The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.
- $P_u$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_d$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.
The BDT Model: Volatility Structure (concluded)

• Corresponding to these two prices are the following yields to maturity,

\[ y_u \triangleq P_u^{-1/(i-1)} - 1, \]
\[ y_d \triangleq P_d^{-1/(i-1)} - 1. \]

• The yield volatility is defined as\(^a\)

\[ \kappa_i \triangleq \frac{\ln(y_u/y_d)}{2}. \]

\(^a\)Recall Eq. (137) on p. 1033.
The BDT Model: Calibration

• The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.

• For economy of expression, all numbers are period based.

• Suppose inductively that we have calculated

\[(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).\]

  – They define the binominal tree up to time \(i - 2\) (thus period \(i - 1\)).

• We now proceed to calculate \(r_i\) and \(v_i\) to extend the tree to time \(i\).
The BDT Model: Calibration (continued)

- Assume the price of the \( i \)-period zero can move to \( P_u \) or \( P_d \) one period from now.
- Let \( y \) denote the current \( i \)-period spot rate, which is known.
- In a risk-neutral economy,

\[
\frac{P_u + P_d}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \tag{155}
\]

- Obviously, \( P_u \) and \( P_d \) are functions of the unknown \( r_i \) and \( v_i \).
The BDT Model: Calibration (continued)

- Viewed from now, the future \((i - 1)\)-period spot rate at time 1 is uncertain.

- Recall that \(y_u\) and \(y_d\) represent the spot rates at the up node and the down node, respectively.\(^a\)

- With \(\kappa_i^2\) denoting their variance, we have

\[
\kappa_i = \frac{1}{2} \ln \left( \frac{P_u^{1/(i-1)} - 1}{P_d^{1/(i-1)} - 1} \right). \tag{156}
\]

\(^a\)Recall p. 1137.
The BDT Model: Calibration (continued)

- Solving Eqs. (155)–(156) for $r$ and $v$ with backward induction takes $O(i^2)$ time.
  - That leads to a cubic-time algorithm.

- We next employ forward induction to derive a quadratic-time calibration algorithm.

- Forward induction figures out, by moving *forward* in time, how much $1 at a node contributes to the price.

- This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

---

\(^{a}\)W. J. Chen (R84526007) & Lyuu (1997); Lyuu (1999).

\(^{b}\)Review p. 1010(a).
The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period $i$ be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i - 1$ be $P_1, P_2, \ldots, P_i$.
- They correspond to rates $r, rv, \ldots, rv^{i-1}$ for period $i$, respectively.
- One dollar at time $i$ has a present value of

$$f(r, v) \triangleq \frac{P_1}{1 + r} + \frac{P_2}{1 + rv} + \frac{P_3}{1 + rv^2} + \cdots + \frac{P_i}{1 + rv^{i-1}}.$$
The BDT Model: Calibration (continued)

- By Eq. (156) on p. 1140, the yield volatility is

\[ g(r, v) \triangleq \frac{1}{2} \ln \left( \frac{\left( \frac{P_{u,1}}{1+rv} + \frac{P_{u,2}}{1+rv^2} + \cdots + \frac{P_{u,i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left( \frac{P_{d,1}}{1+r} + \frac{P_{d,2}}{1+rv} + \cdots + \frac{P_{d,i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right). \]

- Above, \( P_{u,1}, P_{u,2}, \ldots \) denote the state prices at time \( i - 1 \) of the subtree rooted at the up node.\(^a\)

- And \( P_{d,1}, P_{d,2}, \ldots \) denote the state prices at time \( i - 1 \) of the subtree rooted at the down node.\(^b\)

\(^a\)Like \( r_2v_2 \) on p. 1134.
\(^b\)Like \( r_2 \) on p. 1134.
The BDT Model: Calibration (concluded)

• Note that every node maintains three state prices: $P_i, P_{u,i}, P_{d,i}$.

• Now solve

\[
\begin{align*}
    f(r, v) &= \frac{1}{(1 + y)^i}, \\
    g(r, v) &= \kappa_i,
\end{align*}
\]

for $r = r_i$ and $v = v_i$.

• This $O(n^2)$-time algorithm appears on p. 382 of the textbook.
Calibrating the BDT Model with the Differential Tree (in seconds)\textsuperscript{a}

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<th>Number of years</th>
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75MHz Sun SPARCstation 20, one period per year.

\textsuperscript{a}Lyuu (1999).
The BDT Model: Continuous-Time Limit

• The continuous-time limit of the BDT model is\(^a\)

\[
d\ln r = \left( \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dW.
\]

• The short rate volatility \(\sigma(t)\) should be a *declining* function of time for the model to display mean reversion.
  
  – That makes \(\sigma'(t) < 0\).

• In particular, constant \(\sigma(t)\) will not attain mean reversion.

The Black-Karasinski Model\textsuperscript{a}

- The BK model stipulates that the short rate follows

\[ d \ln r = \kappa(t)(\theta(t) - \ln r) \, dt + \sigma(t) \, dW. \]

- This explicitly mean-reverting model depends on time through \( \kappa(\cdot), \theta(\cdot), \) and \( \sigma(\cdot). \)

- The BK model hence has one more degree of freedom than the BDT model.

- The speed of mean reversion \( \kappa(t) \) and the short rate volatility \( \sigma(t) \) are independent.

\textsuperscript{a}Black & Karasinski (1991).
The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

\[
\begin{align*}
t_2 & \triangleq t_1 + \Delta t_1, \\
t_3 & \triangleq t_2 + \Delta t_2.
\end{align*}
\]
\[
\ln r_d(t_2) \quad \ln r_u(t_2) \quad \ln r(t_1) \quad \ln r_{du}(t_3) = \ln r_{ud}(t_3)
\]
The Black-Karasinski Model: Discrete Time (continued)

- Note that

  \[
  \ln r_d(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}, \\
  \ln r_u(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}.
  \]

- To make sure an up move followed by a down move coincides with a down move followed by an up move,

  \[
  \ln r_d(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_d(t_2)) \Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}, \\
  = \ln r_u(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_u(t_2)) \Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}.
  \]
The Black-Karasinski Model: Discrete Time (continued)

- They imply
  \[ \kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1)) \sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \]

- So from \( \Delta t_1 \), we can calculate the \( \Delta t_2 \) that satisfies the combining condition and then iterate.
  \[ t_0 \rightarrow \Delta t_1 \rightarrow t_1 \rightarrow \Delta t_2 \rightarrow t_2 \rightarrow \Delta t_3 \rightarrow \cdots \rightarrow T \]
  (roughly).\(^a\)

\(^a\)As \( \kappa(t), \theta(t), \sigma(t) \) are independent of \( r \), the \( \Delta t_i \) will not depend on \( r \) either.
The Black-Karasinski Model: Discrete Time (concluded)

- Unequal durations $\Delta t_i$ are often necessary to ensure a combining tree.\(^a\)

\(^a\)Amin (1991); C. I. Chen (R98922127) (2011); Lok (D99922028) & Lyuu (2016, 2017).
Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^\pi [M(t)] = \infty$ for any finite $t$ if they model the continuously compounded rate.\(^\text{a}\)

- So periodically compounded rates should be modeled.\(^\text{b}\)

- Another issue is computational.

- Lognormal models usually do not admit of analytical solutions to even basic fixed-income securities.

- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

\(^\text{a}\)Hogan & Weintraub (1993).

\(^\text{b}\)Sandmann & Sondermann (1993).
Problems with Lognormal Models in General (concluded)

• This problem can be somewhat mitigated by adopting variable-duration time steps.\(^a\)
  – Use a fine time step up to the maturity of the short-dated derivative.
  – Use a coarse time step beyond the maturity.

• A down side of this procedure is that it has to be tailor-made for each derivative.

• Finally, empirically, interest rates do not follow the lognormal distribution.

\(^a\)Hull & White (1993).
The Extended Vasicek Model\textsuperscript{a}

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

\[ dr = (\theta(t) - a(t) r) \, dt + \sigma(t) \, dW. \]

- Like the Ho-Lee model, this is a normal model.
- The inclusion of \( \theta(t) \) allows for an exact fit to the current spot rate curve.

\textsuperscript{a}Hull & White (1990).
The Extended Vasicek Model (concluded)

• Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.

• Many European-style securities can be evaluated analytically.

• Efficient numerical procedures can be developed for American-style securities.
The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar) \, dt + \sigma \, dW.$$ 

- When the current term structure is matched,\(^a\)

$$\theta(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}).$$

- Recall that \(f(0, t)\) defines the forward rate curve.

\(^a\)Hull & White (1993).
The Extended CIR Model

• In the extended CIR model the short rate follows

\[ dr = (\theta(t) - a(t)r)\,dt + \sigma(t)\sqrt{r}\,dW. \]

• The functions \( \theta(t), a(t), \) and \( \sigma(t) \) are implied from market observables.

• With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.
The Hull-White Model: Calibration

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given \( a \) and \( \sigma \).
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let \( r_0 \) be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value 
  
  \[ r_0 + j\Delta r \]
  
  for some integer \( j \).

---

\(^a\)Hull & White (1993).
The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at $\Delta t$ apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with
  \[
  t_i \triangleq i\Delta t, \\
  r_j \triangleq r_0 + j\Delta r,
  \]
  as the $(i, j)$ node.
- The short rate at node $(i, j)$, which equals $r_j$, is effective for the time period $[t_i, t_{i+1})$. 

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The Hull-White Model: Calibration (continued)

- Use

\[ \mu_{i,j} \triangleq \theta(t_i) - ar_j \]  \hspace{1cm} (158)

\hspace{1cm} to denote the drift rate\(^a\) of the short rate as seen from node \((i,j)\).

- The three distinct possibilities for node \((i,j)\) with three branches incident from it are displayed on p. 1162.

- The middle branch may be an increase of \(\Delta r\), no change, or a decrease of \(\Delta r\).

\(^a\)Or, the annualized expected change.
The Hull-White Model: Calibration (continued)

\[(i, j) \rightarrow (i + 1, j + 1) \]

\[ (i + 1, j + 1) \rightarrow (i + 1, j + 2) \]

\[ (i, j) \rightarrow (i + 1, j) \]

\[ (i, j) \rightarrow (i + 1, j - 1) \]

\[ (i + 1, j) \rightarrow (i + 1, j - 1) \]

\[ (i + 1, j) \rightarrow (i + 1, j - 2) \]

\[ (i + 1, j - 1) \rightarrow (i + 1, j - 2) \]
The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.

- Define

\[
\begin{align*}
  p_1(i, j) & \triangleq \text{the probability of following the upper branch from node } (i, j), \\
p_2(i, j) & \triangleq \text{the probability of following the middle branch from node } (i, j), \\
p_3(i, j) & \triangleq \text{the probability of following the lower branch from node } (i, j).
\end{align*}
\]

- The root of the tree is set to the current short rate \( r_0 \).

- Inductively, the drift \( \mu_{i,j} \) at node \((i, j)\) is a function of (the still unknown) \( \theta(t_i) \).
  - It describes the expected change from node \((i, j)\).
The Hull-White Model: Calibration (continued)

- Once \( \theta(t_i) \) is available, \( \mu_{i,j} \) can be derived via Eq. (158) on p. 1161.
- This in turn determines the branching scheme at every node \((i,j)\) for each \(j\), as we will see shortly.
- The value of \( \theta(t_i) \) must thus be made consistent with the spot rate \(r(0,t_{i+2})\).\(^a\)

\(^a\)Not \(r(0,t_{i+1})\)!
The Hull-White Model: Calibration (continued)

- The branches emanating from node \((i, j)\) with their probabilities\(^a\) must be chosen to be consistent with \(\mu_{i,j}\) and \(\sigma\).

- This is done by selecting the middle node to be as closest to the current short rate \(r_j\) plus the drift \(\mu_{i,j}\Delta t\).\(^b\)

\(^a\)That is, \(p_1(i, j)\), \(p_2(i, j)\), and \(p_3(i, j)\).

\(^b\)A precursor of Lyuu and C. Wu’s (R90723065) (2003, 2005) mean-tracking idea, which is the precursor of the binomial-trinomial tree of Dai (B82506025, R86526008, D8852600) & Lyuu (2006, 2008, 2010).
The Hull-White Model: Calibration (continued)

- Let $k$ be the number among $\{j - 1, j, j + 1\}$ that makes the short rate reached by the middle branch, $r_k$, closest to $r_j + \mu_{i,j} \Delta t$.

  - But note that $\mu_{i,j}$ is still not computed yet.

- Then the three nodes following node $(i, j)$ are nodes $(i + 1, k + 1), (i + 1, k), (i + 1, k - 1)$.

- See p. 1167 for a possible geometry.

- The resulting tree combines.
The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, $\sigma$, $j$, and $k$:

\[
p_1(i, j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r}, \quad (159)
\]

\[
p_2(i, j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2}, \quad (159')
\]

\[
p_3(i, j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r}, \quad (159'')
\]

where

\[
\eta \overset{\Delta}{=} \mu_{i,j} \Delta t + (j - k) \Delta r.
\]
The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise,\(^a\) certain relations must hold for \(\Delta r\) and \(\Delta t\) to guarantee stability.

- It can be shown that their values must satisfy

\[
\frac{\sigma \sqrt{3}\Delta t}{2} \leq \Delta r \leq 2\sigma \sqrt{\Delta t}
\]

for the probabilities to lie between zero and one.

- For example, \(\Delta r\) can be set to \(\sigma \sqrt{3}\Delta t\).\(^b\)

- Now it only remains to determine \(\theta(t_i)\).

\(^a\)Recall p. 809.

\(^b\)Hull & White (1988).
The Hull-White Model: Calibration (continued)

- At this point at time $t_i$,
  \[ r(0, t_1), r(0, t_2), \ldots, r(0, t_{i+1}) \]
  have already been matched.
- Let $Q(i, j)$ be the state price at node $(i, j)$.
- By construction, the state prices $Q(i, j)$ for all $j$ are known by now.
- We begin with state price $Q(0, 0) = 1$. 
The Hull-White Model: Calibration (continued)

• Let \( \hat{r}(i) \) refer to the short rate value at time \( t_i \).

• The value at time zero of a zero-coupon bond maturing at time \( t_{i+2} \) is then

\[
e^{-r(0,t_{i+2})(i+2)\Delta t} = \sum_j Q(i,j) e^{-r_j \Delta t} E^\pi \left[ e^{-\hat{r}(i+1)\Delta t} \bigg| \hat{r}(i) = r_j \right]. (160)
\]

• The right-hand side represents the value of $1 obtained by holding a zero-coupon bond until time \( t_{i+1} \) and then reinvesting the proceeds at that time at the prevailing short rate \( \hat{r}(i + 1) \), which is stochastic.
The Hull-White Model: Calibration (continued)

- The expectation in Eq. (160) can be approximated by

\[ E^\pi \left[ e^{-\hat{r}(i+1)\Delta t} \middle| \hat{r}(i) = r_j \right] \]

\[ \approx e^{-r_j \Delta t} \left( 1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2} \right). \quad (161) \]

- This solves the chicken-egg problem!

- Substitute Eq. (161) into Eq. (160) and replace \( \mu_{i,j} \) with \( \theta(t_i) - ar_j \) to obtain

\[ \theta(t_i) \approx \frac{\sum_j Q(i, j) e^{-2r_j \Delta t} \left( 1 + ar_j(\Delta t)^2 + \sigma^2(\Delta t)^3 / 2 \right)}{(\Delta t)^2 \sum_j Q(i, j) e^{-2r_j \Delta t}} - e^{-r(0, t_i+2)(i+2) \Delta t}. \]

\(^a\)See Exercise 26.4.2 of the textbook.
The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (161) is actually known analytically by Eq. (28) on p. 175:

\[
E^\pi \left[ e^{\hat{r}(i+1) \Delta t} \left| \hat{r}(i) = r_j \right. \right] = e^{-r_j \Delta t + (-\theta(t_i) + ar_j + \sigma^2 \Delta t/2)(\Delta t)^2}.
\]

- Therefore, alternatively,

\[
\theta(t_i) = \frac{r(0, t_{i+2})(i + 2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i, j) e^{2r_j \Delta t + ar_j(\Delta t)^2}}{(\Delta t)^2}.
\]

- With \( \theta(t_i) \) in hand, we can compute \( \mu_{i,j} \).

---

\( ^a \)See Eq. (158) on p. 1161.
The Hull-White Model: Calibration (concluded)

- With $\mu_{i,j}$ available, we compute the probabilities.\textsuperscript{a}

- Finally the state prices at time $t_{i+1}$:

$$Q(i + 1, j) = \sum_{(i, j^*) \text{ is connected to } (i + 1, j)} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for $j^*$ (why?).

- The total running time is $O(n^2)$.

- The space requirement is $O(n)$ (why?).

\textsuperscript{a}See Eqs. (159) on p. 1168.
Comments on the Hull-White Model

• One can try different values of $a$ and $\sigma$ for each option.

• Or have an $a$ value common to all options but use a different $\sigma$ value for each option.

• Either approach can match all the option prices exactly.

• But suppose the demand is for a single set of parameters that replicate all option prices.

• Then the Hull-White model can be calibrated to all the observed option prices by choosing $a$ and $\sigma$ that minimize the mean-squared pricing error.\(^a\)

\(^a\)Hull & White (1995).
The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form $\sigma r^b$.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1167).  
  - So it is harder to program (for nonprogrammers).
- The second shortcoming is a consequence of the tree’s irregular shape.
The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time $t_i$.

- But without those branches, the tree was not specified, and backward induction on the tree was not possible.

- To avoid this chicken-egg dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (160) on p. 1171 that helps derive $\theta(t_i)$.

- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.
The Hull-White Model: Calibration with Regular Trinomial Trees\textsuperscript{a}

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term $\sigma$.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

\textsuperscript{a}Hull & White (1994).
The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$  

- The tree is dagger-shaped (see p. 1180).
- The number of nodes above the $r_0$-line is $j_{\text{max}}$, and that below the line is $j_{\text{min}}$.
- They will be picked so that the probabilities (159) on p. 1168 are positive for all nodes.
The short rate at node \((0, 0)\) equals \(r_0 = 0\); here \(j_{\text{max}} = 3\) and \(j_{\text{min}} = 2\).
The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- The tree’s branches and probabilities are now in place.
- Phase two fits the term structure.
  - Backward induction is applied to calculate the $\beta_i$ to add to the short rates on the tree at time $t_i$ so that the spot rate $r(0,t_{i+1})$ is matched.\(^a\)

\(^a\)Contrast this with the previous algorithm, where it was $r(0,t_{i+2})$ that was matched!
The Hull-White Model: Calibration

• Set $\Delta r = \sigma \sqrt{3 \Delta t}$ and assume that $a > 0$.

• Node $(i, j)$ is a top node if $j = j_{\text{max}}$ and a bottom node if $j = -j_{\text{min}}$.

• Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j \Delta r$.

• Hence the probabilities in Eqs. (159) on p. 1168 use

  $$\eta \overset{\Delta}{=} -aj \Delta r \Delta t + (j - k) \Delta r.$$ 

• Recall that $k$ denotes the middle branch.
The Hull-White Model: Calibration (continued)

- The probabilities become

\[
\begin{align*}
\bar{p}_1(i, j) &= \frac{1}{6} + \frac{a^2j^2(\Delta t)^2 - 2aj\Delta t(j - k) + (j - k)^2 - a\Delta t + (j - k)}{2}, \quad (162) \\
\bar{p}_2(i, j) &= \frac{2}{3} - \left[ a^2j^2(\Delta t)^2 - 2aj\Delta t(j - k) + (j - k)^2 \right], \quad (163) \\
\bar{p}_3(i, j) &= \frac{1}{6} + \frac{a^2j^2(\Delta t)^2 - 2aj\Delta t(j - k) + (j - k)^2 + a\Delta t - (j - k)}{2}. \quad (164)
\end{align*}
\]

- \( \bar{p}_1 \): up move; \( \bar{p}_2 \): flat move; \( \bar{p}_3 \): down move.
The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node $(i, j)$ is a top node.
  - Let $k = j + 1$ if node $(i, j)$ is a bottom node.
  - Let $k = j$ for the rest of the nodes.

- Note that the probabilities are identical for nodes $(i, j)$ with the same $j$.

- Furthermore, $p_1(i, j) = p_3(i, -j)$. 
The Hull-White Model: Calibration (continued)

• The inequalities

\[ \frac{3 - \sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \]

ensure that all the branching probabilities are positive in the upper half of the tree, that is, \( j > 0 \) (verify this).

• Similarly, the inequalities

\[ -\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3 - \sqrt{6}}{3} \]

ensure that the probabilities are positive in the lower half of the tree, that is, \( j < 0 \).
The Hull-White Model: Calibration (continued)

• To further make the tree symmetric across the $r_0$-line, we let $j_{\text{min}} = j_{\text{max}}$.

• As

$$\frac{3 - \sqrt{6}}{3} \approx 0.184,$$

a good choice is

$$j_{\text{max}} = \lceil 0.184/(a\Delta t) \rceil.$$
The Hull-White Model: Calibration (continued)

• Phase two computes the $\beta_i$'s to fit the spot rates.
• We begin with state price $Q(0, 0) = 1$.
• Inductively, suppose that spot rates
  
  \[ r(0, t_1), r(0, t_2), \ldots, r(0, t_i) \]

  have already been matched.
• By construction, the state prices $Q(i, j)$ for all $j$ are known by now.
The Hull-White Model: Calibration (continued)

• The value of a zero-coupon bond maturing at time \( t_{i+1} \) equals

\[
e^{-r(0,t_{i+1})(i+1)\Delta t} = \sum_j Q(i,j) e^{-(\beta_i + r_j)\Delta t}
\]

by risk-neutral valuation.

• Hence

\[
\beta_i = \frac{r(0,t_{i+1})(i + 1) \Delta t \ln \sum_j Q(i,j) e^{-r_j \Delta t}}{\Delta t}.
\] (166)
The Hull-White Model: Calibration (concluded)

- The short rate at node \((i, j)\) now equals \(\beta_i + r_j\).
- The state prices at time \(t_{i+1}\),
  \[
  Q(i + 1, j), \quad -\min(i + 1, j_{\text{max}}) \leq j \leq \min(i + 1, j_{\text{max}}),
  \]
can now be calculated as before.\(^a\)
- The total running time is \(O(n j_{\text{max}})\).
- The space requirement is \(O(n)\).

\(^a\)Recall p. 1174.
A Numerical Example

• Assume \( a = 0.1, \sigma = 0.01, \) and \( \Delta t = 1 \) (year).

• Immediately, \( \Delta r = 0.0173205 \) and \( j_{\text{max}} = 2 \).

• The plot on p. 1191 illustrates the 3-period trinomial tree after phase one.

• For example, the branching probabilities for node E are calculated by Eqs. (162)–(164) on p. 1183 with \( j = 2 \) and \( k = 1 \).
<table>
<thead>
<tr>
<th>Node</th>
<th>A, C, G</th>
<th>B, F</th>
<th>E</th>
<th>D, H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$ (%)</td>
<td>0.00000</td>
<td>1.73205</td>
<td>3.46410</td>
<td>-1.73205</td>
<td>-3.46410</td>
</tr>
<tr>
<td>$p_1$</td>
<td>0.16667</td>
<td>0.12167</td>
<td>0.88667</td>
<td>0.22167</td>
<td>0.08667</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.66667</td>
<td>0.65667</td>
<td>0.02667</td>
<td>0.65667</td>
<td>0.02667</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.16667</td>
<td>0.22167</td>
<td>0.08667</td>
<td>0.12167</td>
<td>0.88667</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

• Suppose that phase two is to fit the spot rate curve

\[ 0.08 - 0.05 \times e^{-0.18 \times t}. \]

• The annualized continuously compounded spot rates are

\[ r(0, 1) = 3.82365\%, r(0, 2) = 4.51162\%, r(0, 3) = 5.08626\%. \]

• Start with state price \( Q(0, 0) = 1 \) at node A.
A Numerical Example (continued)

• Now, by Eq. (166) on p. 1188,

\[ \beta_0 = r(0, 1) + \ln Q(0, 0) e^{-r_0} = r(0, 1) = 3.82365\%. \]

• Hence the short rate at node A equals

\[ \beta_0 + r_0 = 3.82365\%. \]

• The state prices at year one are calculated as

\[ Q(1, 1) = p_1(0, 0) e^{-(\beta_0 + r_0)} = 0.160414, \]
\[ Q(1, 0) = p_2(0, 0) e^{-(\beta_0 + r_0)} = 0.641657, \]
\[ Q(1, -1) = p_3(0, 0) e^{-(\beta_0 + r_0)} = 0.160414. \]
A Numerical Example (continued)

• The 2-year rate spot rate \( r(0, 2) \) is matched by picking

\[
\beta_1 = r(0, 2) \times 2 + \ln \left[ Q(1, 1) e^{-\Delta r} + Q(1, 0) + Q(1, -1) e^{\Delta r} \right] = 5.20459\%.
\]

• Hence the short rates at nodes B, C, and D equal

\[
\beta_1 + r_j,
\]

where \( j = 1, 0, -1 \), respectively.

• They are found to be 6.93664\%, 5.20459\%, and 3.47254\%. 
A Numerical Example (continued)

- The state prices at year two are calculated as

\[
Q(2, 2) = p_1(1, 1) e^{-(\beta_1 + r_1)} Q(1, 1) = 0.018209,
\]
\[
Q(2, 1) = p_2(1, 1) e^{-(\beta_1 + r_1)} Q(1, 1) + p_1(1, 0) e^{-(\beta_1 + r_0)} Q(1, 0)
= 0.199799,
\]
\[
Q(2, 0) = p_3(1, 1) e^{-(\beta_1 + r_1)} Q(1, 1) + p_2(1, 0) e^{-(\beta_1 + r_0)} Q(1, 0)
+ p_1(1, -1) e^{-(\beta_1 + r_1)} Q(1, -1) = 0.473597,
\]
\[
Q(2, -1) = p_3(1, 0) e^{-(\beta_1 + r_0)} Q(1, 0) + p_2(1, -1) e^{-(\beta_1 + r_1)} Q(1, -1)
= 0.203263,
\]
\[
Q(2, -2) = p_3(1, -1) e^{-(\beta_1 + r_1)} Q(1, -1) = 0.018851.
\]
A Numerical Example (concluded)

• The 3-year rate spot rate \( r(0, 3) \) is matched by picking

\[
\beta_2 = r(0, 3) \times 3 + \ln \left[ Q(2, 2) e^{-2 \times \Delta r} + Q(2, 1) e^{-\Delta r} + Q(2, 0) \\
+ Q(2, -1) e^{\Delta r} + Q(2, -2) e^{2 \times \Delta r} \right] = 6.25359\%.
\]

• Hence the short rates at nodes E, F, G, H, and I equal \( \beta_2 + r_j \), where \( j = 2, 1, 0, -1, -2 \), respectively.

• They are found to be 9.71769\%, 7.98564\%, 6.25359\%, 4.52154\%, and 2.78949\%.

• The figure on p. 1197 plots \( \beta_i \) for \( i = 0, 1, \ldots, 29 \).
The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.
The Heath-Jarrow-Morton (HJM) Model

- This influential model is a forward rate model.
- The HJM model specifies the initial forward rate curve and the forward rate volatility structure.
  - The volatility structure describes the volatility of each forward rate for a given maturity date.
- Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

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*aHeath, Jarrow, & Morton (1992).*
Introduction to Mortgage-Backed Securities
Anyone stupid enough to promise to be responsible for a stranger’s debts deserves to have his own property held to guarantee payment.

— Proverbs 27:13
Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- Suppose the borrower (the mortgagor) defaults, that is, fails to make the contractual payments.
- The lender (the mortgagee) can foreclose the loan by seizing the property.
Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.\textsuperscript{a}

- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.

- The mortgage market has witnessed tremendous innovations in product design.

\textsuperscript{a}They can be traced to 1880s (Levy, 2012).
Mortgage-Backed Securities (concluded)

• The complexity of the products and the prepayment option require advanced models and software techniques.
  – In fact, the mortgage market probably could not have operated efficiently without them.\textsuperscript{a}

• They also consume lots of computing power.

• Our focus will be on residential mortgages.

• But the underlying principles are applicable to other types of assets.

\textsuperscript{a}Merton (1994).
Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
Problems Investing in Mortgages

- The MBS sector is one of the largest in the debt market.\(^a\)

- Individual mortgages are unattractive for many investors.

- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.

- Most investors lack the resources and knowledge to assess the credit risk involved.

\(^a\)See p. 3 of the textbook. The outstanding balance was US$9.3 trillion as of 2017 vs. the US Treasury’s US$14.5 trillion and corporate debt’s US$9.0 trillion (SIFMA, 2018).
Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors.
- That makes the size and the timing of the cash flows unpredictable.
Mortgage Pass-Throughs

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a $25,000 certificate from a $1 million pool is entitled to $2\frac{1}{2}\%$ (or $1/40$th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.

\[ \text{First issued by Ginnie Mae in 1970.} \]
Rule for distribution of cash flows: pro rata

Pass-through: $1 million par pooled mortgage loans
Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.

- Such risk is undesirable from an asset/liability perspective.

- To deal with prepayment uncertainty, CMOs were created.\(^a\)

- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

\(^a\)In June 1983 by Freddie Mac with the help of First Boston, which was acquired by Credit Suisse in 1990.
Collateralized Mortgage Obligations (CMOs) (continued)

- CMOs are *multiple*-maturity, *multiclass* debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.
- The total prepayment risk is now divided among classes of bonds called classes or tranches.
- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

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*a* Tranche is a French word for “slice.”
Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors
  - The outstanding balance of agency CMOs was US$1.1 trillion as of the first quarter of 2015.\(^a\)

\(^a\)SIFMA (2015).
Sequential Tranche Paydown

• In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.

• Each tranche thus has a different effective maturity.

• Each tranche may even have a different coupon rate.