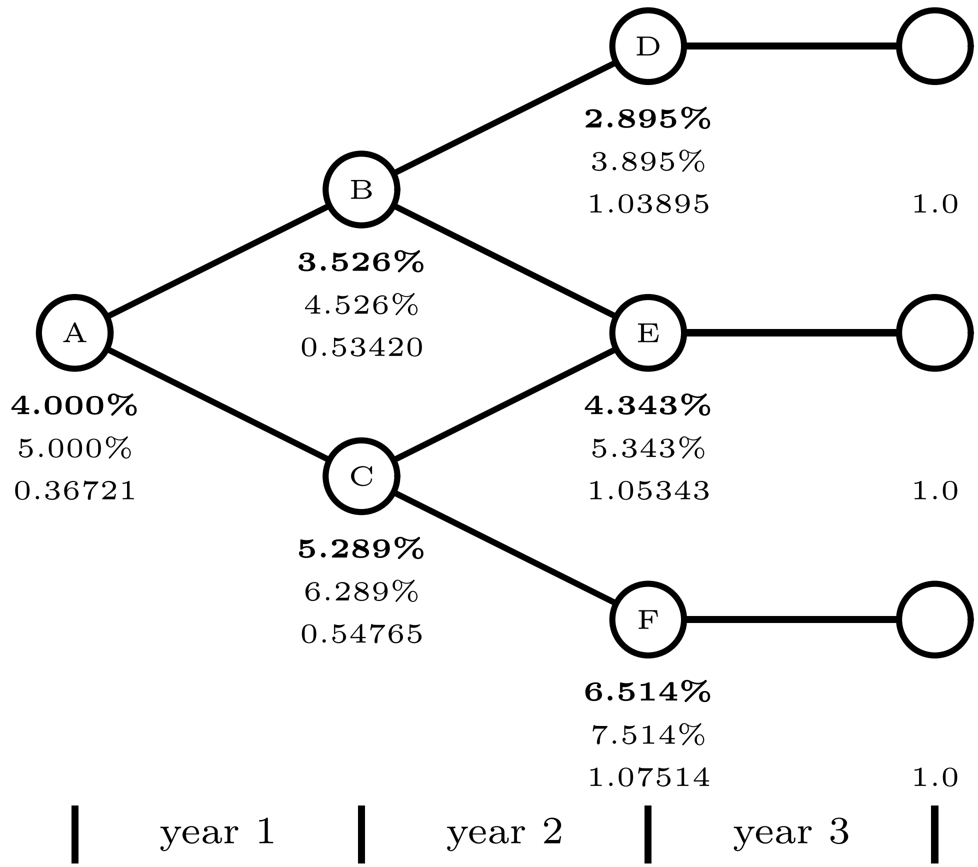
  by Eq. (168) on p. 1248.

- This number agrees with the number derived earlier on p. 47.

Pricing Adjustable-Rate Mortgages

- We turn to ARM pricing as an interesting application of derivatives pricing and the analysis above.
- Consider a 3-year ARM with an interest rate that is 1% above the 1-year T-bill rate at the beginning of the year.
- This 1% is called the margin.
- Assume this ARM carries annual, not monthly, payments.
- The T-bill rates follow the binomial process, in boldface, on p. 1251,[a] and the risk-neutral probability is 0.5.

[a] They are from p. 1012.



Stacked at each node are the T-bill rate, the mortgage rate, and the payment factor for a mortgage initiated at that node and ending at year 3 (based on the mortgage rate at the same node).

Pricing Adjustable-Rate Mortgages (continued)

- How much is the ARM worth to the issuer?
- Each new coupon rate at the reset date determines the level mortgage payment for the months until the next reset date as if the ARM were a fixed-rate loan with the new coupon rate and a maturity equal to that of the ARM.
- For example, for the interest rate tree on p. 1251, the scenario $A \to B \to E$ will leave our three-year ARM with a remaining principal at the end of the second year different from that under the scenario $A \to C \to E$.

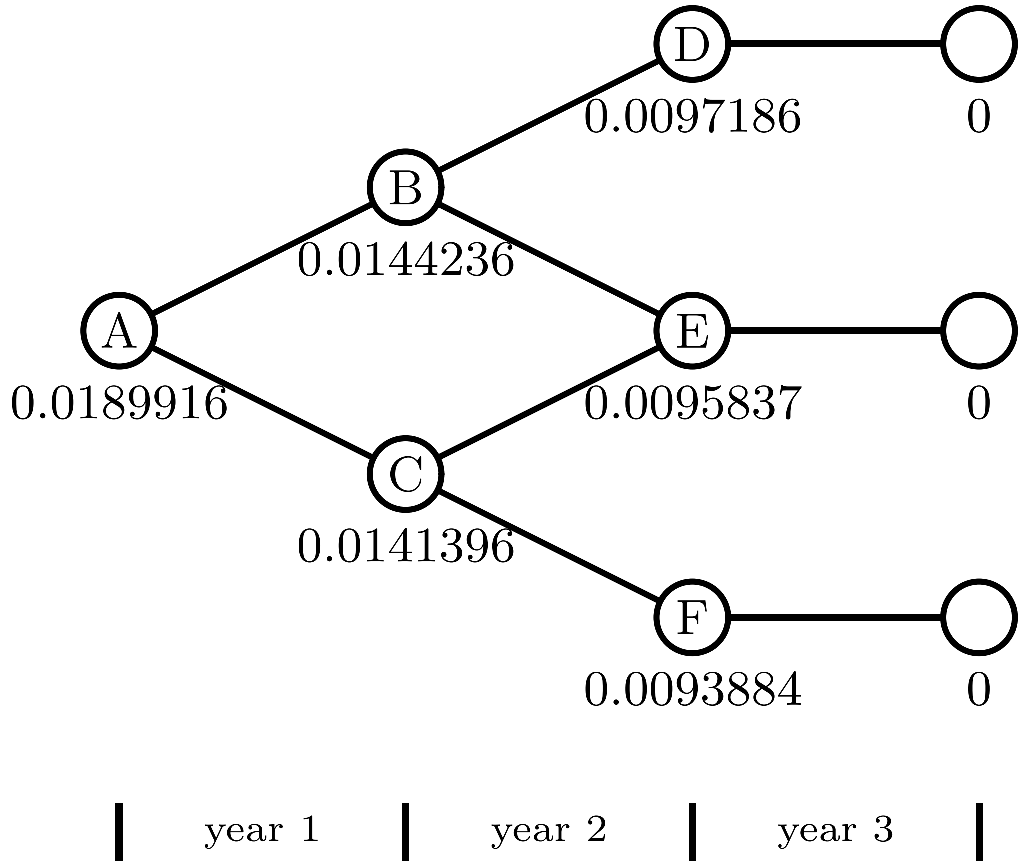
Pricing Adjustable-Rate Mortgages (continued)

- This path dependency calls for care in algorithmic design to avoid exponential complexity.
- Attach to each node on the binomial tree the annual payment per \$1 of principal for a mortgage initiated at that node and ending at year 3.
  - In other words, the payment factor.
- At node B, for example, the annual payment factor can be calculated by Eq. (168) on p. 1248 with $r = 0.04526$, $m = 1$, and $n = 2$ as

$$\frac{0.04526 \times (1.04526)^2}{(1.04526)^2 - 1} = 0.53420.$$

Pricing Adjustable-Rate Mortgages (continued)

- The payment factors for other nodes on p. 1251 are calculated in the same manner.
- We now apply backward induction to price the ARM (see p. 1255).
- At each node on the tree, the *net* value of an ARM of value \$1 initiated at that node and ending at the end of the third year is calculated.
- For example, the value is zero at terminal nodes since the ARM is immediately repaid.


D
0.0097186
0
B
0.0144236
A
0.0189916
E
0.0095837
0
C
0.0141396
F
0.0093884
0
year 1
year 2
year 3

Pricing Adjustable-Rate Mortgages (continued)

- At node D, the value is

$$\frac{1.03895}{1.02895} - 1 = 0.0097186,$$

which is simply the net present value of the payment 1.03895 next year.

  – Recall that the issuer makes a loan of \$1 at D.

- The values at nodes E and F can be computed similarly.

Pricing Adjustable-Rate Mortgages (continued)

- At node B, we first figure out the remaining principal balance after the payment one year hence as

$$1 - (0.53420 - 0.04526) = 0.51106,$$

because \$0.04526 of the payment of \$0.53426 constitutes the interest.

- The issuer will receive \$0.01 above the T-bill rate next year, and the value of the ARM is either \$0.0097186 or \$0.0095837 per \$1, each with probability 0.5.

Pricing Adjustable-Rate Mortgages (continued)

- The ARM's value at node B thus equals

$$\frac{0.51106 \times (0.0097186 + 0.0095837)/2 + 0.01}{1.03526} = 0.0144236.$$

- The values at nodes C and A can be calculated similarly as

$$\begin{aligned} & \frac{(1 - (0.54765 - 0.06289)) \times (0.0095837 + 0.0093884)/2 + 0.01}{1.05289} \\ = \quad & 0.0141396 \\ & \frac{(1 - (0.36721 - 0.05)) \times (0.0144236 + 0.0141396)/2 + 0.01}{1.04} \\ = \quad & 0.0189916, \end{aligned}$$

respectively.

Pricing Adjustable-Rate Mortgages (concluded)

- The value of the ARM to the issuer is hence \$0.0189916 per \$1 of loan amount.

- The above idea of scaling has wide applicability in pricing certain classes of path-dependent securities.[a]

[a] For example, newly issued lookback options in Programming Assignment 11.7.12(2) of the textbook.

More on ARMs

- ARMs are indexed to publicly available indices such as:
  - LIBOR.
  - The constant maturity Treasury rate (CMT).
  - The Cost of Funds Index (COFI).
- COFI is based on an average cost of funds.
- So it moves relatively sluggishly compared with LIBOR.
- Since 1990, the need for securitization gradually shift in LIBOR's favor, but it may change because of scandals.[a]

[a]Morgenson (2012). The LIBOR rate-fixing scandal broke in June 2012. A sample email on August 20, 2007: "ok, i will move the curve down 1bp maybe more if I can".

More on ARMs (continued)

- If the ARM coupon reflects fully and instantaneously current market rates, then the ARM security will be priced close to par and refinancings rarely occur.
- In reality, adjustments are imperfect in many ways.
- At the reset date, a margin is added to the benchmark index to determine the new coupon.

More on ARMs (concluded)

- ARMs often have periodic rate caps that limit the amount by which the coupon rate may increase or decrease at the reset date.
- They also have lifetime caps and floors.
- To attract borrowers, mortgage lenders usually offer a below-market initial rate (the “teaser” rate).
- The reset interval, the time period between adjustments in the ARM coupon rate, is often annual, which is not frequent enough.
- These terms are easy to incorporate into the pricing algorithm.

Expressing Prepayment Speeds

- The cash flow of a mortgage derivative is determined from that of the mortgage pool.
- The single most important factor complicating this endeavor is the unpredictability of prepayments.
- Recall that prepayment represents the principal payment made in excess of the scheduled principal amortization.

Expressing Prepayment Speeds (concluded)

- Compare the amortization factor Bal_t of the pool with the reported factor to determine if prepayments have occurred.
- The amount by which the reported factor is exceeded by the amortization factor is the prepayment amount.

Single Monthly Mortality

- A SMM of ω means $\omega\%$ of the scheduled remaining balance at the end of the month will prepay.[a]
- In other words, the SMM is the percentage of the remaining balance that prepays for the month.
- Suppose the remaining principal balance of an MBS at the beginning of a month is \$50,000, the SMM is 0.5%, and the scheduled principal payment is \$70.
- Then the prepayment for the month is

$$0.005 \times (50{,}000 - 70) \approx 250$$

dollars.

[a]Recall p. 1217.

Single Monthly Mortality (concluded)

- If the same monthly prepayment speed s is maintained since the issuance of the pool, the remaining principal balance at month i will be

$$\mathrm{RB}_i \times \left(1 - \frac{s}{100}\right)^i. \tag{169}$$

- It goes without saying that prepayment speeds must lie between 0% and 100%.

An Example

- Take the mortgage on p. 1246.
- Its amortization factor at the 54th month is 0.824866.
- If the actual factor is 0.8, then the (implied) SMM for the initial period of 54 months is

$$100 \times \left[1 - \left(\frac{0.8}{0.824866} \right)^{1/54} \right] = 0.0566677\%.$$

- In other words, roughly 0.057% of the remaining principal is prepaid per month.

Conditional Prepayment Rate

- The conditional prepayment rate (CPR) is the annualized equivalent of a SMM,

$$\mathrm{CPR} = 100 \times \left[1 - \left(1 - \frac{\mathrm{SMM}}{100} \right)^{12} \right].$$

- Conversely,

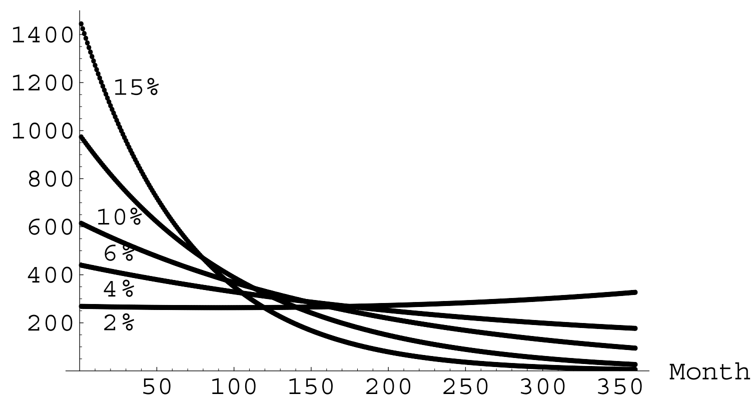
$$\mathrm{SMM} = 100 \times \left[1 - \left(1 - \frac{\mathrm{CPR}}{100} \right)^{1/12} \right].$$

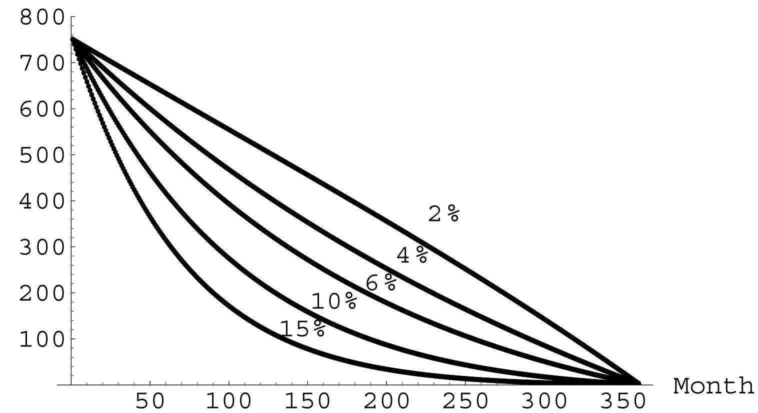
Conditional Prepayment Rate (concluded)

- For example, the SMM of 0.0566677 on p. 1267 is equivalent to a CPR of

$$100 \times \left[1 - \left(1 - \left(\frac{0.0566677}{100} \right)^{12} \right) \right] = 0.677897\%.$$

- Roughly 0.68% of the remaining principal is prepaid annually.
- The figures on 1270 plot the principal and interest cash flows under various prepayment speeds.
- With accelerated prepayments, the principal cash flow is shifted forward in time.





Principal (left) and interest (right) cash flows at various CPRs. The 6% mortgage has 30 years to maturity and an original loan amount of \$100,000.

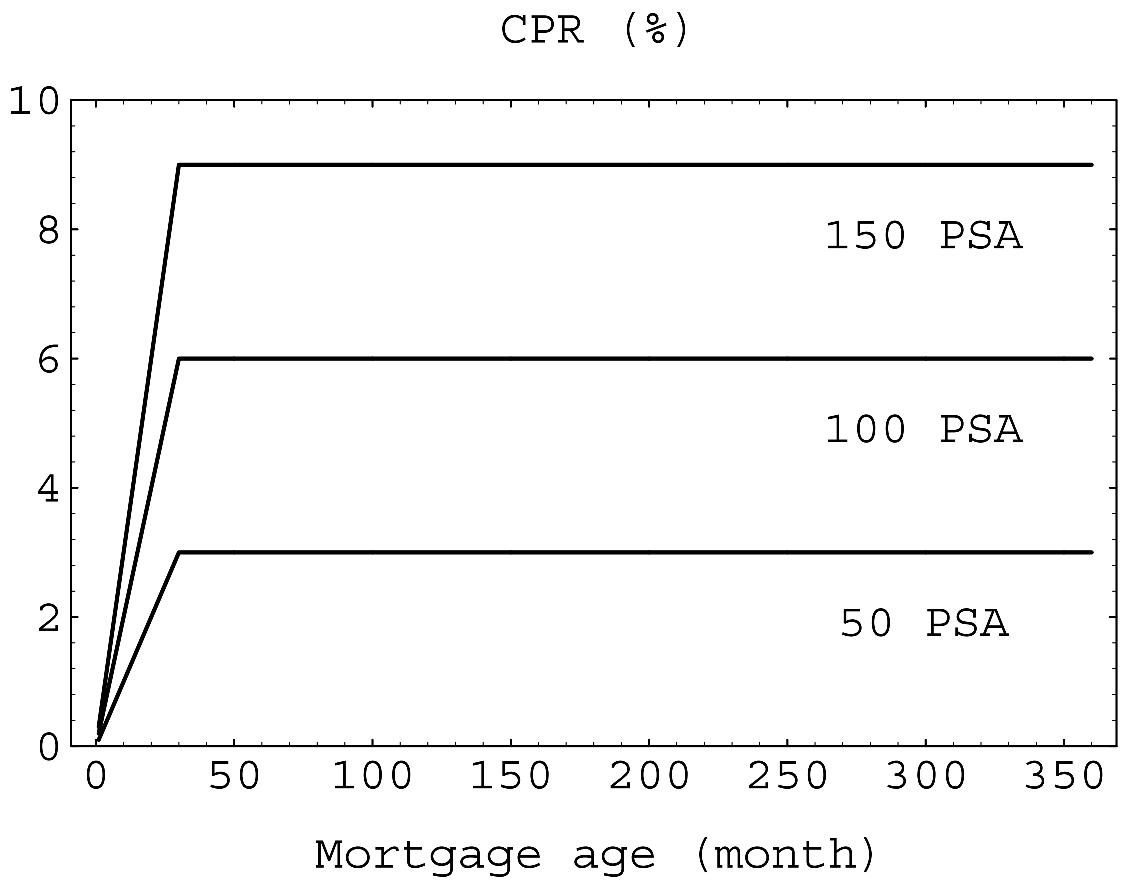
PSA

- In 1985 the Public Securities Association[a] (PSA) standardized a prepayment model.
- The PSA standard is expressed as a monthly series of CPRs.
  - It reflects the increase in CPR that occurs as the pool seasons.
- At the time the PSA proposed its standard, a seasoned 30-year GNMA's typical prepayment speed was $\sim 6\%$ CPR.

[a]It became the Bond Market Association in 1997, which merged with the Securities Industry Association (SIA) to form the Securities Industry and Financial Markets Association (SIFMA) in 2006.

PSA (continued)

- The PSA standard postulates the following prepayment speeds:
  - The CPR is 0.2% for the first month.
  - It increases thereafter by 0.2% per month until it reaches 6% per year for the 30th month.
  - It then stays at 6% for the remaining years.
- The PSA benchmark is also referred to as 100 PSA.
- Other speeds are expressed as some percentage of PSA.
  - 50 PSA means one-half the PSA CPRs.
  - 150 PSA means one-and-a-half the PSA CPRs.


CPR (%)
10
8
6
4
2
0
150 PSA
100 PSA
50 PSA
0
50
100
150
200
250
300
350
Mortgage age (month)

PSA (concluded)

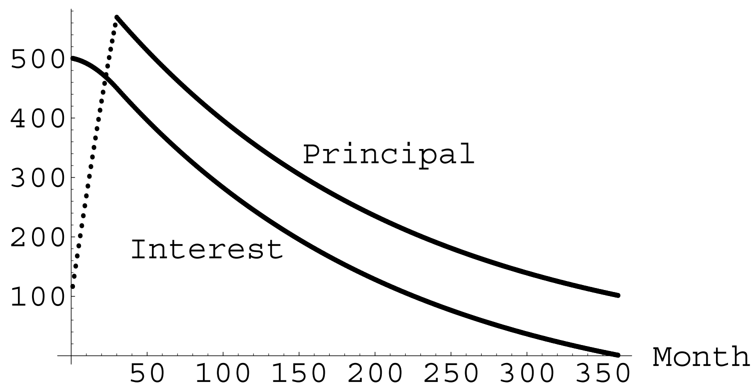
- Mathematically,

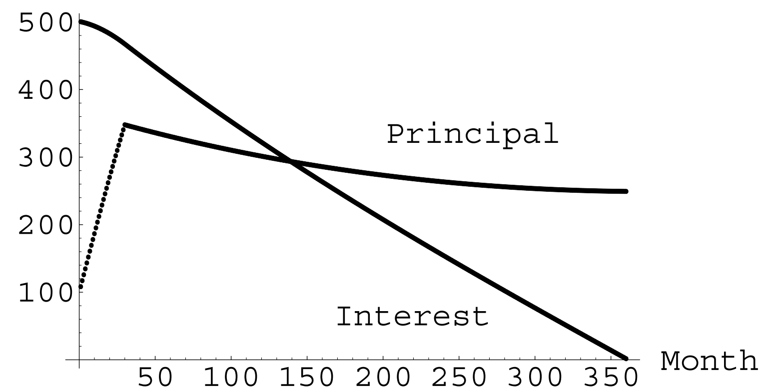
$$\text{CPR} = \begin{cases} 6\% \times \frac{\text{PSA}}{100} & \text{if the pool age exceeds 30 months} \\ \\ 0.2\% \times m \times \frac{\text{PSA}}{100} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

- Conversely,

$$\text{PSA} = \begin{cases} 100 \times \frac{\text{CPR}}{6} & \text{if the pool age exceeds 30 months} \\ \\ 100 \times \frac{\text{CPR}}{0.2 \times m} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

Cash Flows at 50 and 100 PSAs





The 6% mortgage has 30 years to maturity and an original loan amount of \$100,000. The 100 PSA scenario is on the left, and the 50 PSA is on the right.

Prepayment Vector

- The PSA tries to capture how prepayments vary with age.
- But it should be viewed as a market convention rather than a model.
- A vector of PSAs generated by a prepayment model should be used to describe the monthly prepayment speed through time.
- The monthly cash flows can be derived thereof.

Prepayment Vector (continued)

- Similarly, the CPR should be seen purely as a measure of speed rather than a model.
- If one treats a single CPR number as the true prepayment speed, that number will be called the constant prepayment rate.
- This simple model is inconsistent with the empirical fact that pools with new production loans typically prepay at a slower rate than seasoned pools.
- A vector of CPRs should be preferred.

Prepayment Vector (concluded)

- A CPR/SMM vector is easier to work with than a PSA vector because of the lack of dependence on the pool age.
- But they are all equivalent as a CPR vector can always be converted into an equivalent PSA vector and vice versa.

Cash Flow Generation

- Each cash flow is composed of the principal payment, the interest payment, and the principal prepayment.
- Let B_k denote the actual remaining principal balance at month k.
- Then the pool's actual remaining principal balance at time $i-1$ is B_{i-1}.

Cash Flow Generation (continued)

- The scheduled principal and interest payments at time i are

$$\overline{P_i} \triangleq B_{i-1}\left(\frac{\mathrm{Bal}_{i-1}-\mathrm{Bal}_i}{\mathrm{Bal}_{i-1}}\right) \tag{170}$$

$$= B_{i-1}\,\frac{r/m}{(1+r/m)^{n-i+1}-1} \tag{171}$$

$$\overline{I_i} \triangleq B_{i-1}\,\frac{r-\alpha}{m} \tag{172}$$

  - α is the servicing spread (or servicing fee rate).
  - It consists of the servicing fee for the servicer as well as the guarantee fee.

Cash Flow Generation (continued)

- The prepayment at time i is

$$\text{PP}_i = B_{i-1}\,\frac{\text{Bal}_i}{\text{Bal}_{i-1}} \times \text{SMM}_i.$$

  - SMM_i is the prepayment speed for month i.

- If the total principal payment from the pool is $\overline{P_i} + \text{PP}_i$, the remaining principal balance is

$$\begin{aligned} B_i &= B_{i-1} - \overline{P_i} - \text{PP}_i \\ &= B_{i-1}\left[1 - \left(\frac{\text{Bal}_{i-1} - \text{Bal}_i}{\text{Bal}_{i-1}}\right) - \frac{\text{Bal}_i}{\text{Bal}_{i-1}} \times \text{SMM}_i\right] \\ &= \frac{B_{i-1} \times \text{Bal}_i \times (1 - \text{SMM}_i)}{\text{Bal}_{i-1}}. \qquad (173) \end{aligned}$$

Cash Flow Generation (continued)

- Equation (173) can be applied iteratively to yield[a]

$$B_i = \mathrm{RB}_i \times \prod_{j=1}^{i} (1 - \mathrm{SMM}_j). \tag{174}$$

  - The above formula generalizes Eq. (169) on p. 1266.

- Define

$$b_i \stackrel{\Delta}{=} \prod_{j=1}^{i} (1 - \mathrm{SMM}_j).$$

[a] RB_i is defined on p. 1244.

Cash Flow Generation (continued)

- $I'_i \stackrel{\Delta}{=} \mathrm{RB}_{i-1} \times (r - \alpha)/m$ is the scheduled interest payment.
- The scheduled P&I is[a]

$$\overline{P_i} = b_{i-1} P_i \quad \text{and} \quad \overline{I_i} = b_{i-1} I'_i. \tag{175}$$

- The scheduled cash flow and the b_i determined by the prepayment vector are all that are needed to calculate the projected actual cash flows.

---

[a] P_i and I_i are defined on p. 1247.

Cash Flow Generation (concluded)

- If the servicing fees do not exist (that is, $\alpha = 0$), the projected monthly payment *before* prepayment at month i becomes

$$\overline{P_i} + \overline{I_i} = b_{i-1}(P_i + I_i) = b_{i-1}C. \tag{176}$$

  - C is the scheduled monthly payment on the original principal.[a]

- See Figure 29.10 in the text (p. 437) for a linear-time algorithm for generating the mortgage pool's cash flow.

[a]See Eq. (168) on p. 1248.

Cash Flows of Sequential-Pay CMOs

- Take a 3-tranche sequential-pay CMO backed by \$3,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The 3 tranches are called A, B, and Z.
- All three tranches carry the same coupon rate of 12%.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche consists of Z bonds.
  - A Z bond receives no payments until all previous tranches are retired.
  - Although a Z bond carries an explicit coupon rate, the owed interest is accrued and added to the principal balance of that tranche.
  - The Z bond thus protects earlier tranches from extension risk
- When a Z bond starts receiving cash payments, it becomes a pass-through instrument.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche's coupon cash flows are initially used to pay down the tranches preceding it.
- Its existence (as in the ABZ structure here) accelerates the principal repayments of the sequential-pay bonds.
- Assume the ensuing monthly interest rates are 1%, 0.9%, 1.1%, 1.2%, 1.1%, 1.0%.
- Assume that the SMMs are 5%, 6%, 5%, 4%, 5%, 6%.
- We want to calculate the cash flow and then the fair price of each tranche.

Cash Flows of Sequential-Pay CMOs (continued)

- Compute the pool's cash flow by invoking the algorithm in Figure 29.10 in the text (p. 437).
  - $n = 6$, $r = 0.01$, and $\text{SMM} = [\,0.05, 0.06, 0.05, 0.04, 0.05, 0.06\,]$.
- Individual tranches' cash flows and remaining principals thereof can be derived by allocating the pool's principal and interest cash flows based on the CMO structure.
- See the next table for the breakdown.

Month		1	2	3	4	5	6
Interest rate		1.0%	0.9%	1.1%	1.2%	1.1%	1.0%
SMM		5.0%	6.0%	5.0%	4.0%	5.0%	6.0%
Remaining principal (B_i)							
	3,000,000	2,386,737	1,803,711	1,291,516	830,675	396,533	0
A	1,000,000	376,737	0	0	0	0	0
B	1,000,000	1,000,000	783,611	261,215	0	0	0
Z	1,000,000	1,010,000	1,020,100	1,030,301	830,675	396,533	0
Interest ($\overline{I_i}$)		30,000	23,867	18,037	12,915	8,307	3,965
A		20,000	3,767	0	0	0	0
B		10,000	20,100	18,037	2,612	0	0
Z		0	0	0	10,303	8,307	3,965
Principal		613,263	583,026	512,195	460,841	434,142	396,534
A		613,263	376,737	0	0	0	0
B		0	206,289	512,195	261,215	0	0
Z		0	0	0	199,626	434,142	396,534

Cash Flows of Sequential-Pay CMOs (continued)

- Note that the Z tranche's principal is growing at 1% per month until all previous tranches are retired.
- Before that time, the interest due the Z tranche is used to retire A's and B's *principals*.
- For example, the \$10,000 interest due tranche Z at month one is directed to tranche A instead.
- It reduces A's remaining principal from

$$\$1{,}000{,}000 - \$613{,}263 = \$386{,}737$$

by \$10,000 to \$376,737.

Cash Flows of Sequential-Pay CMOs (concluded)

- But it increases Z's from \$1,000,000 to \$1,010,000.
- At month four, the interest amount that goes into tranche Z is \$10,303.
- It is exactly what is required of Z's remaining principal of \$1,030,301.

Pricing Sequential-Pay CMOs

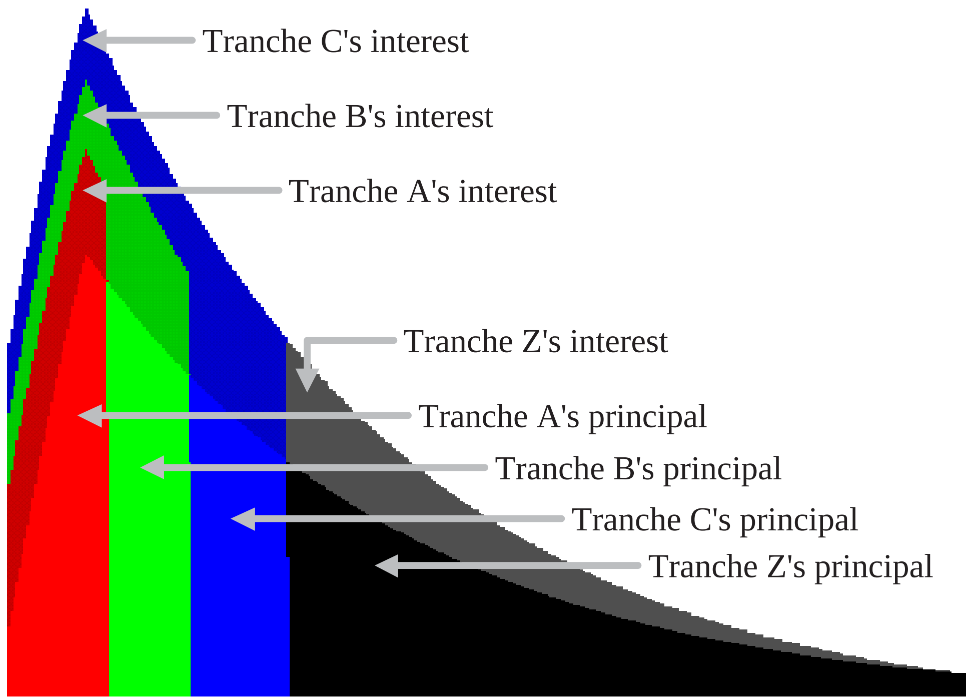
- We now price the tranches:

$$
\begin{aligned}
\text{tranche A} &= \frac{20000 + 613263}{1.01} + \frac{3767 + 376737}{1.01 \times 1.009} = 1000369, \\
\text{tranche B} &= \frac{10000 + 0}{1.01} + \frac{20100 + 206289}{1.01 \times 1.009} + \frac{18037 + 512195}{1.01 \times 1.009 \times 1.011} \\
&\quad + \frac{2612 + 261215}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&= 999719, \\
\text{tranche Z} &= \frac{10303 + 199626}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&\quad + \frac{8307 + 434142}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011} \\
&\quad + \frac{3965 + 396534}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011 \times 1.01} \\
&= 997238.
\end{aligned}
$$

Pricing Sequential-Pay CMOs (concluded)

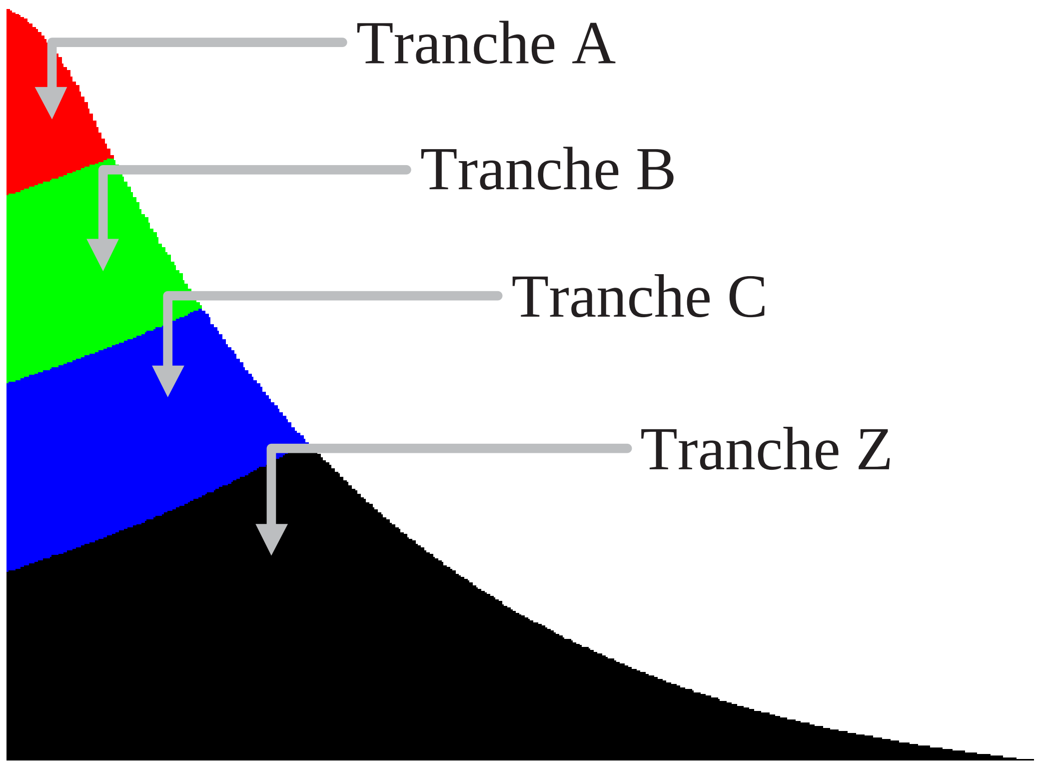
- This CMO has a total theoretical value of \$2,997,326.
- It is slightly less than its par value of \$3,000,000.
- See the algorithm in Figure 29.12 in the text (p. 440) for the cash flow generator.

A 4-Tranche Example: Cash Flows



The mortgage rate is 6%, the actual prepayment speed is 150 PSA, and each tranche has an identical original principal amount.

A 4-Tranche Example: Remaining Principals



Pricing Sequential-Pay CMOs: Methodology

- Suppose we have the interest rate path and the prepayment vector for that interest rate path.
- Then a CMO's cash flow can be calculated and the CMO priced.
- Unfortunately, the remaining principal of a CMO under prepayments is path dependent.
  - For example, a period of high rates before dropping to the current level is not likely to result in the same remaining principal as a period of low rates before rising to the current level.

Pricing Sequential-Pay CMOs: Methodology (concluded)

- If we try to price a 30-year CMO on a binomial interest rate model, there will be $2^{360} \approx 2.35 \times 10^{108}$ paths!
- Hence Monte Carlo simulation is the method of choice.
- First, one interest rate path is generated.
- Based on that path, the prepayment model is applied to generate the pool's principal, prepayment, and interest cash flows.
- Now, the cash flows of individual tranches can be generated and their present values derived.
- Repeat the above procedure over many interest rate scenarios and average the present values.

MBS Valuation Methodologies

1. Static cash flow yield.
2. Option modeling.
3. Option-adjusted spread (OAS).

Cash Flow Yield

- To price an MBS, one starts with its cash flow: The periodic P&I under a static prepayment assumption as given by a prepayment vector.
- The invoice price is now

$$\sum_{i=1}^{n} C_i/(1+r)^{\omega-1+i}.$$

  - C_i is the cash flow at time i.
  - n is the weighted average maturity (WAM).
  - r is the discount rate.
  - ω is the fraction of period from settlement until the first P&I payment date.

Cash Flow Yield (continued)

- The WAM is the weighted average remaining term of the mortgages in the pool, where the weight for each mortgage is the remaining balance.
- The r that equates the above with the market price is called the (static) cash flow yield.
- The static cash flow yield methodology compares the cash flow yield on an MBS with that on "comparable" bonds.
- The implied PSA is the single PSA speed producing the same cash flow yield.[a]

[a]Fabozzi (1991).

Cash Flow Yield (concluded)

- This simple methodology has obvious weaknesses.
  - It is static.
  - The projected cash flow may not be reinvested at the cash flow yield.[a]
  - The MBS may not be held until the final payout date.
  - The actual prepayment behavior is likely to deviate from the assumptions.

---

[a]This deficiency can be remedied somewhat by adopting the static spread methodology on p. 131.

The Option Pricing Methodology

- Virtually all mortgage loans give the homeowner the right to prepay the mortgage at any time.
- The homeowner holds an option to call the mortgage.
- The totality of these rights to prepay constitutes the embedded call option of the pass-through.
- In contrast, the MBS investor is short the embedded call.
- Therefore,

$$\begin{aligned} & \text{pass-through price} \\ = \; & \text{noncallable pass-through price} - \text{call option price}. \end{aligned}$$

The Option Pricing Methodology (continued)

- The option pricing methodology prices the call option by an option pricing model.
- It then estimates the market price of the noncallable pass-through by

$$\begin{aligned} & \text{noncallable pass-through price} \\ = \; & \text{pass-through price} + \text{call option price.} \end{aligned}$$

- The above price is finally used to compute the yield on this theoretical bond which does not prepay.
- This yield is called the option-adjusted yield.

The Option Pricing Methodology (continued)

- The option pricing methodology suffers from several difficulties.
  - The Black-Scholes model is not satisfactory for pricing fixed-income securities.[a]
  - There may not exist a benchmark to compare the option-adjusted yield with to obtain the yield spread.
  - This methodology does not incorporate the shape of the yield curve.

[a]See Section 24.7 of the textbook.

The Option Pricing Methodology (concluded)

- (continued)
  - Prepayment options are often "irrationally" exercised.
  - A partial exercise is possible as the homeowner can prepay a portion of the loan.
    - There is not one option but many, one per homeowner.
  - Valuation of the call option becomes very complicated for CMO bonds.

The OAS Methodology[a]

- The OAS methodology has four major components.
- The interest rate model is the first component.
- The second component is the prepayment model.[b]
  - Deterministic models that are accurate on average seem good enough for pass-throughs, IOs, and POs.[c]
- The OAS methodology is meant to identify investments with the best potential for excess returns.

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[a]The idea can be traced to Waldman and Modzelewski (1985), if not earlier.

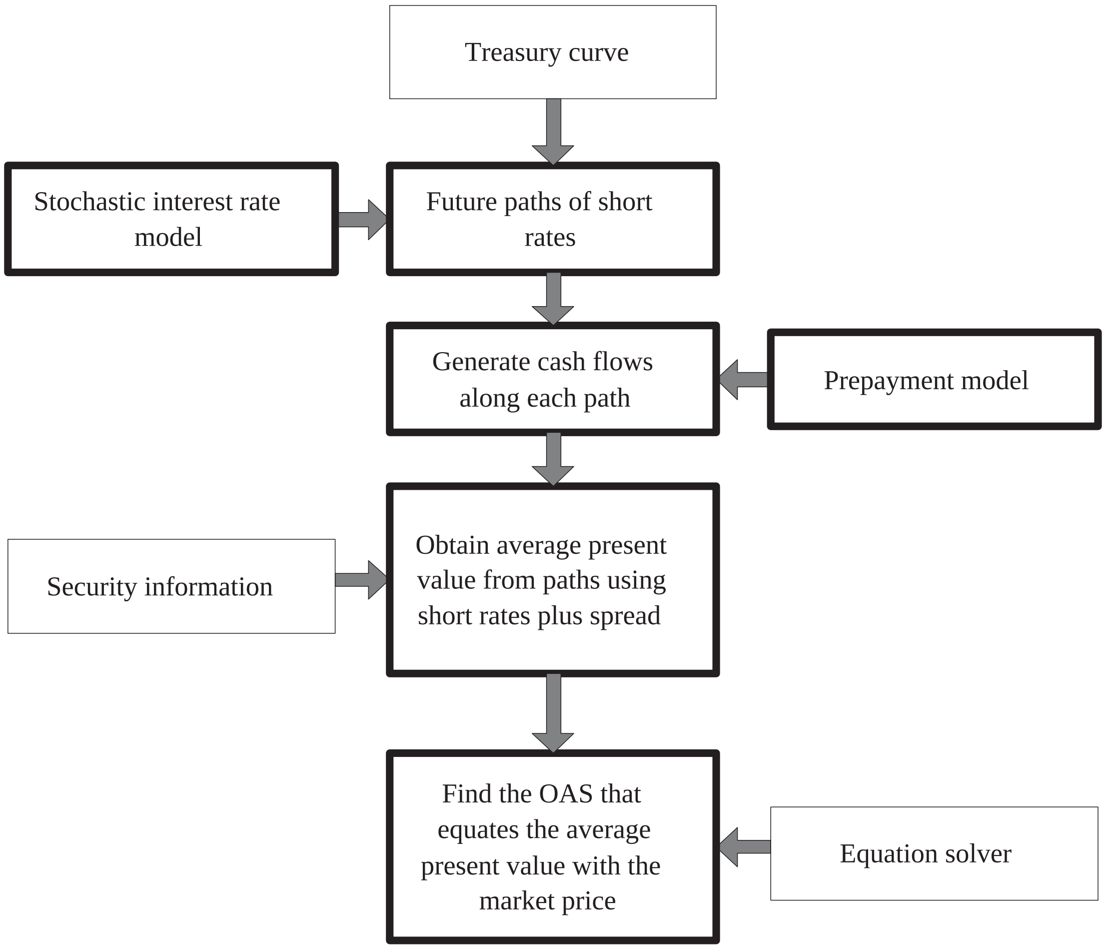
[b]May be the single most important component.

[c]Hayre (1997); Hayre & Rajan (1995).

The OAS Methodology (continued)

- The cash flow generator is the third component.
  - It calculates the current coupon rates for the interest rate paths given by the interest rate model.
  - It then generates the P&I cash flows for the pool as well as allocating them for individual securities based on the prepayment model and security information.
  - Note that the pool cash flow drives many securities.
- Finally, the equation solver calculates the OAS.

The OAS Methodology (continued)



The OAS Methodology (continued)

- The general valuation formula for uncertain cash flows can be written as

$$\text{PV} = \lim_{N \to \infty} \frac{1}{N} \sum_{N \text{ paths } r^*} \sum_n \frac{C_n^*}{(1 + r_1^*)(1 + r_2^*) \cdots (1 + r_n^*)}. \tag{177}$$

  - r^* denotes a risk-neutral interest rate path for which r_i^* is the ith one-period rate.
  - C_n^* is the cash flow at time n under this scenario.

The OAS Methodology (continued)

- The average over scenarios must also match the current spot rate curve, i.e.,

$$\frac{1}{(1+f_1)(1+f_2)\cdots(1+f_n)} = \lim_{N\to\infty}\frac{1}{N}\sum_{N\text{ paths }r^*}\frac{1}{(1+r_1^*)(1+r_2^*)\cdots(1+r_n^*)},$$

  $n = 1, 2, \ldots$, where f_i are the implied forward rates.

- The Monte Carlo calculation of OAS is closely related to Eq. (177) on p. 1309.

The OAS Methodology (continued)

- The interest rate model randomly produces a set of risk-neutral rate paths.
- The cash flow is then generated for each path.
- Finally, we solve for the spread s that makes the average discounted cash flow equal the market price,

$$
\begin{aligned}
& P \\
= \;& \lim_{N \to \infty} \frac{1}{N} \sum_{N \text{ paths } r^*} \sum_n \frac{C_n^*}{(1 + r_1^* + s)(1 + r_2^* + s) \cdots (1 + r_n^* + s)}.
\end{aligned}
$$

- This spread s is the OAS.

The OAS Methodology (continued)

- The implied cost of the embedded option is then calculated as

$$\text{option cost} = \text{static spread} - \text{OAS}.$$

- A common alternative averages the cash flows first and then calculates the OAS as the spread that equates this average cash flow with the market price.
- Although this approach is more efficient, it will generally give a different spread.

The OAS Methodology (continued)

- OAS calculation is very time consuming.
- The majority of the cost lies in generating the cash flows.
- This is because CMOs can become arbitrarily complex in their rules for allocating the cash flows.
- Such complexity requires special care in software design.
- The computational costs are then multiplied by the many runs of Monte Carlo simulation.

The OAS Methodology (concluded)

- Although extremely popular, the OAS methodology suffers from several difficulties.[a]
  - It is model-dependent.
  - It may be difficult to interpret the OAS number.
  - The spread must be a constant.
  - It assumes the cash flow can be reinvested with the same OAS.
  - Dealers may report very different OAS numbers.

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[a] Chernov, Dunn, & Longstaff (2016); Fabozzi & Modigliani (1992).

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