Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

— Roger Lowenstein, 

*When Genius Failed* (2000)
[The] fixed-income traders I knew seemed smarter than the equity trader ⋅⋅⋅ there’s no competitive edge to being smart in the equities business.[.]

Bond market terminology was designed less to convey meaning than to bewilder outsiders.
Terminology

• A period denotes a unit of elapsed time.
  – Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one — unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

\( t \): a point in time.

\( r(t) \): the one-period riskless rate prevailing at time \( t \) for repayment one period later.\(^a\)

\( P(t, T) \): the present value at time \( t \) of one dollar at time \( T \).

\(^a\)Alternatively, the instantaneous spot rate, or short rate, at time \( t \).
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period.$^a$

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

$^a$In other words, the $(T - t)$-period spot rate at time $t$. 
Standard Notations (concluded)

$f(t, T, L)$: the $L$-period forward rate at time $T$ implied at time $t$ stated on a per-period basis and compounded once per period.

$f(t, T)$: the one-period or instantaneous forward rate at time $T$ as seen at time $t$ stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$. 
Fundamental Relations

- The price of a zero-coupon bond equals

\[
P(t, T) = \begin{cases} 
(1 + r(t, T))^{-(T-t)}, & \text{in discrete time,} \\
\exp^{-r(t,T)(T-t)}, & \text{in continuous time.} 
\end{cases} \tag{134}
\]

- \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

- By definition,

\[
f(t, t) = \begin{cases} 
r(t, t + 1), & \text{in discrete time,} \\
r(t, t), & \text{in continuous time.}
\end{cases}
\]
Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \quad (135) \]

- The forward price equals the future value at time \( T \) of the underlying asset.\(^a\)

- Equation (135) holds whether the model is discrete-time or continuous-time.

\(^a\)See Exercise 24.2.1 of the textbook for proof.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by
  \[
  f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1
  \]
  \[\text{(136)}\]
  in discrete time.

- The analog to Eq. (136) under simple compounding is
  \[
  f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right).
  \]
Fundamental Relations (continued)

• In continuous time,

\[
f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L}
\]  

(137)

by Eq. (135) on p. 1039.

• Furthermore,

\[
f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow - \frac{\partial \ln P(t, T)}{\partial T}
\]

\[
= - \frac{\partial P(t, T)/\partial T}{P(t, T)}.
\]
Fundamental Relations (continued)

• So

\[ f(t, T) \triangleq \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \]  

(138)

• Because Eq. (138) is equivalent to

\[ P(t, T) = e^{-\int_t^T f(t, s) \, ds}, \]  

(139)

the spot rate curve is

\[ r(t, T) = \frac{\int_t^T f(t, s) \, ds}{T - t}. \]
Fundamental Relations (concluded)

• The discrete analog to Eq. (139) is

\[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \]

• The short rate and the market discount function are related by

\[ r(t) = - \frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
Risk-Neutral Pricing

- Assume the local expectations theory.

- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
    \[
    \frac{E_t[P(t + 1, T)]}{P(t, T)} = 1 + r(t).
    \] (140)
  - Relation (140) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^{a}\)Theorem 16 on p. 544.
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.

- Equation (140) on p. 1044 can also be expressed as

$$E_t[P(t + 1, T)] = F(t, t + 1, T).$$

- Verify that with, e.g., Eq. (135) on p. 1039.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.$^a$

---

$^a$Under the local expectations theory. But the forward rate is not an unbiased estimator of the expected future short rate (p. 995).
Risk-Neutral Pricing (continued)

• Rewrite Eq. (140) on p. 1044 as

\[
\frac{E^\pi_t [ \frac{P(t+1,T)}{1+r(t)} ]}{1 + r(t)} = P(t,T). \tag{141}
\]

− It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (concluded)

- Apply the above equality iteratively to obtain

\[
P(t, T) = E^\pi_t \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = E^\pi_t \left[ \frac{E^\pi_{t+1} \left[ P(t + 2, T) \right]}{(1 + r(t))(1 + r(t + 1))} \right] = \ldots = E^\pi_t \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right].
\]
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

$$P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \quad (142)$$

• Note that $e^{\int_t^T r(s) \, ds}$ is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

- Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is $c$ per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$. 
Interest Rate Swaps (continued)

• The amount to be paid out at time $t_{i+1}$ is $(f_i - c) \Delta t$ for the floating-rate payer.

• Simple rates are adopted here.

• Hence $f_i$ satisfies

\[
P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.
\]
Interest Rate Swaps (continued)

• The value of the swap at time $t$ is thus

$$\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_t^{t_i} r(s) \, ds} (f_{i-1} - c) \Delta t \right]$$

$$= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_t^{t_i} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_t^{t_i} r(s) \, ds} \left( e^{\int_{t_{i-1}}^{t_i} r(s) \, ds} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right]$$

$$= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present-value calculations.
Swap Rate

- The swap rate, which gives the swap zero value, equals

\[ S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \] (143)

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

- For an ordinary swap, \( P(t, t_0) = 1 \).
The Term Structure Equation\textsuperscript{a}

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow
  \[
  \frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.
  \]
- At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$.

\textsuperscript{a}Vasicek (1977).
The Term Structure Equation (continued)

• The net wealth change follows

\[ -dP(r, t, s_1) + \alpha dP(r, t, s_2) \]

\[ = (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) \, dt \]

\[ + (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) \, dW. \]

• Pick

\[ \alpha \triangleq \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}. \]
The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

\[ -P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2) \]
\[ -P(r, t, s_1) + \alpha P(r, t, s_2) = r. \]

- Simplify the above to obtain

\[ \frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r. \]

- This becomes

\[ \frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)} \]

after rearrangement.
The Term Structure Equation (continued)

• Since the above equality holds for any $s_1$ and $s_2$,

$$\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \Delta \lambda(r, t) \quad (144)$$

for some $\lambda$ independent of the bond maturity $s$.

• As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

• The term $\lambda(r, t)$ is called the market price of risk.

• The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

- Assume a Markovian short rate model,
  \[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]
- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,
  \[
  \mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right) / P,
  \]
  \[ (145) \]
  \[
  \sigma_p = \left( \frac{\sigma(r, t) \partial P}{\partial r} \right) / P,
  \]
  \[ (145') \]
  subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

• Substitute $\mu_p$ and $\sigma_p$ into Eq. (144) on p. 1057 to obtain

$$-rac{\partial P}{\partial T} + \left[ \mu(r, t) - \lambda(r, t) \sigma(r, t) \right] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP.$$  \hspace{1cm} \text{(146)}

• This is called the term structure equation.

• It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.

• Once $P$ is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$
Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:
Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,
  
  \[
  \frac{100}{1.04} = 96.154, \\
  \frac{100}{(1.05)^2} = 90.703.
  \]
- They follow the binomial processes on p. 1062.
Numerical Examples (continued)

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

• The pricing of derivatives can be simplified by assuming investors are risk-neutral.

• Suppose all securities have the same expected one-period rate of return, the riskless rate.

• Then

\[(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4%,\]

where \(p\) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

• Solving the equation leads to $p = 0.319$.

• Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,

\[
C \quad \begin{align*}
0.000 \\
3.039 &= 98.039 - 95
\end{align*}
\]

• To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580. \)

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

• This price is derived without assuming any version of an expectations theory.

• Instead, the arbitrage-free price is derived by replication.

• The price of an interest rate contingent claim does not depend directly on the real-world probabilities.

• The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

\[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

  \[
  F = 92 \ (= 100 - 8) \quad 98 \ (= 100 - 2)
  \]

- As the futures price $F$ is the expected future payoff,\(^a\)

  \[
  F = (1 - p) \times 92 + p \times 98 = 93.914.
  \]

\(^a\)See Exercise 13.2.11 of the textbook or p. 545.
Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is\(^a\)

\[
\frac{90.703}{96.154} = 94.331\%.
\]

- The forward price exceeds the futures price.\(^b\)

\(^{a}\)By Eq. (135) on p. 1039.

\(^{b}\)Recall p. 488.
Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*
Introduction

• We now survey equilibrium models.

• Recall that the spot rates satisfy

\[ r(t, T) = -\frac{\ln P(t, T)}{T - t} \]

by Eq. (134) on p. 1038.

• Hence the discount function \( P(t, T) \) suffices to establish the spot rate curve.

• All models to follow are short rate models.

• Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model\textsuperscript{a}

- The short rate follows
  \[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).

- Since the process is an Ornstein-Uhlenbeck process,
  \[ E[ r(T) \mid r(t) = r ] = \mu + (r - \mu) e^{-\beta(T-t)} \]
  from Eq. (82) on p. 608.

\textsuperscript{a}Vasicek (1977). Vasicek co-founded KMV, which was sold to Moody’s for USD$210 million in 2002.
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[ P(t, T) = A(t, T) e^{-B(t,T) r(t)}, \quad (147) \]

where

\[ A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t,T)-T+t)(\beta^2 \mu - \sigma^2 / 2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0.
\end{cases} \]

and

\[ B(t, T) = \begin{cases} 
\frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0.
\end{cases} \]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve
  \[ (\partial r(t,T)/\partial r) \sigma = \sigma B(t,T)/(T - t). \]
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Higher $\beta$ leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\textsuperscript{a}

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value $1$ and maturing at time $s > T$.

- Its price is given by

$$P(t, s) N(x) - XP(t, T) N(x - \sigma_v).$$

\textsuperscript{a}Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[ x \triangleq \frac{1}{\sigma_v} \ln \left( \frac{P(t,s)}{P(t,T)X} \right) + \frac{\sigma_v}{2}, \]

\[ \sigma_v \equiv v(t,T)B(T,s), \]

\[ v(t,T)^2 \triangleq \begin{cases} 
\sigma^2 \left[ 1 - e^{-2\beta(T-t)} \right] \\
2\beta 
\end{cases}, \text{ if } \beta \neq 0 \\
\sigma^2(T-t), \text{ if } \beta = 0. \]

- By the put-call parity, the price of a European put is

\[ XP(t,T)N(-x + \sigma_v) - P(t,s)N(-x). \]
Binomial Vasicek

• Consider a binomial model for the short rate in the time interval \([0, T]\) divided into \(n\) identical pieces.

• Let \(\Delta t \triangleq T/n\) and

\[
p(r) \triangleq \frac{1}{2} + \frac{\beta(\mu - r)}{2\sigma} \sqrt{\Delta t}.
\]

• The following binomial model converges to the Vasicek model,\(^a\)

\[
r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.
\]

\(^a\)Nelson & Ramaswamy (1990).
Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with
  
  $$\text{Prob}[\xi(k) = 1] = \begin{cases} 
  p(r(k)), & \text{if } 0 \leq p(r(k)) \leq 1 \\
  0, & \text{if } p(r(k)) < 0, \\
  1, & \text{if } 1 < p(r(k)).
  \end{cases}$$

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

- This is consistent with mean reversion.
Binomial Vasicek (concluded)

• The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.

• The binomial tree combines.

• The key feature of the model that makes it happen is its constant volatility, $\sigma$. 
The Cox-Ingersoll-Ross Model\textsuperscript{a}

- It is the following square-root short rate model:

\[ dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \] (148)

- The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).

- The parameter \( \beta \) determines the speed of adjustment.

- The short rate can reach zero only if \( 2\beta \mu < \sigma^2 \).

- See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, & Ross (1985).
Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t = \frac{T}{n}$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

• Instead, consider the transformed process

\[ x(r) \triangleq 2\sqrt{r/\sigma}. \]

• By Ito’s lemma (p. 585),

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \triangleq \frac{2\beta \mu}{(\sigma^2 x)} - \frac{\beta x}{2} - \frac{1}{2x}. \]

• This new process has a constant volatility.

• Thus its binomial tree combines.

---

\(^a\text{See p. 1093ff.}\)
Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation (see next page)

$$r = f(x) \triangleq \frac{x^2 \sigma^2}{4}.$$ 

- But when $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.\(^{a}\)

\(^{a}\)Nawalkha & Beliaeva (2007).
Binomial CIR (continued)

- The probability of an up move at each node $r$ is

$$p(r) \triangleq \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}.$$ 

- $r^+ \triangleq f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \triangleq f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Binomial CIR (concluded)

• It can be shown that

\[ p(r) = \left( \beta \mu - \frac{\sigma^2}{4} \right) \sqrt{\frac{\Delta t}{r}} - B \sqrt{r \Delta t} + C, \]

for some \( B \geq 0 \) and \( C > 0 \).\(^a\)

• If \( \beta \mu - (\sigma^2/4) \geq 0 \), the up-move probability \( p(r) \) decreases if and only if short rate \( r \) increases.

• Even if \( \beta \mu - (\sigma^2/4) < 0 \), \( p(r) \) tends to decrease as \( r \) increases and decrease as \( r \) declines.

• This phenomenon agrees with mean reversion.

\(^a\)Thanks to a lively class discussion on May 28, 2014.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2\) (year) for the binomial approximation.

• See p. 1091(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (concluded)

• Consider the node which is the result of an up move from the root.

• Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

• Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

• Once the short rates are in place, computing the probabilities is easy.

• Convergence is quite good.\(^\text{a}\)

\(^{\text{a}}\)See p. 369 of the textbook.
A General Method for Constructing Binomial Models\(^a\)

- We are given a continuous-time process,

\[
dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW.
\]

- Need to make sure the binomial model’s drift and diffusion converge to the above process.

- Set the probability of an up move to

\[
\frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}.
\]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

\(^a\)Nelson & Ramaswamy (1990).
A General Method (continued)

- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$
\sigma(y, t)\sqrt{\Delta t} - \sigma(y_u, t + \Delta t)\sqrt{\Delta t}
\neq
-\sigma(y, t)\sqrt{\Delta t} + \sigma(y_d, t + \Delta t)\sqrt{\Delta t}
$$

in general.

- When $\sigma(y, t)$ is a constant independent of $y$, equality holds and the tree combines.
A General Method (continued)

• To achieve this, define the transformation

\[ x(y, t) \triangleq \int_{\sigma(z, t)^{-1}}^{y} \sigma(z, t)^{-1} \, dz. \]

• Then \( x \) follows

\[ dx = m(y, t) \, dt + dW \]

for some \( m(y, t) \).\(^a\)

• The diffusion term is now a constant, and the binomial tree for \( x \) combines.

\(^a\)See Exercise 25.2.13 of the textbook.
A General Method (concluded)

• The transformation is unique.\(^a\)

• The probability of an up move remains

\[
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
\]

where \(y(x, t)\) is the inverse transformation of \(x(y, t)\) from \(x\) back to \(y\).

• Note that

\[
y_u(x, t) \triangleq y(x + \sqrt{\Delta t}, t + \Delta t),
\]

\[
y_d(x, t) \triangleq y(x - \sqrt{\Delta t}, t + \Delta t).
\]

\(^a\)H. Chiu (R98723059) (2012).
Examples

• The transformation is

\[ \int_r^r (\sigma \sqrt{z})^{-1} \, dz = \frac{2\sqrt{r}}{\sigma} \]

for the CIR model.

• The transformation is

\[ \int_S^S (\sigma z)^{-1} \, dz = \frac{\ln S}{\sigma} \]

for the Black-Scholes model.

• The familiar BOPM and CRR discretize \( \ln S \) not \( S \).
On One-Factor Short Rate Models

• By using only the short rate, they ignore other rates on the yield curve.

• Such models also restrict the volatility to be a function of interest rate levels only.

• The prices of all bonds move in the same direction at the same time (their magnitudes may differ).

• The returns on all bonds thus become highly correlated.

• In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.

- Derivatives whose values depend on the correlation structure will be mispriced.

- The calibrated models may not generate term structures as concave as the data suggest.

- The term structure empirically changes in slope and curvature as well as makes parallel moves.

- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multifactor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.
Options on Coupon Bonds\textsuperscript{a}

• Assume the market discount function $P$ is a monotonically decreasing function of the short rate $r$.
  – Such as a one-factor short rate model.

• The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.

• Consider a European call expiring at time $T$ on a bond with par value $1$.

• Let $X$ denote the strike price.

\textsuperscript{a}Jamshidian (1989).
Options on Coupon Bonds (continued)

• The bond has cash flows \( c_1, c_2, \ldots, c_n \) at times \( t_1, t_2, \ldots, t_n \), where \( t_i > T \) for all \( i \).

• The payoff for the option is

\[
\max \left\{ \left[ \sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - X, 0 \right\}.
\]

• At time \( T \), there is a unique value \( r^* \) for \( r(T) \) that renders the coupon bond’s price equal the strike price \( X \).
Options on Coupon Bonds (continued)

- This $r^*$ can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for $r$.

- Let

$$X_i \triangleq P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$.

- Note that $P(r, T, t_i) \geq X_i$ if and only if $r \leq r^*$. 
Options on Coupon Bonds (concluded)

• As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\max \left\{ \left[ \sum_{i=1}^n c_i P(r(T), T, t_i) \right] - \left[ \sum_{i=1}^n c_i X_i \right], 0 \right\}$$

$$= \sum_{i=1}^n c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

• Thus the call is a package of $n$ options on the underlying zero-coupon bond.

• Why can’t we do the same thing for Asian options?\footnote{Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.}
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)
Motivations

• Recall the difficulties facing equilibrium models mentioned earlier.
  – They usually require the estimation of the market price of risk.\(^a\)
  – They cannot fit the market term structure.
  – But consistency with the market is often mandatory in practice.

\(^a\)Recall p. 1057.
No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.

- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.

- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.

- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

---

\[a\] T. Ho & S. B. Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to Bloomberg on May 26, 2012.
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.
The Ho-Lee Model

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

---

*T. Ho & S. B. Lee (1986).*
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices \( P(t, t+1), P(t, t+2), \ldots \) at time \( t \) identified with the root of the tree.

- Let the discount factors in the next period be
  \[
P_d(t+1, t+2), P_d(t+1, t+3), \ldots, \quad \text{if short rate moves down},
  \]
  \[
P_u(t+1, t+2), P_u(t+1, t+3), \ldots, \quad \text{if short rate moves up}.
  \]

- By backward induction, it is not hard to see that for \( n \geq 2 \),
  \[
  P_u(t+1, t+n) = P_d(t+1, t+n) e^{-(v_2+\cdots+v_n)}.
  \]

\[(149)\]

\( a \)See p. 376 of the textbook.
The Ho-Lee Model (continued)

- It is also not hard to check that the \( n \)-period zero-coupon bond has yields

\[
y_d(n) \triangleq - \frac{\ln P_d(t + 1, t + n)}{n - 1}
\]

\[
y_u(n) \triangleq - \frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}
\]

- The volatility of the yield to maturity for this bond is therefore

\[
\kappa_n \triangleq \sqrt{p y_u(n)^2 + (1 - p) y_d(n)^2 - [p y_u(n) + (1 - p) y_d(n)]^2}
\]

\[
= \sqrt{p(1 - p)} (y_u(n) - y_d(n))
\]

\[
= \sqrt{p(1 - p)} \frac{v_2 + \cdots + v_n}{n - 1}.
\]
The Ho-Lee Model (concluded)

• In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1 - p)} \, v_2. \quad (150)$$

• The volatility of the short rate therefore equals

$$\sqrt{p(1 - p) \, (r_u - r_d)},$$

where $r_u$ and $r_d$ are the two successor rates.\textsuperscript{a}

\textsuperscript{a}Contrast this with the lognormal model (127) on p. 979.
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

  \( \kappa_2, \kappa_3, \ldots \). \n
  - The volatility structure is supplied by the market.
  - For the Ho-Lee model, it is independent of

    \( r_2, r_3, \ldots \).

- It is easy to compute the \( v_i \)s from the volatility structure, and vice versa.\textsuperscript{a}

- The \( r_i \)s can be computed by forward induction.

\textsuperscript{a}Review p. 1113.
The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = \left[pP_u(t+1, t+n) + (1-p) P_d(t+1, t+n)\right] P(t, t+1). \]

- Combine the above with Eq. (149) on p. 1112 and assume \( p = 1/2 \) to obtain

\[
\begin{align*}
P_d(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \\
P_u(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}.
\end{align*}
\]

---

\( ^a \) Recall Eq. (141) on p. 1046.

\( ^b \) In the limit, only the volatility matters; the first formula is similar to multiple logistic regression.
The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.\(^a\)

- Suppose all \(v_i\) equal some constant \(v\) and \(\delta \Delta = e^v > 0\).

- Then

\[
P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}},
\]
\[
P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.
\]

- Short rate volatility \(\sigma = v/2\) by Eq. (150) on p. 1114.

- Price derivatives by taking expectations under the risk-neutral probability.

\(^a\)See Exercise 26.2.3 of the textbook.
Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.

- But it can actually be calibrated in $O(n)$ time.
  - Derive the $v_i$’s in linear time.
  - Derive the $r_i$’s in linear time.\(^a\)

\(^a\)See Programming Assignment 26.2.6 of the textbook.
The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an \( n \)-period zero-coupon bond is
  \[
  r(t, t + n) \triangleq \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).
  \]

- Its two possible value are
  \[
  \ln \frac{P_d(t + 1, t + n)}{P(t, t + n)} \quad \text{and} \quad \ln \frac{P_u(t + 1, t + n)}{P(t, t + n)}.
  \]

- Thus the variance of return is
  \[
  \text{Var}[r(t, t + n)] = p(1 - p)((n - 1)\sigma)^2 = (n - 1)^2\sigma^2.
  \]
The Ho-Lee Model: Yields and Their Covariances 
(concluded)

- The covariance between $r(t, t + n)$ and $r(t, t + m)$ is$^a$
  \[(n - 1)(m - 1) \sigma^2.\]

- As a result, the correlation between any two one-period 
rates of return is one.

- Strong correlation between rates is inherent in all 
one-factor Markovian models.

$^a$See Exercise 26.2.7 of the textbook.
The Ho-Lee Model: Short Rate Process

• The continuous-time limit of the Ho-Lee model is\(^a\)

\[
dr = \theta(t) \, dt + \sigma \, dW.
\]

• This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

• A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

\[
dr = \theta(t) \, dt + \sigma(t) \, dW.
\]

• This corresponds to the discrete-time model in which \(v_i\) are not all identical.

\(^a\)See Exercise 26.2.10 of the textbook.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
- It has all the problems associated with a one-factor model.\textsuperscript{a}

\textsuperscript{a}Recall pp. 1098ff. See T. Ho & S. B. Lee (2004) for a multifactor Ho-Lee model.
Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.
- Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born every day.
Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.

- Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.
The Black-Derman-Toy Model\textsuperscript{a}

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 975ff.\textsuperscript{b}
- The volatility structure\textsuperscript{c} is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with the baseline rates $r_i$.

\textsuperscript{a}Black, Derman, & Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

\textsuperscript{b}Repeated on next page.

\textsuperscript{c}Recall Eq. (133) on p. 1025.
The Black-Derman-Toy Model (concluded)

• Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.

• Lognormal models preclude negative short rates.