Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?

- Define price volatility by

\[- \frac{\partial P}{\partial y} \frac{1}{P}. \]  \hspace{1cm} (14)
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[
- \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

– \( F \) is the par value.
– \( C \) is the coupon payment per period.
– Formula can be simplified a bit with \( C = Fc/m \).

For the above bond,

\[
- \frac{\partial P}{\partial y} > 0.
\]
Macaulay Duration\textsuperscript{a}

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.
- The weights are the cash flows’ PVs divided by the asset’s price.
- Formally,
  \[
  MD \triangleq \frac{1}{P} \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} i.
  \]
- The Macaulay duration, in periods, is equal to
  \[
  MD = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \tag{15}
  \]

\textsuperscript{a}Macaulay (1938).
MD of Bonds

• The MD of a level-coupon bond is

\[ MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \]  \hspace{1cm} (16)

• It can be simplified to

\[ MD = \frac{c(1+y)[(1+y)^n-1]}{cy[(1+y)^n-1]+y^2} + n(y-c), \]

where \( c \) is the period coupon rate.

• The MD of a zero-coupon bond equals \( n \), its term to maturity.

• The MD of a level-coupon bond is less than \( n \).
Remarks

• Equations (15) on p. 93 and (16) on p. 94 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility$^a$ may decrease.

\[\text{\textsuperscript{a}As originally defined in Eq. (14) on p. 91.}\]
How Not To Think about MD

• The MD has its origin in measuring the length of time a bond investment is outstanding.

• But it should be seen mainly as measuring price volatility.

• Duration of a security can be longer than its maturity or negative!

• Neither makes sense under the maturity interpretation.

• Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

• For the MD to be year-based, modify Eq. (16) on p. 94 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \left( \frac{C}{1 + \frac{y}{k}} \right)^i + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Equation (15) on p. 93 also becomes

\[
MD = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \( \frac{MD \text{ (in periods)}}{k} \).
Modified Duration

- Modified duration is defined as

\[ \text{modified duration} \triangleq -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \] (17)

- By the Taylor expansion,

percent price change \( \approx -\text{modified duration} \times \text{yield change}. \)
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\%.\]
Modified Duration of a Portfolio

- The modified duration of a portfolio equals

\[ \sum_{i} \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[
\frac{P_- - P_+}{P_0(y_+ - y_-)}.
\]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
$P_+$
$P_0$
$P_-$

$y_-$  $y$  $y_+$
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.

- An alternative is to use

\[
\frac{P_0 - P_+}{P_0 \Delta y}.
\]

  - More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^a\)

• The percentage price change expressed in percentage terms is then approximated by

\[-D\% \times \Delta r\]

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\(^a\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as
  \[ \text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}. \]

- The approximate dollar price change is
  \[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

- One can hedge a bond portfolio with a dollar duration \( D \) by bonds with a dollar duration \( -D \).
Convexity

• Convexity is defined as

\[ \text{convexity (in periods)} \triangleq \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

• The convexity of a level-coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

- The approximation $\frac{\Delta P}{P} \approx - \text{duration} \times \text{yield change}$ works for small yield changes.

- For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$

- Recall the figure on p. 107.
The Practices

- Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$ 

- $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

• The effective convexity is defined as

\[
\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
\]

- \(P_-\) is the price if the yield is decreased by \(\Delta y\).
- \(P_+\) is the price if the yield is increased by \(\Delta y\).
- \(P_0\) is the initial price, \(y\) is the initial yield.
- \(\Delta y\) is small.

• Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

• Numerically, choosing the right \(\Delta y\) is a delicate matter.
Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2$ and $2$:

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 825ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

• Concerned with how interest rates change with maturity.

• The set of yields to maturity for bonds form the term structure.
  – The bonds must be of equal quality.
  – They differ solely in their terms to maturity.

• The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots the yields to maturity of coupon bonds against maturity.
- A par yield curve is constructed from bonds trading near par.
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is by definition
  
  \[ [1 + S(i)]^{-i}. \]

  - It is the price of an $i$-period zero-coupon bond.\(^a\)
- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity:
  
  \[ S(1), S(2), \ldots , S(n). \]

\(^a\)Recall Eq. (9) on p. 66.
Problems with the PV Formula

• In the bond price formula (3) on p. 39,
\[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]
every cash flow is discounted at the same yield \( y \).

• Consider two riskless bonds with different yields to maturity because of their different cash flow streams:
\[ PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \]
\[ PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

![Diagram showing cash flows at times 1, 2, 3, ..., n]
Spot Rate Discount Methodology (concluded)

• So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \]  (18)

• This pricing method incorporates information from the term structure.

• It discounts each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

– Above, $d(i) \triangleq [1 + S(i)]^{-i}, i = 1, 2, \ldots, n$, are called the discount factors.

– $d(i)$ is the PV of one dollar $i$ periods from now.

– This formula—now just a definition—will be justified on p. 215.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.

- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and
  
  $$S(1), S(2), \ldots, S(n-1).$$

- Then $S(n)$ can be computed from Eq. (18) on p. 124, repeated below,

  $$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$ 

- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).

- Some maturities might be missing from the data points (the incompleteness problem).

- Treasuries might not be of the same quality.

- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\(^a\)

\(^a\)Often without economic justifications.
Which One?
Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.
Static Spread

- Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift \textit{in parallel} to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

- Unlike the yield spread, the static spread incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

• $y_k$: yield to maturity for the $k$-period coupon bond.
• $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
• $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
• $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
• $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

• If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.

- Invest $1 for $j$ periods to end up with $[1 + S(j)]^j$ dollars at time $j$.
  - The maturity strategy.

- Invest $1$ in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.

- Will have $[1 + S(i)]^i[1 + S(i, j)]^{j-i}$ dollars at time $j$.
  - $S(i, j)$: $(j - i)$-period spot rate $i$ periods from now.
  - The rollover strategy.
Forward Rates (concluded)

• When $S(i, j)$ equals

$$f(i, j) \triangleq \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$  \hspace{1cm} (19)

we will end up with $[1 + S(j)]^j$ dollars again.

• By definition, $f(0, j) = S(j)$.

• $f(i, j)$ is called the (implied) forward rates.
  
  – More precisely, the $(j-i)$-period forward rate $i$ periods from now.
Time Line

\[ f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \]

Time 0

\[ S(1) \quad S(2) \quad S(3) \quad S(4) \]
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.

• $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

• When the spot rate curve is normal, the forward rate dominates the spot rates,

\[ f(i, j) > S(j) > \cdots > S(i). \]

• When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

\[ f(i, j) < S(j) < \cdots < S(i). \]
(a) Forward rate curve, spot rate curve, yield curve

(b) Yield curve, spot rate curve, forward rate curve
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)]. \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves
(concluded)

• Since they are identical,

\[ S(n) = \left\{ \left[ 1 + S(1) \right]\left[ 1 + f(1, 2) \right] \right. \]

\[ \left. \cdots \left[ 1 + f(n - 1, n) \right] \right\}^{1/n} - 1. \]  \hspace{1cm} (20)

• Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

• Other equivalencies can be derived similarly, such as

\[ f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \]  \hspace{1cm} (21)
Locking in the Forward Rate \( f(n, m) \)

- Buy one \( n \)-period zero-coupon bond for \( 1/(1 + S(n))^n \) dollars.
- Sell \( (1 + S(m))^m/(1 + S(n))^n \) \( m \)-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: \( 1/(1 + S(n))^n \).
- At time \( n \) there will be a cash inflow of $1.
- At time \( m \) there will be a cash outflow of \( (1 + S(m))^m/(1 + S(n))^n \) dollars.
Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the rate $f(n, m)$ between times $n$ and $m$. 

\[
(1 + S(m))^m / (1 + S(n))^n
\]
Forward Loans

• We had generated the cash flow of a type of forward contract called the forward loan.

• Agreed upon today, it enables one to
  – Borrow money at time \( n \) in the future, and
  – Repay the loan at time \( m > n \) with an interest rate equal to the forward rate

\[
f(n, m).
\]

• Can the spot rate curve be an arbitrary curve?\(^a\)

\(^a\)Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.
Synthetic Bonds

• We had seen that

\[
\text{forward loan} = n\text{-period zero} - [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• Thus

\[
\text{n-period zero} = \text{forward loan} + [1 + f(n, m)]^{m-n} \times m\text{-period zero}.
\]

• We have created a synthetic zero-coupon bond with forward loans and other zero-coupon bonds.

• Very useful if the \( n \)-period zero is unavailable or illiquid.