Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.

— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader [$\cdots$]
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later.[a]

$P(t, T)$**:** the present value at time t of one dollar at time T.

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[a] Alternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

$r(t, T)$**:** the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.[a]

$F(t, T, M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

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[a] In other words, the $(T - t)$-period spot rate at time t.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1 + r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases} \tag{134}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t, t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t,T,M) = \frac{P(t,M)}{P(t,T)}, \quad T \leq M. \tag{135}$$

  - The forward price equals the future value at time T of the underlying asset.[a]

- Equation (135) holds whether the model is discrete-time or continuous-time.

[a] See Exercise 24.2.1 of the textbook for proof.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)} \right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)} \right)^{1/L} - 1 \tag{136}$$

  in discrete time.

- The analog to Eq. (136) under simple compounding is

$$f(t,T,L) = \frac{1}{L} \left(\frac{P(t,T)}{P(t,T+L)} - 1 \right).$$

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{137}$$

by Eq. (135) on p. 1039.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \stackrel{\Delta}{=} \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \le T. \tag{138}$$

- Because Eq. (138) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{139}$$

the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s)\,ds}{T-t}.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (139) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
  $$\frac{E_t[\, P(t+1, T)\,]}{P(t, T)} = 1 + r(t). \tag{140}$$
  - Relation (140) in fact follows from the risk-neutral valuation principle.[a]

[a]Theorem 16 on p. 544.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Equation (140) on p. 1044 can also be expressed as

$$E_t[\, P(t+1,T)\,] = F(t,t+1,T).$$

  – Verify that with, e.g., Eq. (135) on p. 1039.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.[a]

[a] Under the local expectations theory. But the forward rate is *not* an unbiased estimator of the expected future short rate (p. 995).

Risk-Neutral Pricing (continued)

- Rewrite Eq. (140) on p. 1044 as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T). \tag{141}$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (concluded)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t,T) \\
= \;& E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \;& E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \;& E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right].
\end{aligned}
$$

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies
$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{142}$$
- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the *floating-rate payer*.
- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} (f_{i-1} - c)\, \Delta t \right] \\
= \; & \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= \; & \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} \left(e^{\int_{t_{i-1}}^{t_i} r(s)\, ds} - (1 + c\Delta t) \right) \right] \\
= \; & \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= \; & P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present-value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{143}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Term Structure Equation[a]

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

[a] Vasicek (1977).

The Term Structure Equation (continued)

- The net wealth change follows

$$
\begin{aligned}
& -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\
= \quad & (-P(r,t,s_1)\, \mu_p(r,t,s_1) + \alpha P(r,t,s_2)\, \mu_p(r,t,s_2))\, dt \\
& + (-P(r,t,s_1)\, \sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\, \sigma_p(r,t,s_2))\, dW.
\end{aligned}
$$

- Pick

$$
\alpha \triangleq \frac{P(r,t,s_1)\, \sigma_p(r,t,s_1)}{P(r,t,s_2)\, \sigma_p(r,t,s_2)}.
$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1) + \alpha P(r,t,s_2)} = r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2) - \sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1) - \sigma_p(r,t,s_2)} = r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2) - r}{\sigma_p(r,t,s_2)} = \frac{\mu_p(r,t,s_1) - r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \stackrel{\Delta}{=} \lambda(r,t) \tag{144}$$

  for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} \right) / P, \tag{145}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r} \right) / P, \tag{145$'$}$$

subject to $P(\,\cdot\,,T,T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (144) on p. 1057 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{146}$$

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

$$
\begin{tikzcd}[row sep=small]
 & 8\% \\
4\% \arrow[ur] \arrow[dr] & \\
 & 2\%
\end{tikzcd}
$$

Numerical Examples (continued)

- *No* real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 1062.

Numerical Examples (continued)

$$
90.703 \begin{array}{l} \nearrow \; 92.593 \; (= 100/1.08) \\ \searrow \; 98.039 \; (= 100/1.02) \end{array}
\qquad\qquad
96.154 \begin{array}{l} \nearrow \; 100 \\ \searrow \; 100 \end{array}
$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$C \begin{array}{l} \nearrow \;\; 0.000 \\ \\ \searrow \;\; 3.039 \; (= 98.039 - 95) \end{array}$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p)\times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:

$$\begin{tikzcd}[row sep=small] & 92\ (= 100 - 8) \\ F \arrow[ur] \arrow[dr] & \\ & 98\ (= 100 - 2) \end{tikzcd}$$

- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

[a] See Exercise 13.2.11 of the textbook or p. 545.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

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[a]By Eq. (135) on p. 1039.

[b]Recall p. 488.

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}$$

  by Eq. (134) on p. 1038.

- Hence the discount function $P(t, T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\,r(T) \,|\, r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$

from Eq. (82) on p. 608.

[a]Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody's for USD$210 million in 2002.

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{147}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta}\right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6}\right] & \text{if } \beta = 0. \end{cases}$$

and

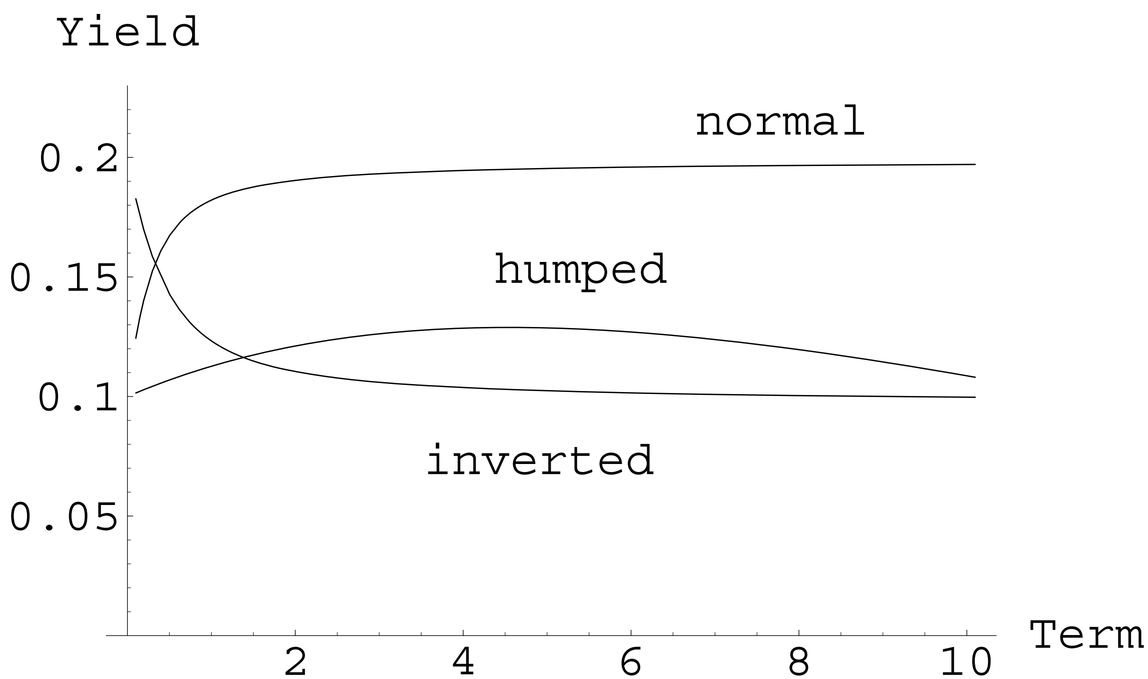
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve

$$(\partial r(t,T)/\partial r)\,\sigma = \sigma B(t,T)/(T-t).$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Higher β leads to greater attenuation of volatility with maturity.


Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
Term
2
4
6
8
10

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\stackrel{\Delta}{=} \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\stackrel{\Delta}{=} \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
X P(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into n identical pieces.
- Let $\Delta t \stackrel{\Delta}{=} T/n$ and

$$p(r) \stackrel{\Delta}{=} \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

---

[a]Nelson & Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0, & \text{if } p(r(k)) < 0, \\ 1, & \text{if } 1 < p(r(k)). \end{cases}$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \tag{148}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, & Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[\,0, T\,]$.
- Divide it into n periods of duration $\Delta t \stackrel{\Delta}{=} T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process[a]

$$x(r) \stackrel{\Delta}{=} 2\sqrt{r}/\sigma.$$

- By Ito's lemma (p. 585),

$$dx = m(x)\,dt + dW,$$

where

$$m(x) \stackrel{\Delta}{=} 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a *constant* volatility.
- Thus its binomial tree combines.

---

[a]See p. 1093ff.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

---

[a] Nawalkha & Beliaeva (2007).

$$
\begin{array}{cccccccccc}
 & & & & x + 2\sqrt{\Delta t} & & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & & & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & & & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
x & & & & x & f(x) & & & & f(x) \\
 & \searrow & & \nearrow & & & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & & & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & & & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t} & & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (continued)

- The probability of an up move at each node r is

$$p(r) \stackrel{\Delta}{=} \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}.$$

  - $r^+ \stackrel{\Delta}{=} f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \stackrel{\Delta}{=} f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Binomial CIR (concluded)

- It can be shown that
$$p(r) = \left(\beta\mu - \frac{\sigma^2}{4}\right)\sqrt{\frac{\Delta t}{r}} - B\sqrt{r\Delta t} + C,$$
for some $B \geq 0$ and $C > 0$.[a]
- If $\beta\mu - (\sigma^2/4) \geq 0$, the up-move probability $p(r)$ decreases if and only if short rate r increases.
- Even if $\beta\mu - (\sigma^2/4) < 0$, $p(r)$ *tends* to decrease as r increases and decrease as r declines.
- This phenomenon agrees with mean reversion.

[a] Thanks to a lively class discussion on May 28, 2014.

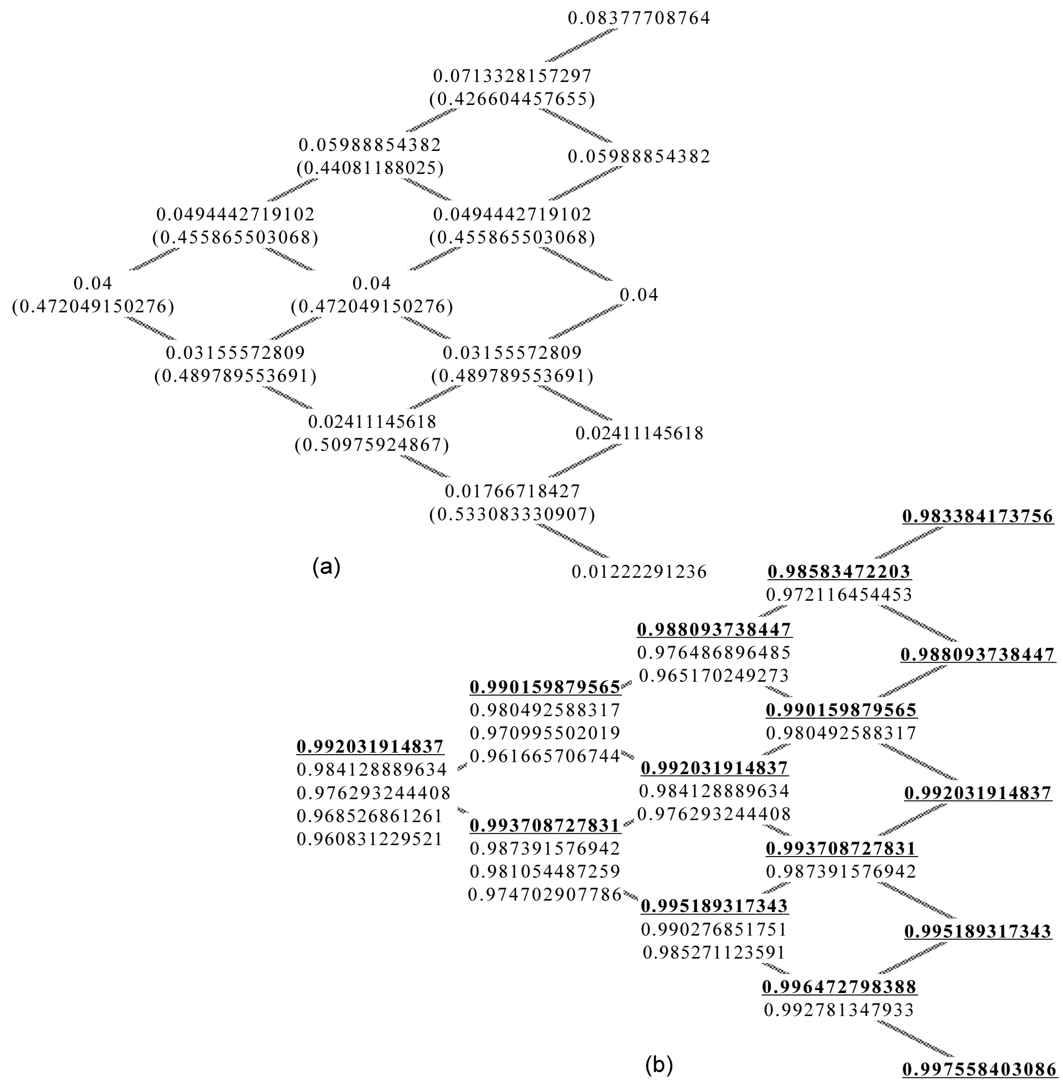
Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 1091(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.


0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
(0.44081188025)
0.05988854382
0.0494442719102
(0.455865503068)
0.0494442719102
(0.455865503068)
0.04
(0.472049150276)
0.04
(0.472049150276)
0.04
0.03155572809
(0.489789553691)
0.03155572809
(0.489789553691)
0.02411145618
(0.50975924867)
0.02411145618
0.01766718427
(0.533083330907)
0.01222291236
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
0.965170249273
0.988093738447
0.990159879565
0.980492588317
0.970995502019
0.961665706744
0.990159879565
0.980492588317
0.992031914837
0.984128889634
0.976293244408
0.968526861261
0.960831229521
0.992031914837
0.984128889634
0.976293244408
0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086

(a)

(b)

Numerical Examples (concluded)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

- Once the short rates are in place, computing the probabilities is easy.
- Convergence is quite good.[a]

[a] See p. 369 of the textbook.

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \stackrel{\Delta}{=} y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \stackrel{\Delta}{=} y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.

[a]Nelson & Ramaswamy (1990).

A General Method (continued)

- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y, t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t + \Delta t)\sqrt{\Delta t} \\ \neq \quad & -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t + \Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.
- When $\sigma(y, t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \triangleq \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows

$$dx = m(y,t)\, dt + dW$$

for some $m(y,t)$.[a]

- The diffusion term is now a constant, and the binomial tree for x combines.

[a] See Exercise 25.2.13 of the textbook.

A General Method (concluded)

- The transformation is unique.[a]
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.

- Note that

$$\begin{aligned} y_{\mathrm{u}}(x,t) &\stackrel{\Delta}{=} y(x+\sqrt{\Delta t}, t+\Delta t), \\ y_{\mathrm{d}}(x,t) &\stackrel{\Delta}{=} y(x-\sqrt{\Delta t}, t+\Delta t). \end{aligned}$$

[a] H. Chiu (R98723059) (2012).

Examples

- The transformation is
$$\int^{r} (\sigma\sqrt{z})^{-1}\, dz = \frac{2\sqrt{r}}{\sigma}$$
for the CIR model.
- The transformation is
$$\int^{S} (\sigma z)^{-1}\, dz = \frac{\ln S}{\sigma}$$
for the Black-Scholes model.
- The familiar BOPM and CRR discretize $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multifactor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.

Options on Coupon Bonds[a]

- Assume the market discount function P is a monotonically decreasing function of the short rate r.
  - Such as a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.

[a]Jamshidian (1989).

Options on Coupon Bonds (continued)

- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left\{\left[\sum_{i=1}^{n} c_i P(r(T), T, t_i)\right] - X, 0\right\}.$$

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.

Options on Coupon Bonds (continued)

- This r^* can be obtained by solving
$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$
numerically for r.

- Let
$$X_i \stackrel{\Delta}{=} P(r^*, T, t_i),$$
the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

- Note that $P(r, T, t_i) \geq X_i$ if and only if $r \leq r^*$.

Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\begin{aligned} & \max\left\{ \left[\sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - \left[\sum_{i=1}^{n} c_i X_i \right], 0 \right\} \\ = \; & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0). \end{aligned}$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a] Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.[a]
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

---

[a] Recall p. 1057.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

---

[a] T. Ho & S. B. Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] T. Ho & S. B. Lee (1986).

$$
\begin{array}{cccccc}
 & & & & & \nearrow \\
 & & & & r_3 & \\
 & & & \nearrow & & \searrow \\
 & & r_2 & & & \\
 & \nearrow & & \searrow & & \nearrow \\
r_1 & & & & r_3 + v_3 & \\
 & \searrow & & \nearrow & & \searrow \\
 & & r_2 + v_2 & & & \\
 & & & \searrow & & \nearrow \\
 & & & & r_3 + 2v_3 & \\
 & & & & & \searrow
\end{array}
$$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$
\begin{array}{ll}
P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots, & \text{if short rate moves down,} \\
P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots, & \text{if short rate moves up.}
\end{array}
$$

- By backward induction, it is not hard to see that for $n \geq 2$,[a]

$$P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2+\cdots+v_n)}. \tag{149}$$

[a] See p. 376 of the textbook.

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$
\begin{aligned}
y_{\mathrm{d}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{d}}(t+1, t+n)}{n-1} \\
y_{\mathrm{u}}(n) &\stackrel{\Delta}{=} -\frac{\ln P_{\mathrm{u}}(t+1, t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1}
\end{aligned}
$$

- The volatility of the yield to maturity for this bond is therefore

$$
\begin{aligned}
\kappa_n &\stackrel{\Delta}{=} \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\
&= \sqrt{p(1-p)}\; (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\
&= \sqrt{p(1-p)}\; \frac{v_2 + \cdots + v_n}{n-1}.
\end{aligned}
$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{150}$$

- The volatility of the short rate therefore equals

$$\sqrt{p(1-p)}\,(r_{\mathrm{u}} - r_{\mathrm{d}}),$$

where r_{u} and r_{d} are the two successor rates.[a]

---

[a]Contrast this with the lognormal model (127) on p. 979.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

$$\kappa_2, \kappa_3, \ldots .$$

  - The volatility structure is supplied by the market.
  - For the Ho-Lee model, it is independent of

$$r_2, r_3, \ldots .$$

- It is easy to compute the v_is from the volatility structure, and vice versa.[a]
- The r_is can be computed by forward induction.

---

[a]Review p. 1113.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy[a]

$$P(t, t+n) = [\, pP_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n)\,]\, P(t, t+1).$$

- Combine the above with Eq. (149) on p. 1112 and assume $p = 1/2$ to obtain[b]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \, \frac{2 \times \exp[\, v_2 + \cdots + v_n\,]}{1 + \exp[\, v_2 + \cdots + v_n\,]},$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \, \frac{2}{1 + \exp[\, v_2 + \cdots + v_n\,]}.$$

---

[a] Recall Eq. (141) on p. 1046.

[b] In the limit, only the volatility matters; the first formula is similar to multiple logistic regression.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.[a]
- Suppose all v_i equal some constant v and $\delta \stackrel{\Delta}{=} e^v > 0$.
- Then

$$\begin{aligned} P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\ P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}. \end{aligned}$$

- Short rate volatility $\sigma = v/2$ by Eq. (150) on p. 1114.
- Price derivatives by taking expectations under the risk-neutral probability.

[a] See Exercise 26.2.3 of the textbook.

Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in $O(n)$ time.
  - Derive the v_i's in linear time.
  - Derive the r_i's in linear time.[a]

[a] See Programming Assignment 26.2.6 of the textbook.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \stackrel{\Delta}{=} \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its two possible value are

$$\ln \frac{P_{\mathrm{d}}(t+1, t+n)}{P(t, t+n)} \quad \text{and} \quad \ln \frac{P_{\mathrm{u}}(t+1, t+n)}{P(t, t+n)}.$$

- Thus the variance of return is

$$\operatorname{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2 \sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is[a]

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is one.
- Strong correlation between rates is inherent in all one-factor Markovian models.

[a]See Exercise 26.2.7 of the textbook.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is[a]

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\, dt + \sigma(t)\, dW.$$

- This corresponds to the discrete-time model in which v_i are not all identical.

[a]See Exercise 26.2.10 of the textbook.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.[a]

---

[a]Recall pp. 1098ff. See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born every day.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 975ff.[b]
- The volatility structure[c] is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with the baseline rates r_i.

---

[a] Black, Derman, & Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

[b] Repeated on next page.

[c] Recall Eq. (133) on p. 1025.

$$
\begin{array}{ccccccc}
 & & & & & & r_4 \\
 & & & & & \nearrow & \\
 & & & & r_3 & & \\
 & & & \nearrow & & \searrow & \\
 & & r_2 & & & & r_4 v_4 \\
 & \nearrow & & \searrow & & \nearrow & \\
r_1 & & & & r_3 v_3 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & r_2 v_2 & & & & r_4 v_4^2 \\
 & & & \searrow & & \nearrow & \\
 & & & & r_3 v_3^2 & & \\
 & & & & & \searrow & \\
 & & & & & & r_4 v_4^3
\end{array}
$$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
- Lognormal models preclude negative short rates.