Time-Dependent Short Rates

• Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.

• The riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.

• In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i \tau}{\tau},$$

where $r_i$ is the continuously compounded short rate measured in periods for period $i$.\(^a\)

• Will the binomial tree fail to combine?

\(^a\)That is, one-period forward rate.
Trading Days and Calendar Days

- Interest accrues based on the calendar day.

- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.$^a$

- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?$^b$

---

$^a$Fama (1965); K. French (1980); K. French & Roll (1986).

$^b$Recall p. 158 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the *annualized* volatility of stock price *one year from now*.

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with\(^a\)

\[
\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}
\]

\(^a\)D. French (1984).
Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?\textsuperscript{a}

\textsuperscript{a}Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.
Options on a Stock That Pays Dividends

• Early exercise must be considered.

• Proportional dividend payout model is tractable (see text).
  – The dividend amount is a constant proportion of the prevailing stock price.

• In general, the corporate dividend policy is a complex issue.
Known Dividends

- Constant dividends introduce complications.
- Use $D$ to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D) u$, $(Su - D) d$, $(Sd - D) u$, $(Sd - D) d$.
  - The binomial tree no longer combines.
\[(Su - D)u\]
\[(Su - D)\]
\[(Su - D)d\]
\[S\]
\[(Sd - D)u\]
\[(Sd - D)\]
\[(Sd - D)d\]
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.$^a$

- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.

- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, $\sigma$ is the volatility of the process followed by the risky component.

- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

---

$^a$Roll (1977).
An Ad-Hoc Approximation (concluded)

• Start with the current stock price minus the PV of future dividends before expiration.

• Develop the binomial tree for the new stock price as if there were no dividends.

• Then add to each stock price on the tree the PV of all future dividends before expiration.

• American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 312 (Step 1)

\[(S - D/R)u^2\]
The Ad-Hoc Approximation vs. P. 312 (Step 2)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R) + D/R = S\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 312

- The trees are different.
- The stock prices at maturity are also different.
  - \((Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d\) (p. 312).
  - \((S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2\) (ad hoc).
- Note that, as \(d < R < u\),
  \[
  (Su - D)u > (S - D/R)u^2, \\
  (Sd - D)d < (S - D/R)d^2,
  \]

---

\(^{a}\)Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.
The Ad-Hoc Approximation vs. P. 312 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach\textsuperscript{a}

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 750ff).
- Other approaches include adjusting $\sigma$ and approximating the known dividend with a dividend yield.\textsuperscript{b}

\textsuperscript{a}Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

\textsuperscript{b}Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).
Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.

- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_{\tau}$ with a continuous dividend yield of $q$ would grow from $S$ to $S_{\tau}e^{q\tau}$ without the dividends.

- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, only the distribution of $S_{\tau}$ matters.
Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$.

\[
C = Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \\
P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x), \\
\]

where

\[
x \triangleq \ln(S/X) + \left(r - q + \sigma^2/2\right)\tau \\
\sigma\sqrt{\tau}.
\]

- Formulas (41) and (41') remain valid as long as the dividend yield is predictable.

\[\text{aMerton (1973).}\]
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace \( u \) with \( ue^{-q\Delta t} \) and \( d \) with \( de^{-q\Delta t} \), where \( \Delta t \triangleq \tau/n \).
  - The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, \( p \) should use the original \( u \) and \( d! \).

\(^a\)Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.
Continuous Dividend Yields (concluded)

• Alternatively, pick the risk-neutral probability as

\[
e^{(r-q)\Delta t - d} \quad \frac{e^{(r-q)\Delta t} - d}{u - d},
\]

(42)

where \( \Delta t \triangleq \tau/n \).

– The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

• The \( u \) and \( d \) remain unchanged.

• Other than the change in Eq. (42), binomial tree algorithms stay the same as if there were no dividends.
Exercise Boundaries of American Options (in the Continuous-Time Model)\(^a\)

- The exercise boundary is a nondecreasing function of \( t \) for American puts (see the plot next page).

- The exercise boundary is a nonincreasing function of \( t \) for American calls.

\(^a\)See Section 9.7 of the textbook for the tree analog.
Risk Reversals\textsuperscript{a}

- From formulas (41) and (41′) on p. 321, one can verify that $C = P$ when

$$X = Se^{(r-q)\tau}.$$

- A risk reversal consists of a short out-of-the-money put and a long out-of-the-money call with the same maturity.\textsuperscript{b}

- Furthermore, the portfolio has zero value.

- A short risk reversal position is also called a collar.\textsuperscript{c}

\textsuperscript{a}Neftci (2008).
\textsuperscript{b}Thus their strike prices must be distinct.
\textsuperscript{c}Bennett (2014).
Sensitivity Analysis of Options
Cleopatra’s nose, had it been shorter, the whole face of the world would have been changed.
— Blaise Pascal (1623–1662)
Sensitivity Measures ("The Greeks")

• How the value of a security changes relative to changes in a given parameter is key to hedging.
  – Duration, for instance.

• Let \( x \triangleq \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} \) (recall p. 291).

• Recall that

\[
N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,
\]

the density function of standard normal distribution.
Delta

- Defined as
  \[ \Delta \triangleq \frac{\partial f}{\partial S} \]
  - \( f \) is the price of the derivative.
  - \( S \) is the price of the underlying asset.

- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.\(^a\)

- The delta used in the BOPM (p. 238) is the discrete analog.

- The delta of a long stock is apparently 1.

\(^a\)Elementary calculus.
Delta (continued)

• The delta of a European call on a non-dividend-paying stock equals
  \[ \frac{\partial C}{\partial S} = N(x) > 0. \]

• The delta of a European put equals
  \[ \frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0. \]

• So the deltas of a call and an otherwise identical put cancel each other when \( N(x) = 1/2 \), i.e., when \(^{a} \)
  \[ X = S e^{(r+\sigma^2/2) \tau}. \] (43)

\(^{a}\) The straddle (p. 206) \( C + P \) then has zero delta!
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curves: out-of-the-money calls or in-the-money puts.
Delta (continued)

• Suppose the stock pays a continuous dividend yield of $q$.

• Let

$$x \triangleq \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}$$  \hspace{1cm} (44)$$

(recall p. 321).

• Then

$$\frac{\partial C}{\partial S} = e^{-q\tau} N(x) > 0,$$

$$\frac{\partial P}{\partial S} = -e^{-q\tau} N(-x) < 0.$$
Delta (continued)

- Consider an $X_1$-strike call and an $X_2$-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let
  \[
  x_i \triangleq \frac{\ln(S/X_i) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. 
  \]
  (45)
- Then their deltas sum to zero when $x_1 = -x_2$.
- That implies
  \[
  \frac{S}{X_1} = \frac{X_2}{S} e^{-(2r-2q+\sigma^2) \tau}. 
  \]
  (46)

\[a\] The strangle (p. 208) $C + P$ then has zero delta!
Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then
  \[ X = S e^{(r-q+\sigma^2/2)\tau} \]
  leads to a straddle with zero delta.
  - This generalizes Eq. (43) on p. 331.
- When $C(X_1)$’s delta and $P(X_2)$’s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.
Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.

- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and \(-\Delta\) shares of stock is delta-neutral.
  - Short \(\Delta\) shares of stock to hedge a long call.
  - Long \(\Delta\) shares of stock to hedge a short call.

- In general, hedge a position in a security with delta \(\Delta_1\) by shorting \(\Delta_1/\Delta_2\) units of a security with delta \(\Delta_2\).
Theta (Time Decay)

- Defined as the rate of change of a security’s value with respect to time, or $\Theta \triangleq -\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial t}$.

- For a European call on a non-dividend-paying stock,

  \[ \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0. \]

  - The call loses value with the passage of time.

- For a European put,

  \[ \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}). \]

  - Can be negative or positive.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curve: out-of-the-money call or in-the-money put.
Theta (concluded)

- Suppose the stock pays a continuous dividend yield of $q$.
- Define $x$ as in Eq. (44) on p. 333.
- For a European call, add an extra term to the earlier formula for the theta:
  \[ \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) + qSe^{-q\tau}N(x). \]
- For a European put, add an extra term to the earlier formula for the theta:
  \[ \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - qSe^{-q\tau}N(-x). \]
Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or \( \Gamma \triangleq \frac{\partial^2 \Pi}{\partial S^2} \).
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs to be rebalanced more often to maintain delta neutrality.
- Roughly, delta \( \sim \) duration, and gamma \( \sim \) convexity.
- The gamma of a European call or put on a non-dividend-paying stock is
  \[
  \frac{N'(x)}{(S\sigma\sqrt{\tau})} > 0.
  \]
Dotted lines: in-the-money call or out-of-the-money put.
Solid lines: at-the-money option.
Dashed lines: out-of-the-money call or in-the-money put.
Vega\(^a\) (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security’s value with respect to the volatility of the underlying asset
  \[ \Lambda \triangleq \frac{\partial f}{\partial \sigma}. \]
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is \( S \sqrt{\tau} N'(x) > 0. \)
  - So higher volatility always increases the option value.

\(^a\)Vega is not Greek.
Vega (continued)

• Note that

\[ \Lambda = \tau \sigma S^2 \Gamma. \]

• If the stock pays a continuous dividend yield of \( q \), then

\[ \Lambda = Se^{-q\tau} \sqrt{\tau} N'(x), \]

where \( x \) is defined in Eq. (44) on p. 333.

• Vega is maximized when \( x = 0 \), i.e., when

\[ S = X e^{-(r-q+\sigma^2/2) \tau}. \]

• Vega declines very fast as \( S \) moves away from that peak.

---

\(^a\text{Reiss & Wystup (2001).}\)
Vega (continued)

• Now consider a portfolio consisting of an $X_1$-strike call $C$ and a short $X_2$-strike put $P$, $X_1 \geq X_2$.

• The options’ vegas cancel out when

\[ x_1 = -x_2, \]

where $x_i$ are defined in Eq. (45) on p. 334.

• This leads to Eq. (46) on p. 334.
  – Recall the same condition led to zero delta for the strangle $C + P$ (p. 334).
Vega (concluded)

• Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 296 for the Black-Scholes model).

• The Black-Scholes formula (p. 291) implies

$$C \to S,$$

$$P \to Xe^{-r\tau},$$

as $\sigma \to \infty$.

• These boundary conditions may be handy for certain numerical methods.
Dotted curve: in-the-money call or out-of-the-money put.  
Solid curves: at-the-money option.  
Dashed curve: out-of-the-money call or in-the-money put.
Variance Vega\textsuperscript{a}

- Defined as the rate of change of a security’s value with respect to the variance (square of volatility) of the underlying asset

\[ V \triangleq \frac{\partial f}{\partial \sigma^2} . \]

- It is easy to verify that

\[ V = \frac{\Lambda}{2\sigma} . \]

\textsuperscript{a}Demeterfi, Derman, Kamal, & Zou (1999).
Rho

• Defined as the rate of change in its value with respect to interest rates

\[ \rho \triangleq \frac{\partial f}{\partial r}. \]

• The rho of a European call on a non-dividend-paying stock is

\[ X \tau e^{-r \tau} N(x - \sigma \sqrt{\tau}) > 0. \]

• The rho of a European put on a non-dividend-paying stock is

\[ -X \tau e^{-r \tau} N(-x + \sigma \sqrt{\tau}) < 0. \]
Dotted curves: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curves: out-of-the-money call or in-the-money put.
Numerical Greeks

• Needed when closed-form formulas do not exist.

• Take delta as an example.

• A standard method computes the finite difference,

\[
\frac{f(S + \Delta S) - f(S - \Delta S)}{2 \Delta S}.
\]

• The computation time roughly doubles that for evaluating the derivative security itself.
An Alternative Numerical Delta

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, $f_u$ and $f_d$ are computed.
- These values correspond to derivative values at stock prices $S_u$ and $S_d$, respectively.
- Delta is approximated by

\[ \frac{f_u - f_d}{S_u - S_d}. \]  

(47)

- Almost zero extra computational effort.

\[ ^{a}\text{Pelsser & Vorst (1994).} \]
Numerical Gamma

• At the stock price \((S_{uu} + S_{ud})/2\), delta is approximately \((f_{uu} - f_{ud})/(S_{uu} - S_{ud})\).

• At the stock price \((S_{ud} + S_{dd})/2\), delta is approximately \((f_{ud} - f_{dd})/(S_{ud} - S_{dd})\).

• Gamma is the rate of change in deltas between \((S_{uu} + S_{ud})/2\) and \((S_{ud} + S_{dd})/2\), that is,

\[
\frac{f_{uu} - f_{ud}}{S_{uu} - S_{ud}} - \frac{f_{ud} - f_{dd}}{S_{ud} - S_{dd}} \cdot \frac{1}{(S_{uu} - S_{dd})/2}.
\]

(48)

• Alternative formulas exist (p. 654).
Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives
  \[
  \frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.
  \]

- It does not work (see text for the reason).

- In general, calculating gamma is a hard problem numerically.

- But why did the binomial tree version work?
Other Numerical Greeks

• The theta can be computed as

\[ \frac{f_{ud} - f}{2(\tau/n)}. \]

  – In fact, the theta of a European option can be derived from delta and gamma (p. 653).

• The vega of a European option can be derived from gamma (p. 343).

• For rho, there seems no alternative but to run the binomial tree algorithm twice.\(^a\)

\(^a\)But see p. 822 and pp. 1004ff.
Extensions of Options Theory
Data! data! data!

— *The Adventures of Sherlock Holmes* (1892)

As I never learnt mathematics,  
so I have had to think.

— Joan Robinson (1903–1983)
Pricing Corporate Securities\(^a\)

- Interpret the underlying asset as the total value of the firm.

- The option pricing methodology can be applied to price corporate securities.
  - The result is called the structural model.

- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

\(^a\)Black & Scholes (1973); Merton (1974).
Risky Zero-Coupon Bonds and Stock

• Consider XYZ.com.

• Capital structure:
  – $n$ shares of its own common stock, $S$.
  – Zero-coupon bonds with an aggregate par value of $X$.

• What is the value of the bonds, $B$?

• What is the value of the XYZ.com stock?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds’ maturity date, suppose the total value of the firm $V^*$ is less than the bondholders’ claim $X$.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

<table>
<thead>
<tr>
<th>$V^*$ $\leq$ $X$</th>
<th>$V^* &gt; X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>$V^*$</td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
</tr>
</tbody>
</table>
Risky Zero-Coupon Bonds and Stock (continued)

• The stock has the same payoff as a call!

• It is a call on the total value of the firm with a strike price of $X$ and an expiration date equal to the bonds’.
  – This call provides the limited liability for the stockholders.

• The bonds are a covered call\(^{a}\) on the total value of the firm.

• Let $V$ stand for the total value of the firm.

• Let $C$ stand for a call on $V$.

\(^{a}\)See p. 197.
Risky Zero-Coupon Bonds and Stock (continued)

• Thus

\[ nS = C, \]
\[ B = V - C. \]

• Knowing \( C \) amounts to knowing how the value of the firm is divided between stockholders and bondholders.

• Whatever the value of \( C \), the total value of the stock and bonds at maturity remains \( V^* \).

• The relative size of debt and equity is irrelevant to the firm’s current value \( V \).
Risky Zero-Coupon Bonds and Stock (continued)

• From Theorem 10 (p. 291) and the put-call parity,

\[
\begin{align*}
\nu S &= VN(x) - X e^{-r\tau} N(x - \sigma \sqrt{\tau}), \\
B &= VN(-x) + X e^{-r\tau} N(x - \sigma \sqrt{\tau}).
\end{align*}
\]

(49) (50)

– Above,

\[
x \triangleq \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma \sqrt{\tau}}.
\]

• The continuously compounded yield to maturity of the firm’s bond is

\[
\frac{\ln(X/B)}{\tau}.
\]

\(^a\)Merton (1974).
Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

\[
\frac{\ln(X/B)}{\tau} - r = \frac{1}{\tau} \ln \left( N(-z) + \frac{1}{\omega} N(z - \sigma \sqrt{\tau}) \right).
\]

- \( \omega \triangleq X e^{-r\tau}/V. \)

- \( z \triangleq (\ln \omega)/(\sigma \sqrt{\tau}) + (1/2) \sigma \sqrt{\tau} = -x + \sigma \sqrt{\tau}. \)

- Note that \( \omega \) is the debt-to-total-value ratio.
Risky Zero-Coupon Bonds and Stock (concluded)

• In general, suppose the firm has a dividend yield at rate $q$ and the bankruptcy costs are a constant proportion $\alpha$ of the remaining firm value.

• Then Eqs. (49)–(50) on p. 363 become, respectively,

\[
\begin{align*}
nS & = Ve^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \\
B & = (1 - \alpha)Ve^{-q\tau}N(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}).
\end{align*}
\]

– Above,

\[
x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]
A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.
- XYZ.com’s securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$. 
<table>
<thead>
<tr>
<th>Option</th>
<th>Strike</th>
<th>Exp.</th>
<th>Vol.</th>
<th>Last</th>
<th>Vol.</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merck</td>
<td>30</td>
<td>Jul</td>
<td>328</td>
<td>151/4</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>441/2</td>
<td>35</td>
<td>Jul</td>
<td>150</td>
<td>91/2</td>
<td>10</td>
<td>1/16</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Apr</td>
<td>887</td>
<td>43/4</td>
<td>136</td>
<td>1/16</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Jul</td>
<td>220</td>
<td>51/2</td>
<td>297</td>
<td>1/4</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Oct</td>
<td>58</td>
<td>6</td>
<td>10</td>
<td>1/2</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Apr</td>
<td>3050</td>
<td>7/8</td>
<td>100</td>
<td>11/8</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>May</td>
<td>462</td>
<td>13/8</td>
<td>50</td>
<td>13/8</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Jul</td>
<td>883</td>
<td>115/16</td>
<td>147</td>
<td>13/4</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Oct</td>
<td>367</td>
<td>23/4</td>
<td>188</td>
<td>21/16</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for $15.25$.
- So XYZ.com’s stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

dollars.
- Or $975$ per bond.
A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $X$ par value plus $n$ written European puts on Merck at a strike price of $30$.
  - By the put-call parity.\textsuperscript{a}

- The difference between $B$ and the price of the default-free bond is the value of these puts.

- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts $X$.

\textsuperscript{a}See p. 220.
<table>
<thead>
<tr>
<th>Promised payment to bondholders $X$</th>
<th>Current market value of bonds $B$</th>
<th>Current market value of stock $nS$</th>
<th>Current total value of firm $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30,000</td>
<td>29,250.0</td>
<td>15,250.0</td>
<td>44,500</td>
</tr>
<tr>
<td>35,000</td>
<td>35,000.0</td>
<td>9,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>40,000</td>
<td>39,000.0</td>
<td>5,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>45,000</td>
<td>42,562.5</td>
<td>1,937.5</td>
<td>44,500</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for $115/16$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.
A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option’s terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such as volatility, dividend, and strike price are under partial control of the stockholders or their boards.
A Numerical Example (continued)

• Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.

• The total debt is now $X = 45,000$ dollars.

• The table on p. 370 says the total market value of the bonds should be $42,562.5$.

• The new bondholders pay

\[42,562.5 \times (15/45) = 14,187.5\]

dollars.

• The remaining stock is worth $1,937.5$. 
A Numerical Example (continued)

- The stockholders therefore gain

\[ 14,187.5 + 1,937.5 - 15,250 = 875 \] dollars.

- The original bondholders lose an equal amount,

\[ 29,250 - \frac{30}{45} \times 42,562.5 = 875. \]

- This is called claim dilution.\(^a\)

\(^{a}\)Fama & Miller (1972).
A Numerical Example (continued)

• Suppose the stockholders sell \((1/3) \times n\) Merck shares to fund a $14,833.3 cash dividend.

• They now have $14,833.3 in cash plus a call on \((2/3) \times n\) Merck shares.

• The strike price remains \(X = 30,000\).

• This is equivalent to owning \(2/3\) of a call on \(n\) Merck shares with a strike price of $45,000.

• \(n\) such calls are worth $1,937.5 (p. 370).

• So the total market value of the XYZ.com stock is \((2/3) \times 1,937.5 = 1,291.67\) dollars.
A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

\[(2/3) \times n \times 44.5 - 1,291.67 = 28,375\]

dollars.

- Hence the stockholders gain

\[14,833.3 + 1,291.67 - 15,250 \approx 875\]

dollars.

- The bondholders watch their value drop from $29,250 to $28,375, a loss of $875.
Further Topics

• Other Examples:
  – Subordinated debts as bull call spreads.
  – Warrants as calls.
  – Callable bonds as American calls with 2 strike prices.
  – Convertible bonds.

• Securities with a complex liability structure must be solved by trees.\(^a\)

\(^a\)Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).
Barrier Options

- Their payoff depends on whether the underlying asset’s price reaches a certain price level $H$ throughout its life.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier $H$ is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a \textit{down-and-out} option if $H < S$.
- A put knock-out option is sometimes called an \textit{up-and-out} option when $H > S$.

\footnote{A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.}
Barrier Options (continued)

- A knock-in (KI) option comes into existence if a certain barrier is reached.

- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.

- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.

- Formulas exist for all the possible barrier options mentioned above.\(^a\)

\(^a\)Haug (2006).
Barrier Options (concluded)

- Knock-in puts are the most popular barrier options.\textsuperscript{a}
- Knock-out puts are the second most popular barrier options.\textsuperscript{b}
- Knock-out calls are the most popular among barrier call options.\textsuperscript{c}

\textsuperscript{a}Bennett (2014).
\textsuperscript{b}Bennett (2014).
\textsuperscript{c}Bennett (2014).
A Formula for Down-and-In Calls\textsuperscript{a}

- Assume \( X \geq H \).
- The value of a European down-and-in call on a stock paying a dividend yield of \( q \) is
\[
Se^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(x) - Xe^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}),
\]
\[(51)\]

\[
- x \triangleq \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]

\[
- \lambda \triangleq \frac{(r - q + \sigma^2/2)}{\sigma^2}.
\]

- A European down-and-out call can be priced via the in-out parity (see text).

\textsuperscript{a}Merton (1973). See Exercise 17.1.6 of the textbook for a proof.
A Formula for Up-and-In Puts\textsuperscript{a}

- Assume $X \leq H$.
- The value of a European up-and-in put is
  \[
  X e^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda - 2} N(-x + \sigma \sqrt{\tau}) - S e^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(-x).
  \]
- Again, a European up-and-out put can be priced via the in-out parity.

\textsuperscript{a}Merton (1973).
Are American Options Barrier Options?\textsuperscript{a}

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

\textsuperscript{a}Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.
Interesting Observations

• Assume $H < X$.

• Replace $S$ in the Merton pricing formula Eq. (41) on p. 321 for the call with $H^2/S$.

• Equation (51) on p. 382 for the down-and-in call becomes Eq. (41) when $r - q = \sigma^2/2$.

• Equation (51) becomes $S/H$ times Eq. (41) when $r - q = 0$. 

©2018 Prof. Yuh-Dauh Lyuu, National Taiwan University
Interesting Observations (concluded)

• Replace $S$ in the pricing formula for the down-and-in call, Eq. (51), with $H^2/S$.

• Equation (51) becomes Eq. (41) when $r - q = \sigma^2/2$.

• Equation (51) becomes $H/S$ times Eq. (41) when $r - q = 0$.\(^a\)

• Why?\(^b\)

\(^a\)Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.
\(^b\)Apply the reflection principle (p. 683), Eq. (40) on p. 284, and Lemma 9 (p. 289).
Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.

- Pricing down-and-in options is subtler.
$S = 8, \ X = 6, \ H = 4, \ R = 1.25, \ u = 2, \text{ and } d = 0.5.$

Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25.$
Binomial Tree Algorithms (continued)

- But convergence is erratic because $H$ is not at a price level on the tree (see plot on next page).\textsuperscript{a}
  - The barrier $H$ is moved lower (or higher) to a closeby node price.
  - This “effective barrier” thus changes as $n$ increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.\textsuperscript{b}
- Solutions will be presented later.

\textsuperscript{a}Boyle & Lau (1994).
\textsuperscript{b}J. Lin (R95221010) (2008).
Binomial Tree Algorithms (concluded)\textsuperscript{a}

\textbf{Down-and-in call value}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig}
\end{figure}

\textsuperscript{a}Lyuu (1998).
Daily Monitoring

- Many barrier options monitor the barrier only for daily closing prices.

- If so, only nodes at the end of a day need to check for the barrier condition.

- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into $d$ periods.

- Does this save time or space?\textsuperscript{a}

\textsuperscript{a}Contributed by Ms. Chen, Tzu-Chun (R94922003) and others on April 12, 2006.
A Heptanomial Tree (6 Periods Per Day)
Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.\(^a\)
- They are equally popular for indices.\(^b\)

\(^a\text{Bennett (2014).}\)
\(^b\text{Bennett (2014).}\)