Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.
- The riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,

$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$

where r_i is the continuously compounded short rate measured in periods for period i.[a]

- Will the binomial tree fail to combine?

[a] That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  – Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a]Fama (1965); K. French (1980); K. French & Roll (1986).

[b]Recall p. 158 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma\sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

---

[a] D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

[a] Contributed by Mr. Lu, Zheng-Liang (`D00922011`) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.

$$
\begin{tikzcd}[column sep=small]
 & & (Su-D)\,u \\
 & Su-D \arrow[ur] \arrow[dr] & \\
S \arrow[ur] \arrow[ddr] & & (Su-D)\,d \\
 & & (Sd-D)\,u \\
 & Sd-D \arrow[ur] \arrow[dr] & \\
 & & (Sd-D)\,d
\end{tikzcd}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

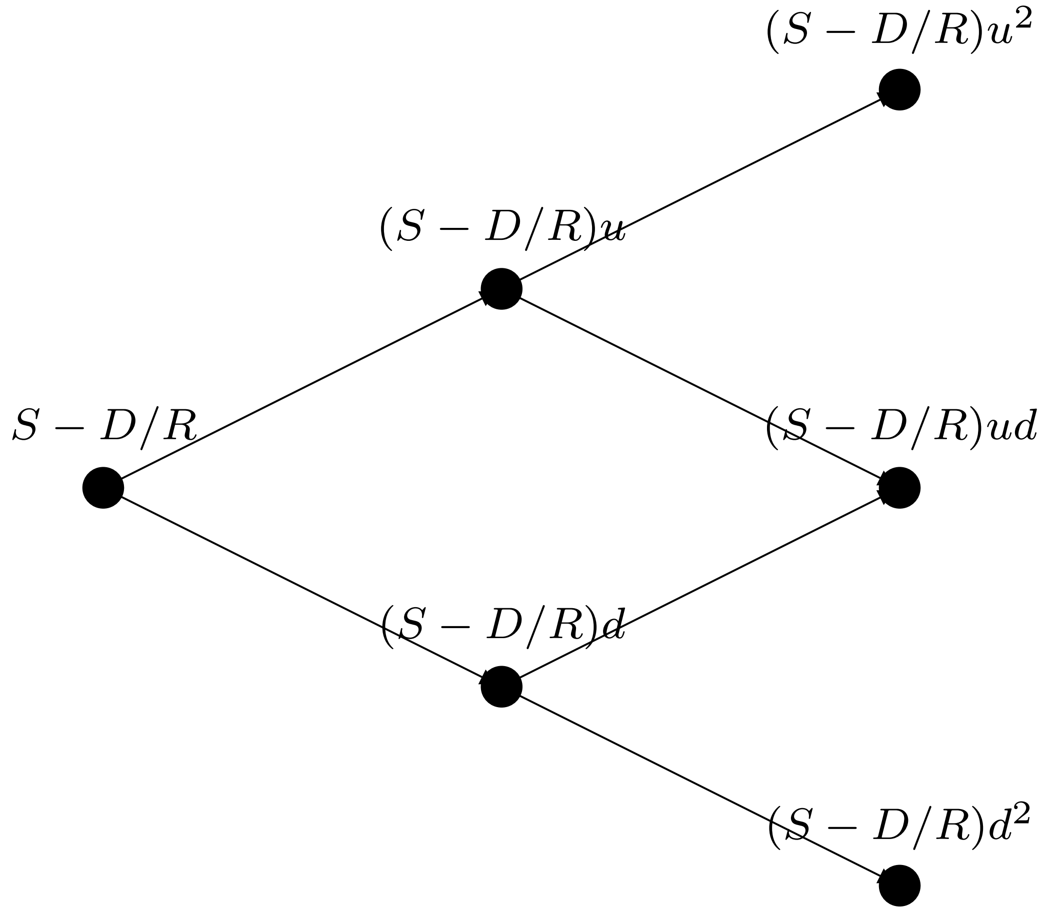
---

[a]Roll (1977).

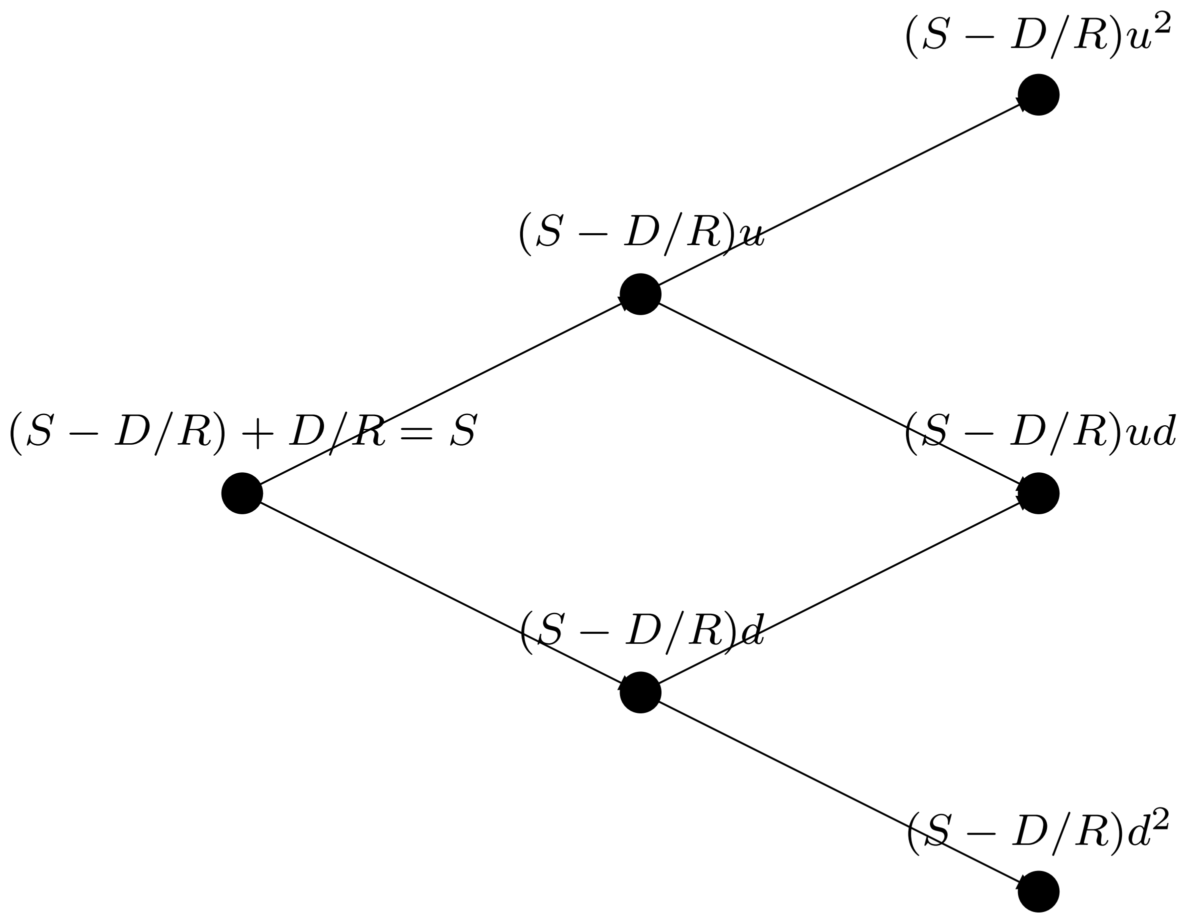
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 312 (Step 1)



The Ad-Hoc Approximation vs. P. 312 (Step 2)



The Ad-Hoc Approximation vs. P. 312[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\,u, (Su - D)\,d, (Sd - D)\,u, (Sd - D)\,d$ (p. 312).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$
\begin{aligned}
(Su - D)\,u &> (S - D/R)u^2, \\
(Sd - D)\,d &< (S - D/R)d^2,
\end{aligned}
$$

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 312 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually increased when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 750ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

---

[a] Dai (B82506025, R86526008, D8852600) & Lyuu (2004). Also Arealy & Rodrigues (2013).

[b] Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{41}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{41$'$}$$

where

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (41) and (41′) remain valid as long as the dividend yield is predictable.

[a]Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \stackrel{\Delta}{=} \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

[a] Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{42}$$

where $\Delta t \stackrel{\Delta}{=} \tau/n$.

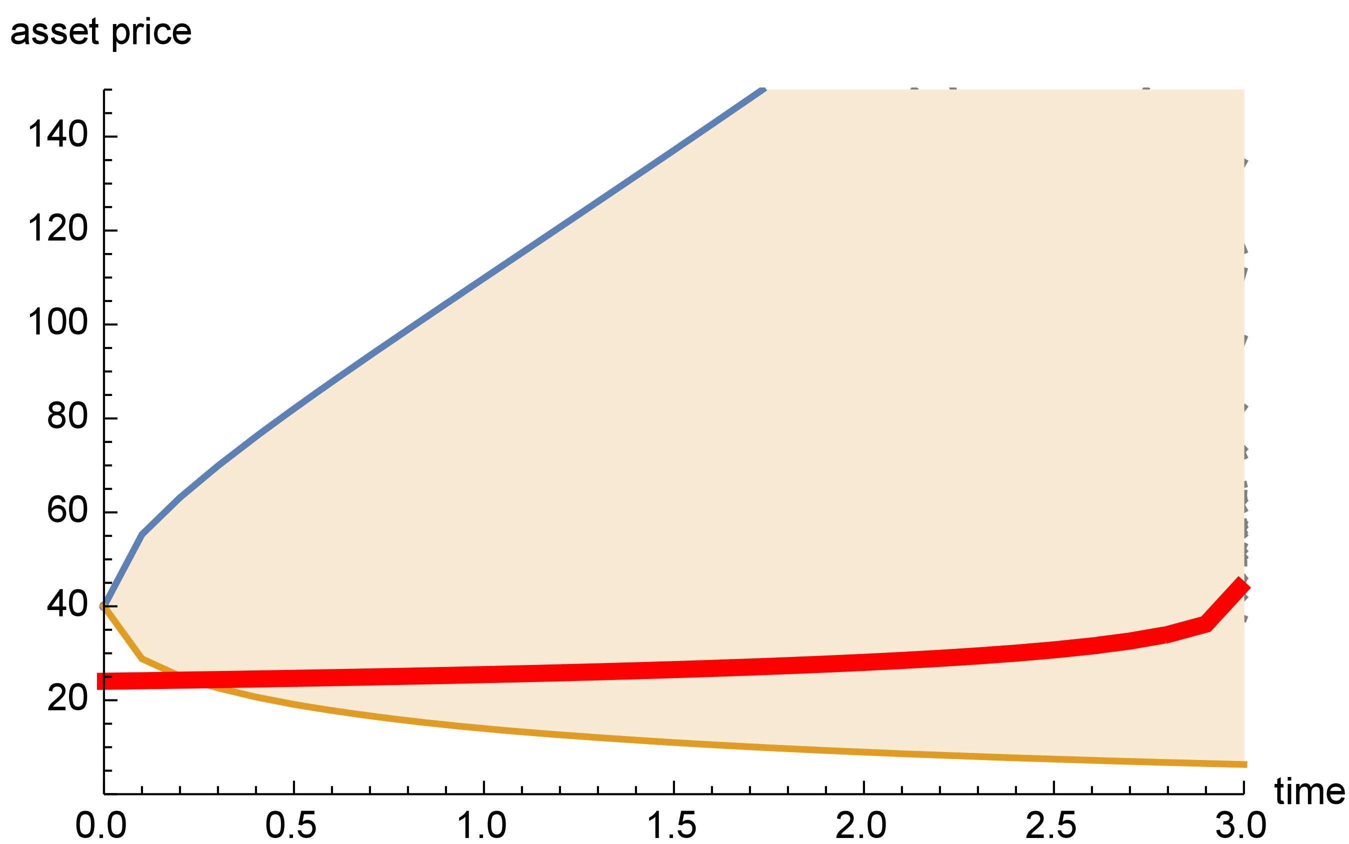
  – The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Other than the change in Eq. (42), binomial tree algorithms stay the same *as if there were no dividends*.

Exercise Boundaries of American Options (in the Continuous-Time Model)[a]

- The exercise boundary is a nondecreasing function of t for American puts (see the plot next page).
- The exercise boundary is a nonincreasing function of t for American calls.

[a]See Section 9.7 of the textbook for the tree analog.


asset price
140
120
100
80
60
40
20
time
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Risk Reversals[a]

- From formulas (41) and (41′) on p. 321, one can verify that $C = P$ when
$$X = Se^{(r-q)\tau}.$$
- A risk reversal consists of a short out-of-the-money put and a long out-of-the-money call with the same maturity.[b]
- Furthermore, the portfolio has zero value.
- A short risk reversal position is also called a collar.[c]

---

[a] Neftci (2008).
[b] Thus their strike prices must be distinct.
[c] Bennett (2014).

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \triangleq \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 291).
- Recall that
$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$
the density function of standard normal distribution.

Delta

- Defined as

$$\Delta \triangleq \frac{\partial f}{\partial S}.$$

  - f is the price of the derivative.
  - S is the price of the underlying asset.

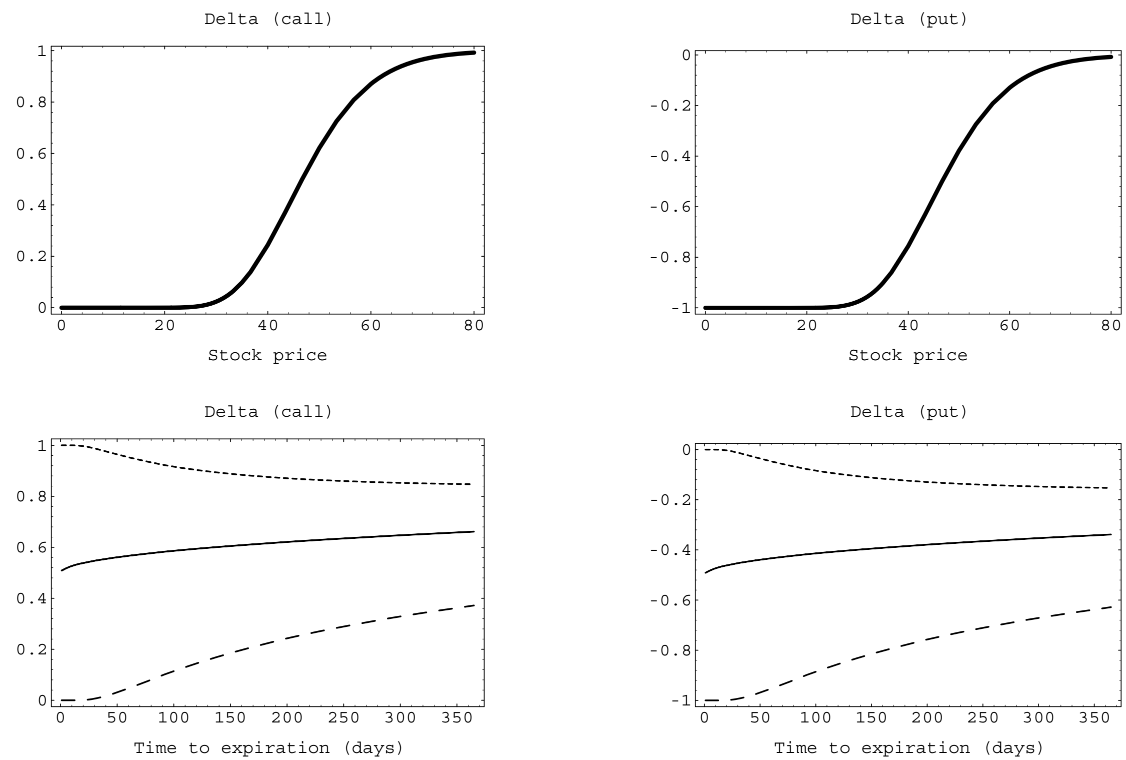
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM (p. 238) is the discrete analog.
- The delta of a long stock is apparently 1.

[a]Elementary calculus.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.$$
- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]
$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{43}$$

[a]The straddle (p. 206) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let

$$x \triangleq \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{44}$$

  (recall p. 321).
- Then

$$\begin{aligned}\frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0,\\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0.\end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \triangleq \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right) \tau}{\sigma\sqrt{\tau}}. \tag{45}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r - 2q + \sigma^2)\, \tau}. \tag{46}$$

[a]The strangle (p. 208) $C + P$ then has zero delta!

Delta (concluded)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then

$$X = Se^{(r-q+\sigma^2/2)\,\tau}$$

  leads to a straddle with zero delta.
  - This generalizes Eq. (43) on p. 331.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?
- In general, no.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

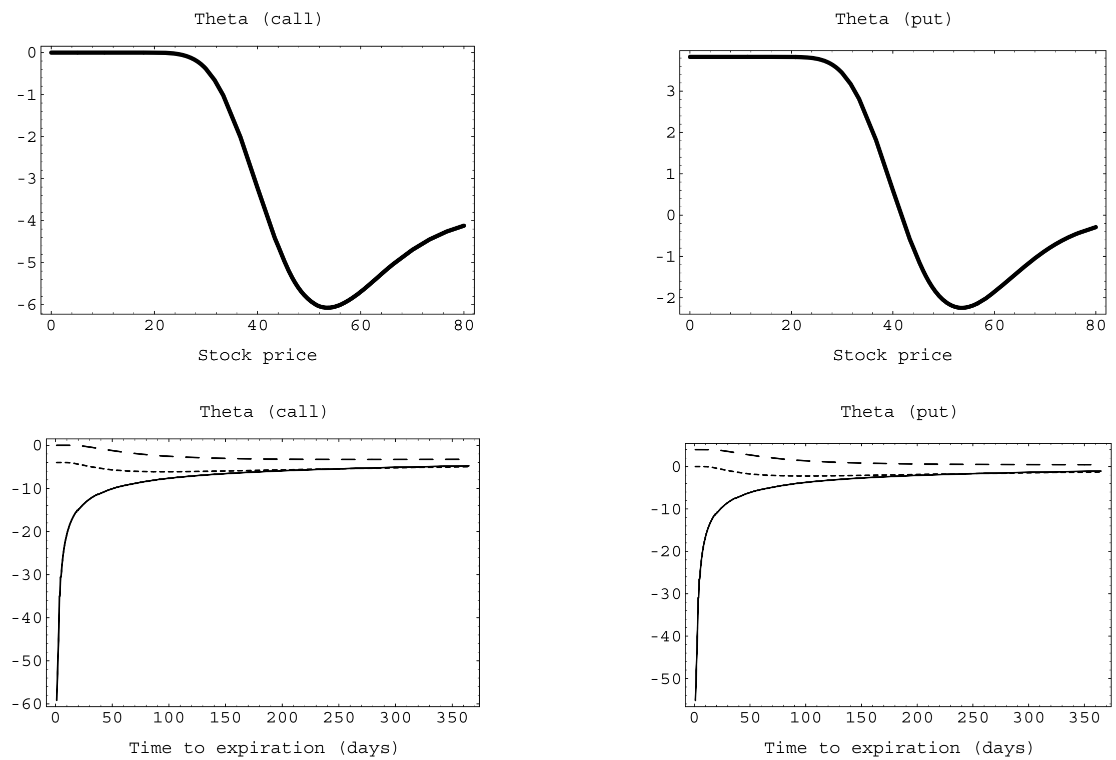
- Defined as the rate of change of a security's value with respect to time, or $\Theta \stackrel{\Delta}{=} -\partial f/\partial \tau = \partial f/\partial t$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau} N(x - \sigma\sqrt{\tau}) < 0.$$

  - The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau} N(-x + \sigma\sqrt{\tau}).$$

  - Can be negative or positive.



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Theta (concluded)

- Suppose the stock pays a continuous dividend yield of q.
- Define x as in Eq. (44) on p. 333.
- For a European call, add an extra term to the earlier formula for the theta:

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau} N(x - \sigma\sqrt{\tau}) + qSe^{-q\tau} N(x).$$

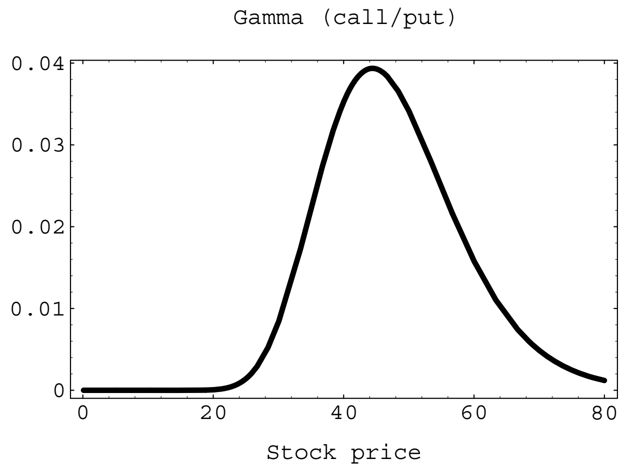
- For a European put, add an extra term to the earlier formula for the theta:

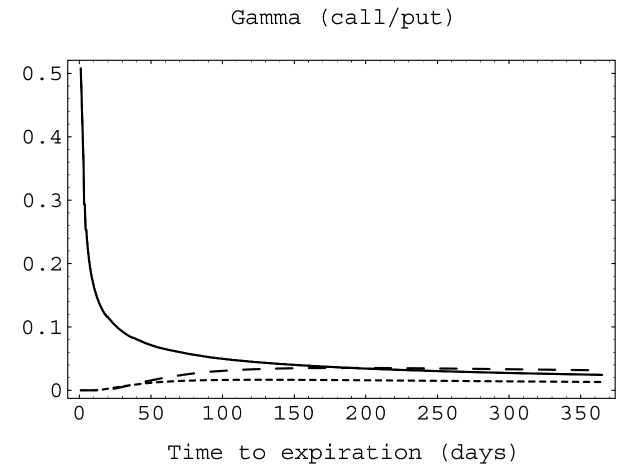
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - qSe^{-q\tau} N(-x).$$

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \triangleq \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$





Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

$$\Lambda \triangleq \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\,N'(x) > 0$.
  - So higher volatility always increases the option value.

[a] Vega is not Greek.

Vega (continued)

- Note that[a]

$$\Lambda = \tau \sigma S^2 \Gamma.$$

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = S e^{-q\tau} \sqrt{\tau}\, N'(x),$$

where x is defined in Eq. (44) on p. 333.

- Vega is maximized when $x = 0$, i.e., when

$$S = X e^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

[a]Reiss & Wystup (2001).

Vega (continued)

- Now consider a portfolio consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when

$$x_1 = -x_2,$$

  where x_i are defined in Eq. (45) on p. 334.
- This leads to Eq. (46) on p. 334.
  - Recall the same condition led to zero delta for the strangle $C + P$ (p. 334).

Vega (concluded)

- Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

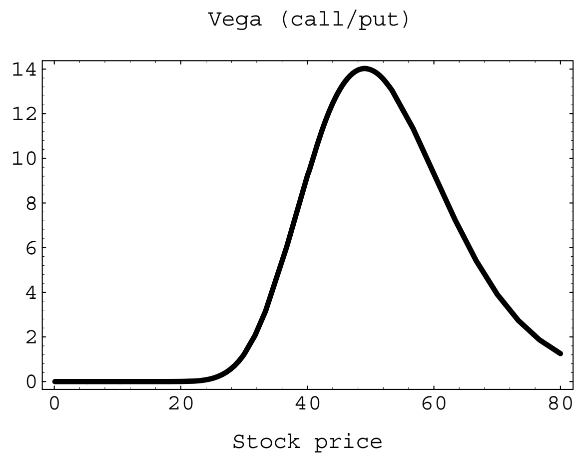
(which answers the question on p. 296 for the Black-Scholes model).

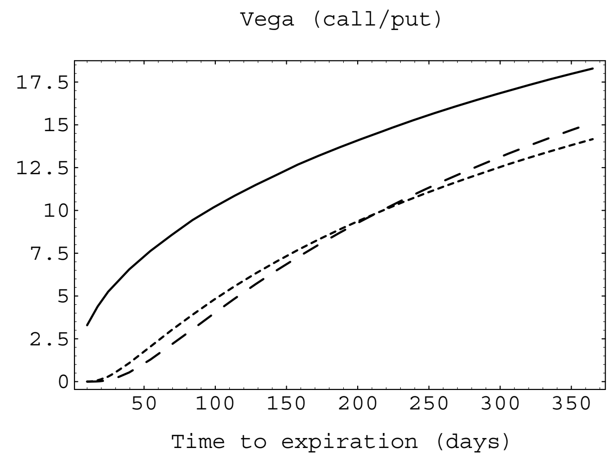
- The Black-Scholes formula (p. 291) implies

$$\begin{aligned} C &\to S, \\ P &\to X e^{-r\tau}, \end{aligned}$$

as $\sigma \to \infty$.

- These boundary conditions may be handy for certain numerical methods.





Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Variance Vega[a]

- Defined as the rate of change of a security's value with respect to the variance (square of volatility) of the underlying asset

$$V \triangleq \frac{\partial f}{\partial \sigma^2}.$$

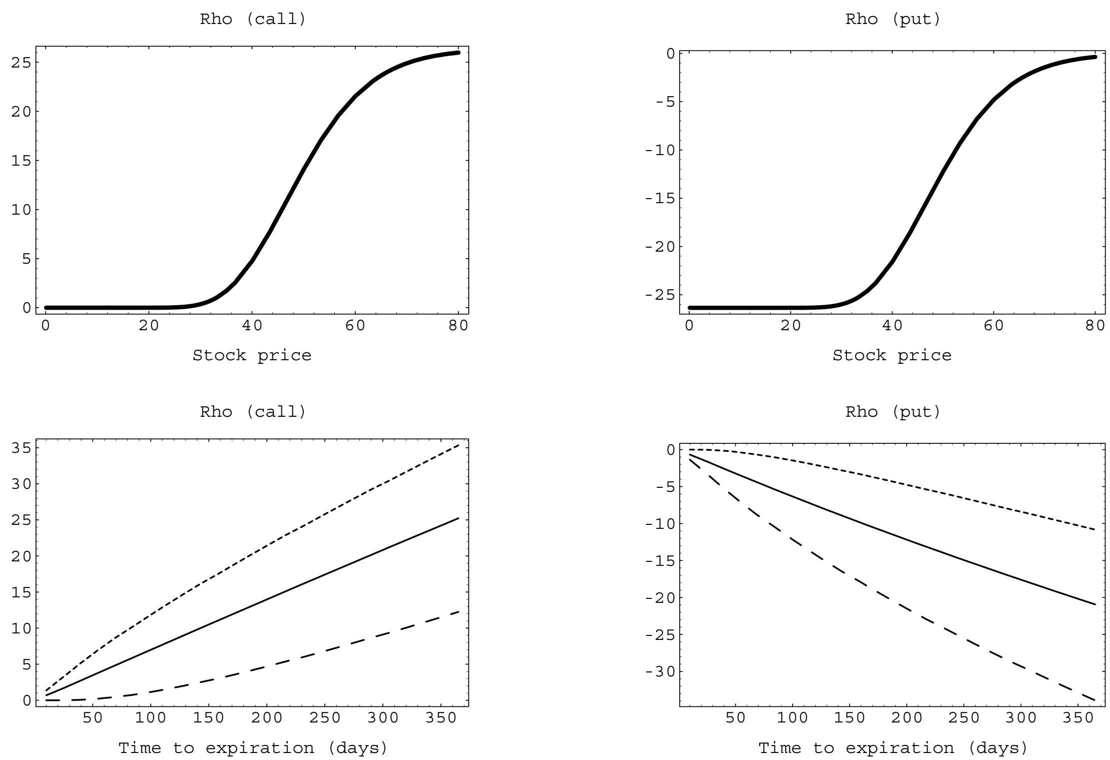
- It is easy to verify that

$$V = \frac{\Lambda}{2\sigma}.$$

[a]Demeterfi, Derman, Kamal, & Zou (1999).

Rho

- Defined as the rate of change in its value with respect to interest rates
$$\rho \stackrel{\Delta}{=} \frac{\partial f}{\partial r}.$$
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curves: out-of-the-money call or in-the-money put.

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$
- The computation time roughly doubles that for evaluating the derivative security itself.

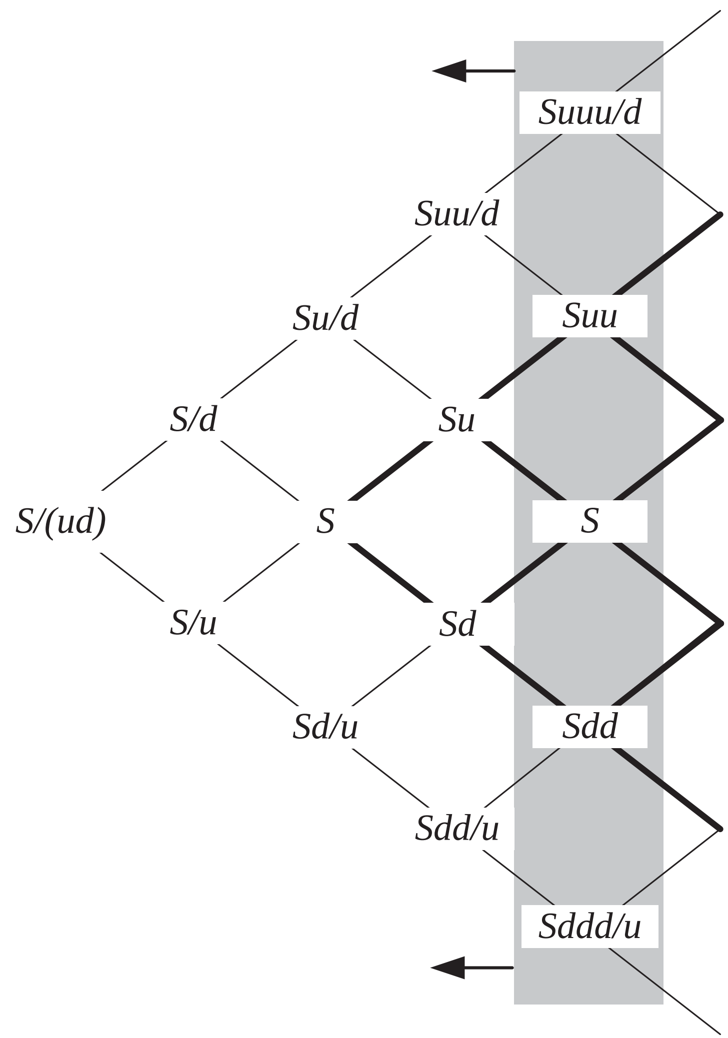
An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}. \tag{47}$$

- Almost zero extra computational effort.

[a] Pelsser & Vorst (1994).


Suuu/d
Suu/d
Su/d
Suu
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}. \tag{48}$$

- Alternative formulas exist (p. 654).

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$

- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.
- But why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option can be derived from delta and gamma (p. 653).

- The vega of a European option can be derived from gamma (p. 343).

- For rho, there seems no alternative but to run the binomial tree algorithm twice.[a]

[a]But see p. 822 and pp. 1004ff.

Extensions of Options Theory

Data! data! data!

— *The Adventures of Sherlock Holmes* (1892)

As I never learnt mathematics,
so I have had to think.

— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to price corporate securities.
  - The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a]Black & Scholes (1973); Merton (1974).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call[a] on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V.

---

[a]See p. 197.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned} nS &= C, \\ B &= V - C. \end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 10 (p. 291) and the put-call parity,[a]

$$\begin{aligned} nS &= VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \quad (49) \\ B &= VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \quad (50) \end{aligned}$$

  - Above,

$$x \stackrel{\Delta}{=} \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

[a] Merton (1974).

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned} & \frac{\ln(X/B)}{\tau} - r \\ = \quad & -\frac{1}{\tau} \ln\left(N(-z) + \frac{1}{\omega} N(z - \sigma\sqrt{\tau}) \right). \end{aligned}$$

  - $\omega \stackrel{\Delta}{=} Xe^{-r\tau}/V$.
  - $z \stackrel{\Delta}{=} (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of the remaining firm value.
- Then Eqs. (49)–(50) on p. 363 become, respectively,

$$\begin{aligned} nS &= Ve^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ B &= (1-\alpha)Ve^{-q\tau} N(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  - Above,

$$x \triangleq \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

  dollars.
  - Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.[a]
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

[a] See p. 220.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility, dividend, and strike price are under partial control of the stockholders or their boards.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 370 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

  dollars.
- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$

  dollars.

- The *original* bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$

  - This is called claim dilution.[a]

[a] Fama & Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a \$14,833.3 cash dividend.
- They now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning 2/3 of a call on n Merck shares with a strike price of \$45,000.
- n such calls are worth \$1,937.5 (p. 370).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

  dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

  dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

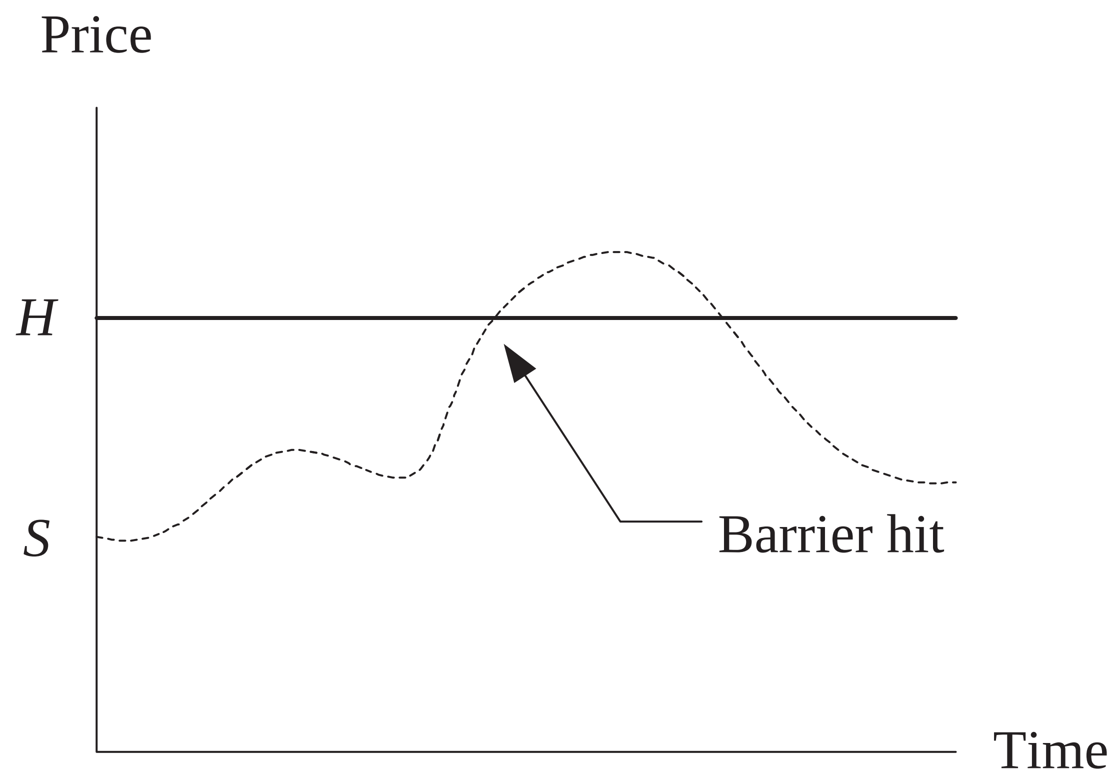
- Other Examples:
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
- Securities with a complex liability structure must be solved by trees.[a]

[a] Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H *throughout its life*.
- A knock-out (KO) option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

[a]A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.


Price
H
S
Barrier hit
Time

Barrier Options (continued)

- A knock-in (KI) option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all the possible barrier options mentioned above.[a]

[a] Haug (2006).

Barrier Options (concluded)

- Knock-in puts are the most popular barrier options.[a]
- Knock-out puts are the second most popular barrier options.[b]
- Knock-out calls are the most popular among barrier call options.[c]

[a]Bennett (2014).
[b]Bennett (2014).
[c]Bennett (2014).

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(x-\sigma\sqrt{\tau}), \tag{51}$$

  - $x \stackrel{\Delta}{=} \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \stackrel{\Delta}{=} (r-q+\sigma^2/2)/\sigma^2$.
- A European down-and-out call can be priced via the in-out parity (see text).

[a]Merton (1973). See Exercise 17.1.6 of the textbook for a proof.

A Formula for Up-and-In Puts[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- Again, a European up-and-out put can be priced via the in-out parity.

[a] Merton (1973).

Are American Options Barrier Options?[a]

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.

Interesting Observations

- Assume $H < X$.
- Replace S in the Merton pricing formula Eq. (41) on p. 321 for the call with H^2/S.
- Equation (51) on p. 382 for the down-and-in call becomes Eq. (41) when $r - q = \sigma^2/2$.
- Equation (51) becomes S/H times Eq. (41) when $r - q = 0$.

Interesting Observations (concluded)

- Replace S in the pricing formula for the down-and-in call, Eq. (51), with H^2/S.
- Equation (51) becomes Eq. (41) when $r - q = \sigma^2/2$.
- Equation (51) becomes H/S times Eq. (41) when $r - q = 0$.[a]
- Why?[b]

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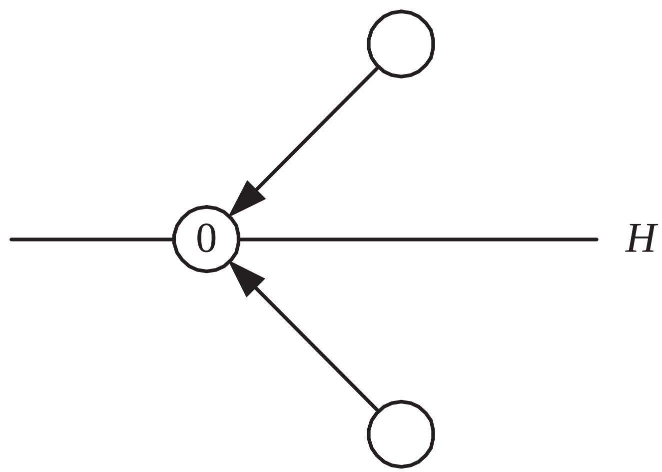
[a] Contributed by Mr. Chou, Ming-Hsin (`R02723073`) on April 24, 2014.

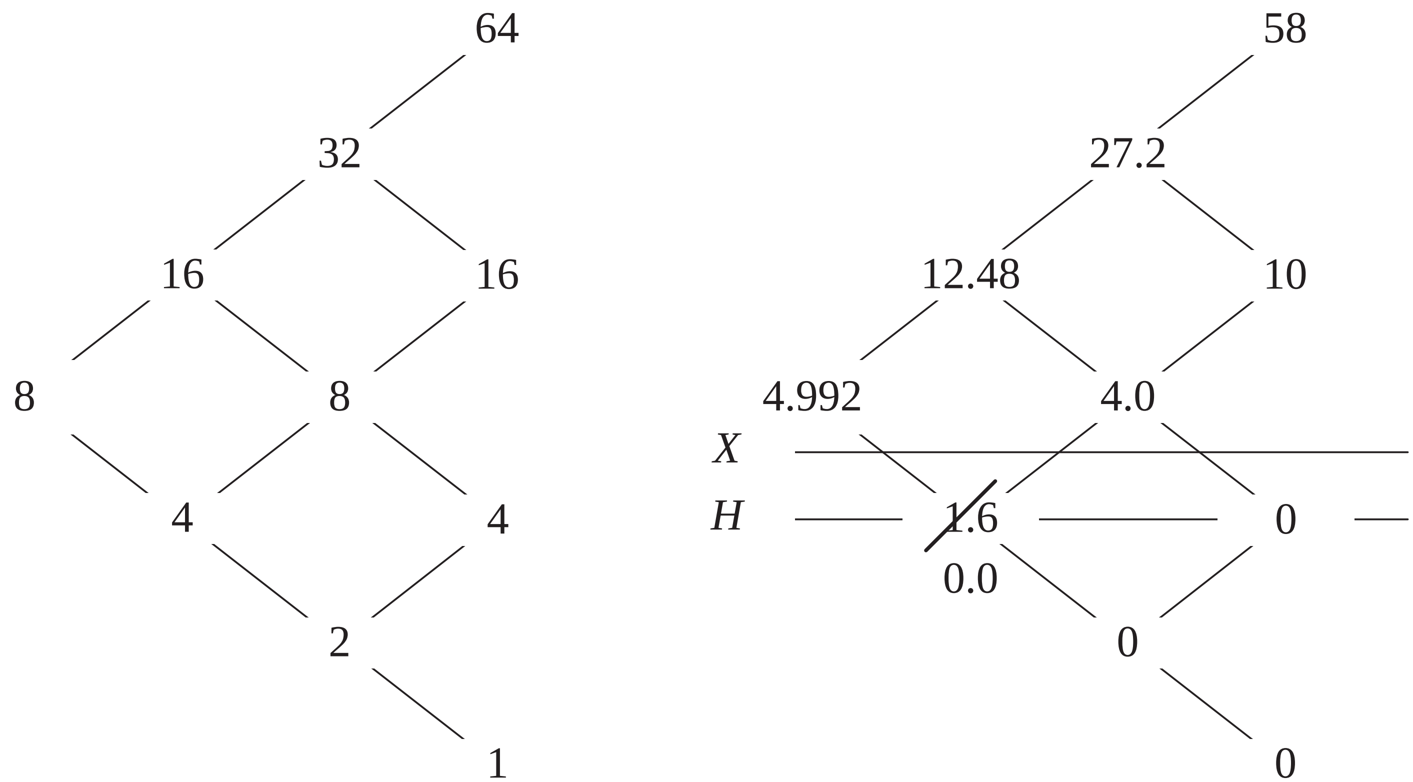
[b] Apply the reflection principle (p. 683), Eq. (40) on p. 284, and Lemma 9 (p. 289).

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.



- Pricing down-and-in options is subtler.



$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.

Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (continued)

- But convergence is erratic because H is not at a price level on the tree (see plot on next page).[a]
  - The barrier H is moved lower (or higher) to a closeby node price.
  - This "effective barrier" thus changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]
- Solutions will be presented later.

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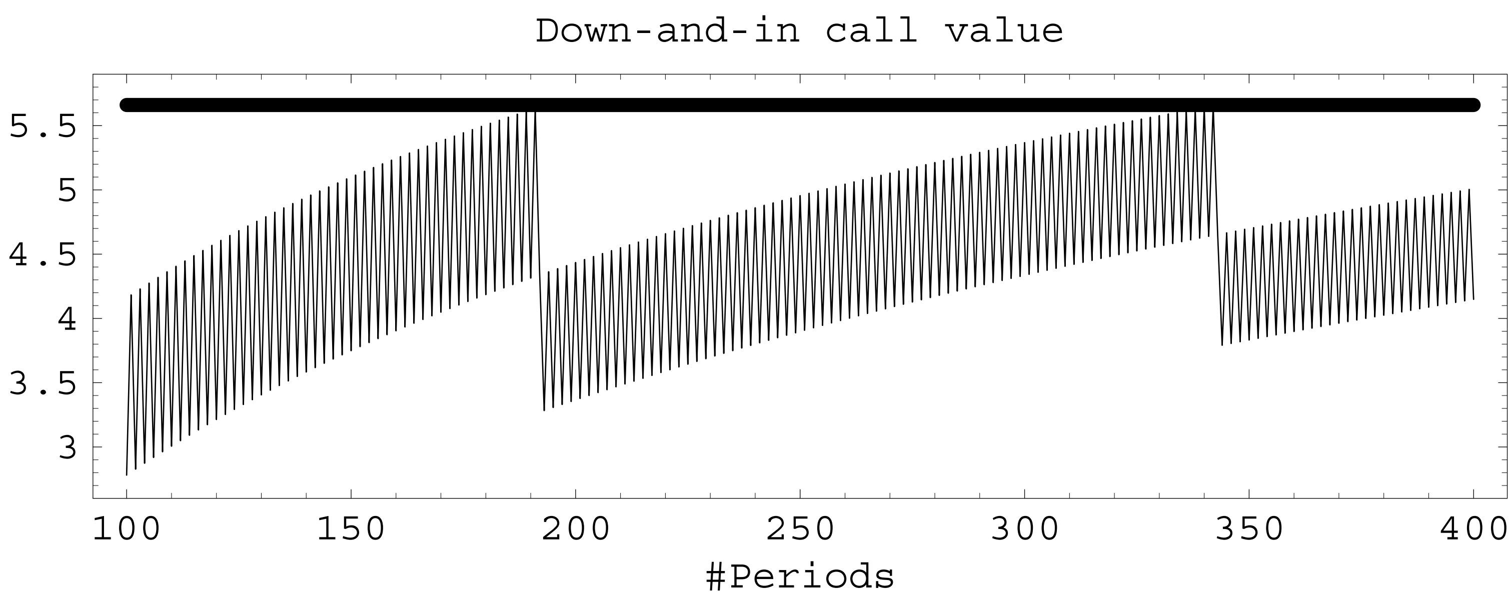
[a] Boyle & Lau (1994).
[b] J. Lin (`R95221010`) (2008).

Binomial Tree Algorithms (concluded)[a]



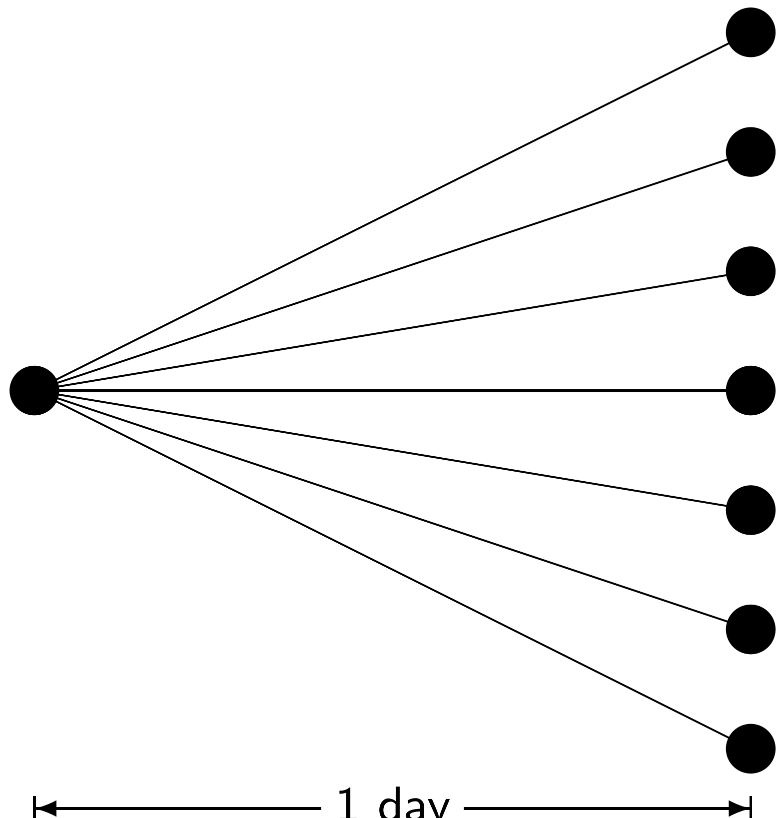
[a]Lyuu (1998).

Daily Monitoring

- Many barrier options monitor the barrier only for daily *closing prices*.
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d+1$ nodes if each day is partitioned into d periods.
- Does this save time or space?[a]

[a]Contributed by Ms. Chen, Tzu-Chun (`R94922003`) and others on April 12, 2006.

A Heptanomial Tree (6 Periods Per Day)



Discrete Monitoring vs. Continuous Monitoring

- Discrete barriers are more expensive for knock-out options than continuous ones.
- But discrete barriers are less expensive for knock-in options than continuous ones.
- Discrete barriers are far less popular than continuous ones for individual stocks.[a]
- They are equally popular for indices.[b]

[a] Bennett (2014).
[b] Bennett (2014).