Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
  - This was called calibration (the reverse of pricing).
- Assume the short rate volatility is such that

$$v \stackrel{\Delta}{=} \frac{r_{\mathrm{h}}}{r_{\ell}} = 1.5,$$

independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

An Approximate Calibration Scheme

- Start with the implied one-period forward rates.
- Then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let f_j denote the forward rate in period j.
- This forward rate can be derived from the market discount function via

$$f_j = \frac{d(j)}{d(j+1)} - 1$$

(see Exercise 5.6.3 in text).

An Approximate Calibration Scheme (continued)

- Since the ith short rate $r_j v_j^{i-1}$, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left(\frac{2}{1+v_j} \right)^{j-1} f_j. \tag{125}$$

- This binomial interest rate tree is trivial to set up, in $O(n)$ time.

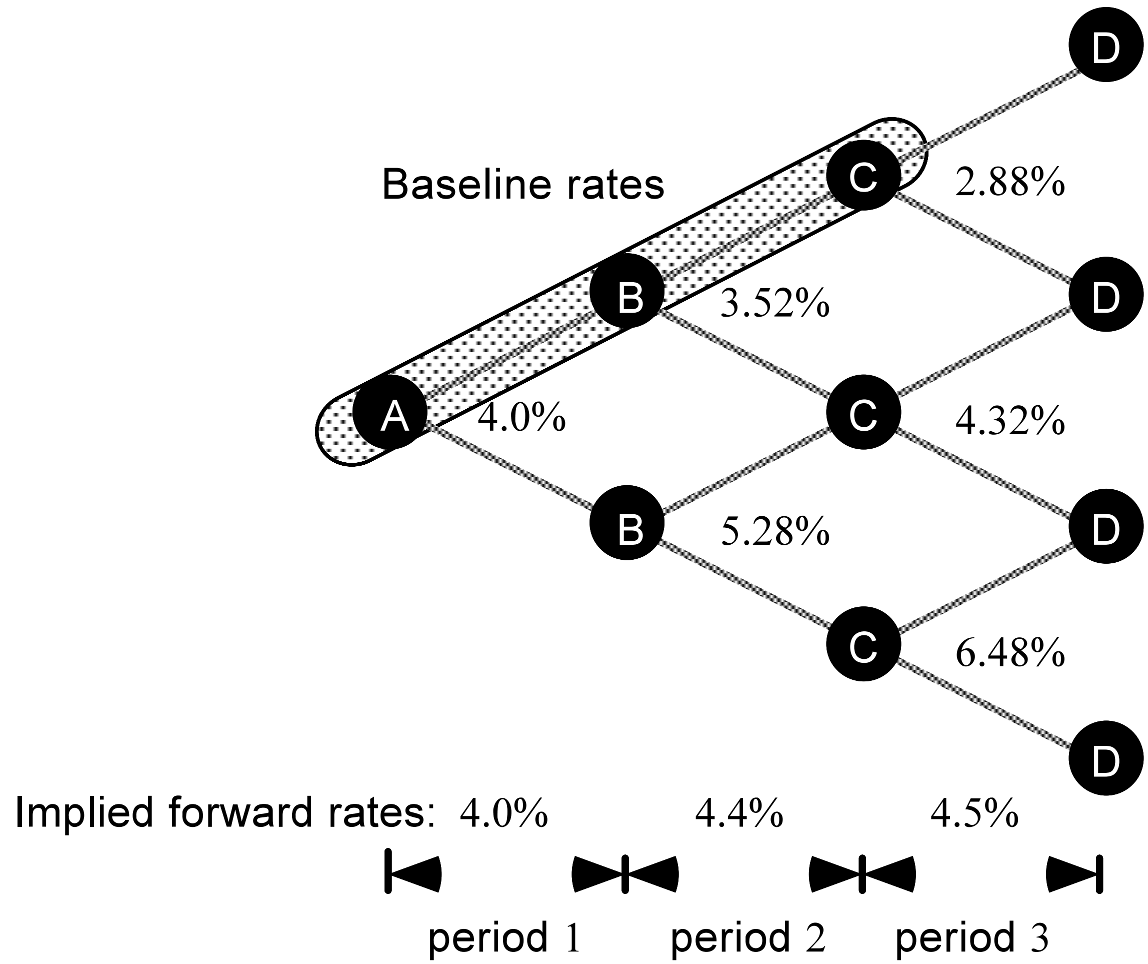
An Approximate Calibration Scheme (continued)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

$$\frac{1}{4} \times \frac{1}{1.04} \times \left(\frac{1}{1.0352} \times \left(\frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left(\frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

  or 0.88155, which exceeds discount factor 0.88135.

- The tree is thus *not* calibrated.


Baseline rates
D
C
2.88%
B
3.52%
D
A
4.0%
C
4.32%
B
5.28%
D
C
6.48%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree *overprices* the bonds.[a]
- Suppose we replace the baseline rates r_j by $r_j v_j$.
- Then the resulting tree *underprices* the bonds.[b]
- The true baseline rates are thus bounded between r_j and $r_j v_j$.

---

[a] See Exercise 23.2.4 in text.
[b] Lyuu & C. Wang (F95922018) (2009, 2011).

Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the m-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate r_m for period m.
- A root-finding method is applied to solve $f(r_m) = P$ for r_m given the zero's price P and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in $O(n^3)$ time.

Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.[a]
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price, the Arrow-Debreu price, or Green's function.
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 0 to time n.

[a] Jamshidian (1991).

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at *time* j and there are $j+1$ nodes.
  - The unknown baseline rate for *period* j is $r \triangleq r_j$.
  - The multiplicative ratio is $v \triangleq v_j$.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at *earlier* time $j-1$.
  - They correspond to rates $r, rv, \ldots, rv^{j-1}$ for period j (recall p. 949).
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.
- We want to find r based on $P_1, P_2, \ldots, P_j$ and the price of the j-period zero-coupon bond.

Binomial Interest Rate Tree Calibration (continued)

- One dollar at time j has a known market value of $1/[\,1 + S(j)\,]^j$, where $S(j)$ is the j-period spot rate.
- Alternatively, this dollar has a present value of

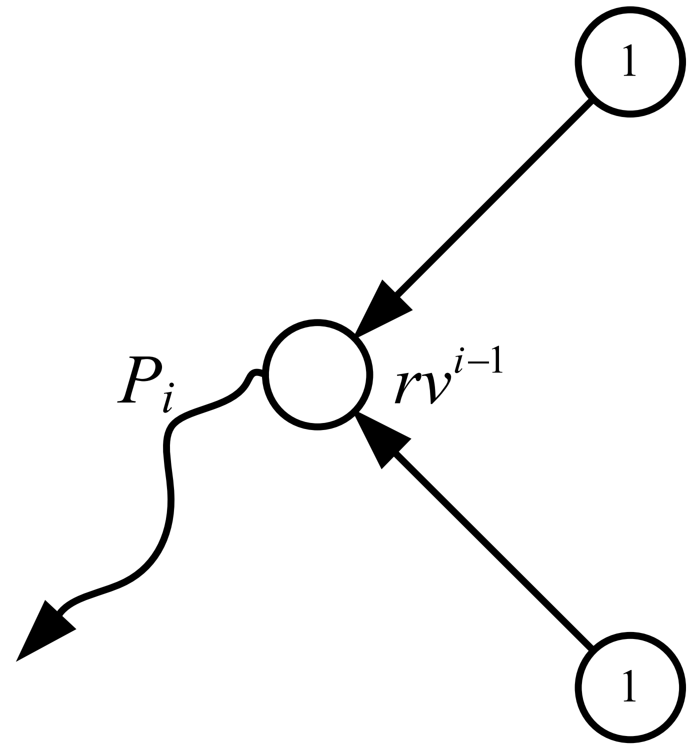
$$g(r) \triangleq \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}$$

(see next plot).

- So we solve

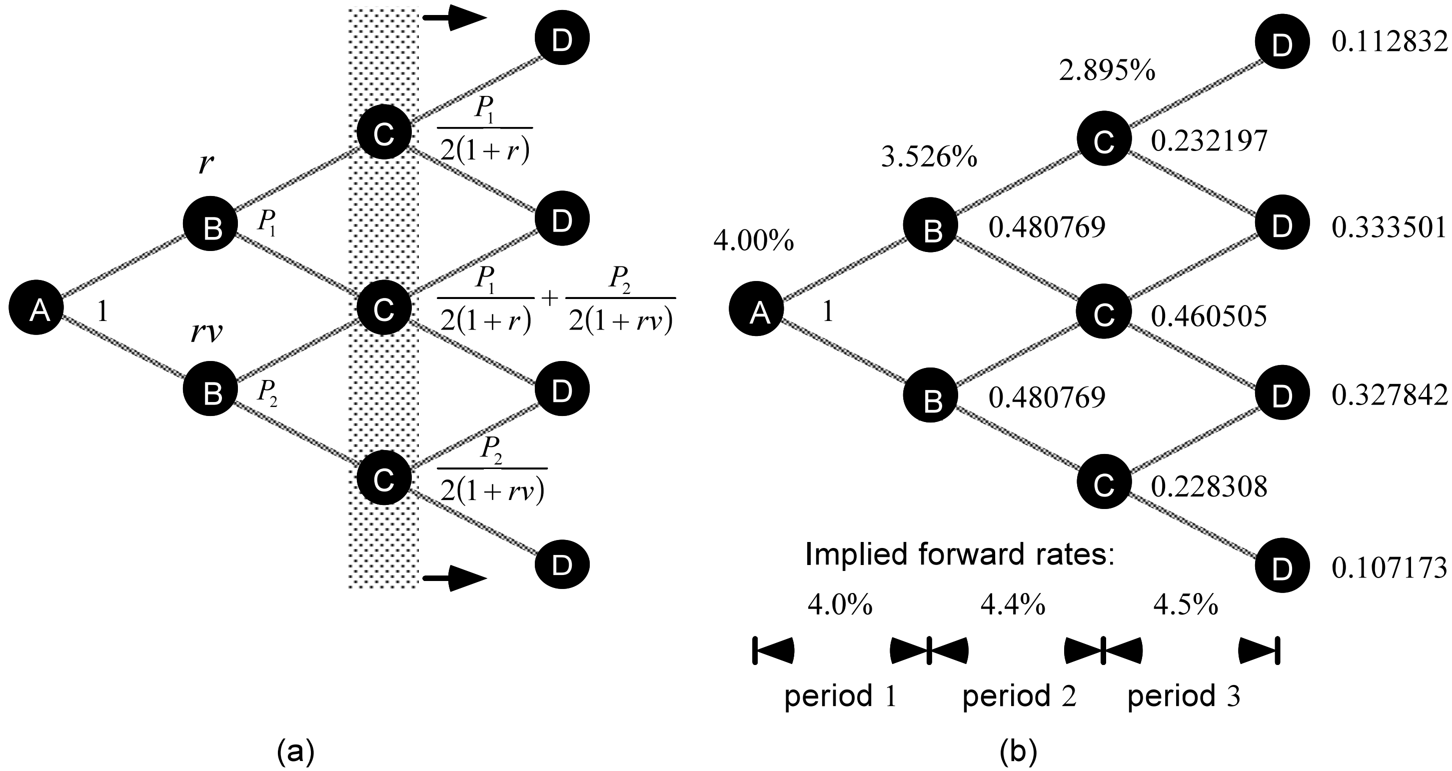
$$g(r) = \frac{1}{[\,1+S(j)\,]^j} \tag{126}$$

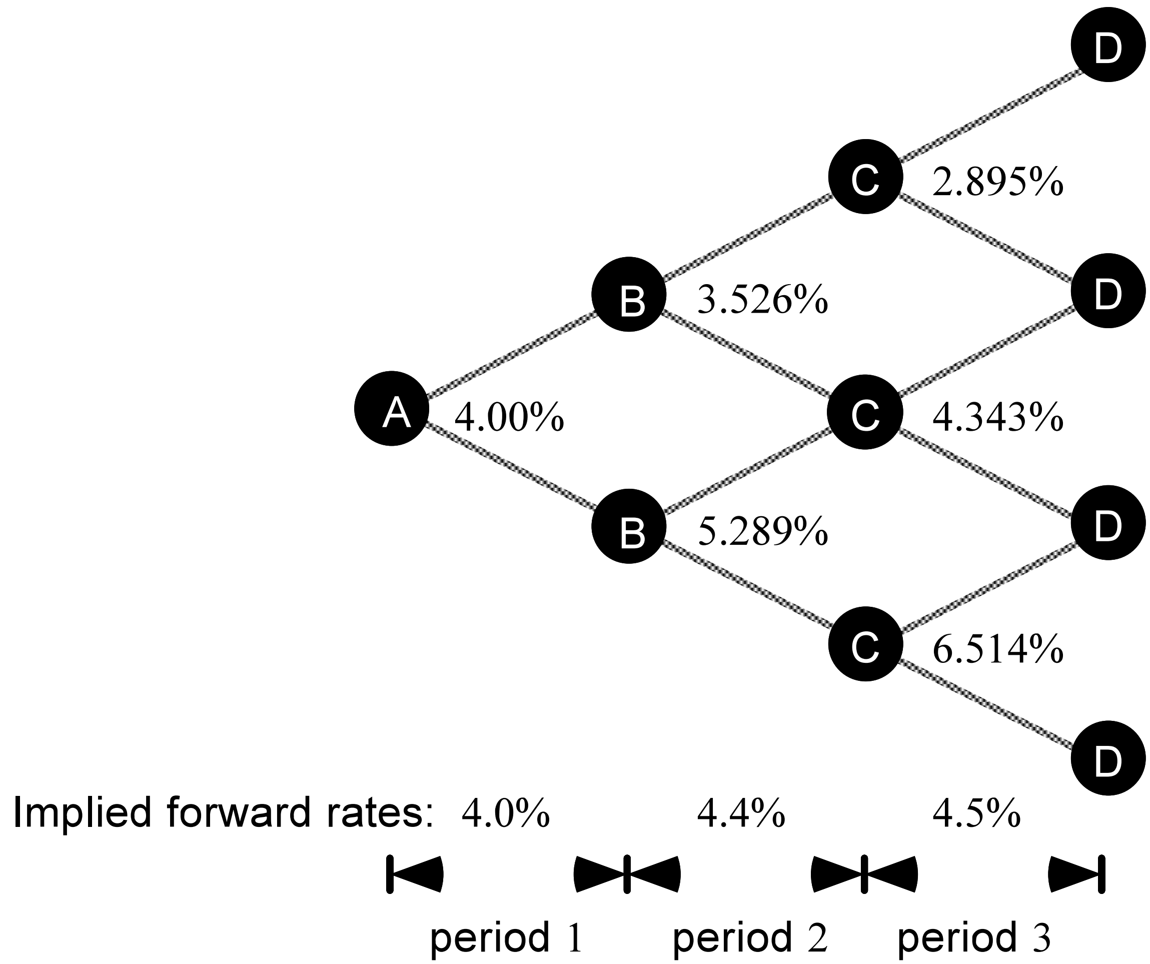
for r.


1
P_i
rv^{i-1}
1

Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for r is guaranteed.
- The state prices at time j can now be calculated (see panel (a) next page).
- We call a tree with these state prices a binomial state price tree (see panel (b) next page).
- The calibrated tree is depicted on p. 970.


D
C
r
B
P_1
A
1
rv
B
P_2
C
C
D
D
D
D
$\frac{P_1}{2(1+r)}$
$\frac{P_1}{2(1+r)} + \frac{P_2}{2(1+rv)}$
$\frac{P_2}{2(1+rv)}$
(a)
4.00%
A
1
3.526%
B
0.480769
B
0.480769
2.895%
C
0.232197
C
0.460505
C
0.228308
D
0.112832
D
0.333501
D
0.327842
D
0.107173
Implied forward rates:
4.0%
4.4%
4.5%
period 1
period 2
period 3
(b)


D
C
2.895%
B
3.526%
D
A
4.00%
C
4.343%
B
5.289%
D
C
6.514%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the r in Eq. (126) on p. 966 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The total running time is $O(n^2)$, as each root-finding routine consumes $O(j)$ time.
- With a good initial guess,[a] the Newton-Raphson method converges in only a few steps.[b]

---

[a]Such as the $r_j = (\frac{2}{1+v_j})^{j-1} f_j$ on p. 959.

[b]Lyuu (1999).

A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, r_2, satisfies

$$\frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.$$

- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in panel (b) on p. 969 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

A Numerical Example (concluded)

- The baseline rate for the third period, r_3, satisfies

$$\frac{0.232197}{1+r_3}+\frac{0.460505}{1+1.5\times r_3}+\frac{0.228308}{1+(1.5)^2\times r_3}=0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 960 using the new rates:

$$\frac{1}{4}\times\frac{1}{1.04}\times[\frac{1}{1.03526}\times(\frac{1}{1.02895}+\frac{1}{1.04343})+\frac{1}{1.05289}\times(\frac{1}{1.04343}+\frac{1}{1.06514})],$$

which equals 0.88135, an exact match.

- The tree on p. 970 prices without bias the benchmark securities.

Spread of Nonbenchmark Bonds

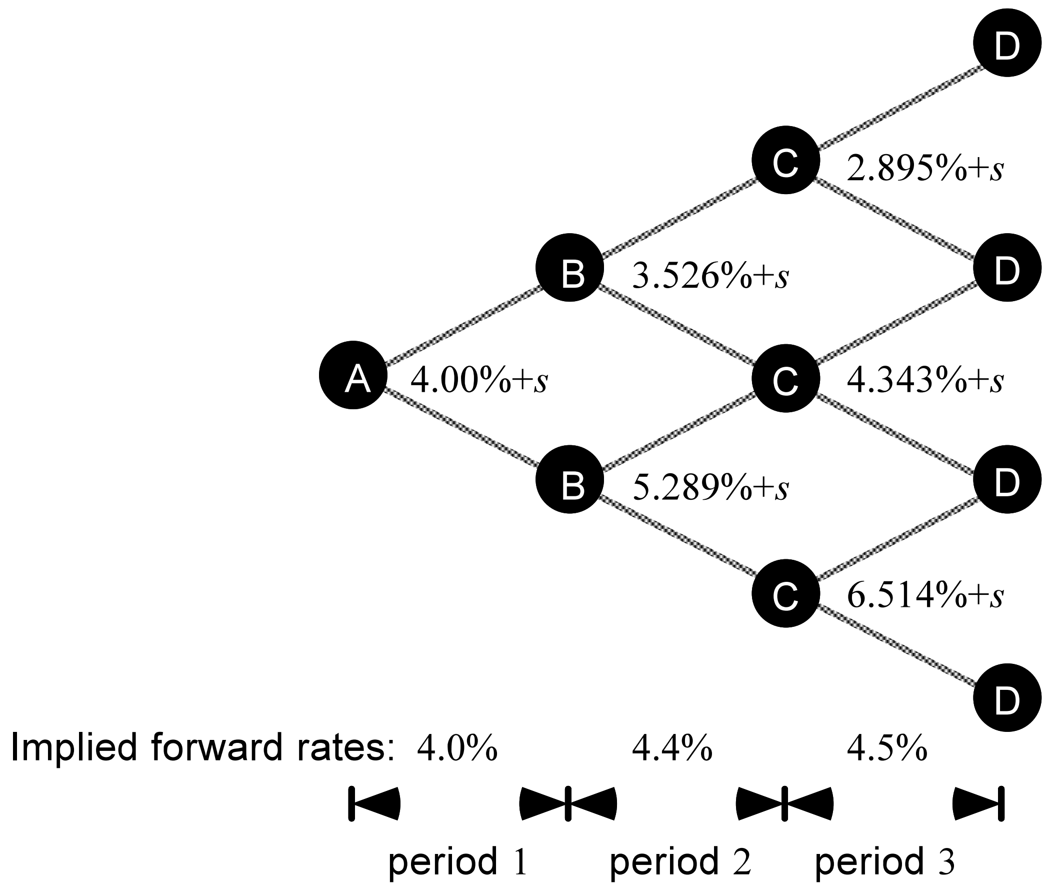
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 976.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.


D
C
2.895%+s
B
3.526%+s
D
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- We will employ the Newton-Raphson root-finding method to solve
$$p(s) - P = 0$$
for s.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

Spread of Nonbenchmark Bonds (continued)

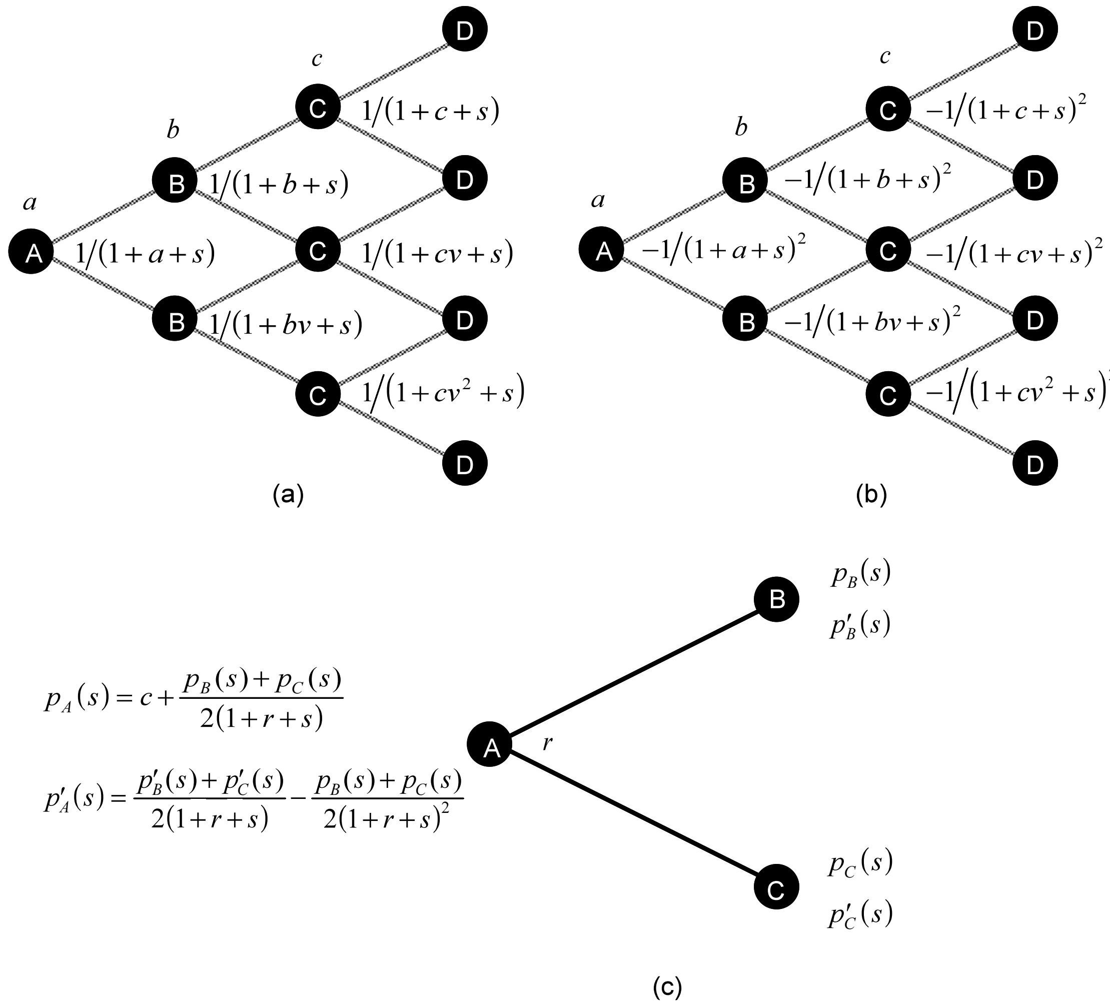
- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1+r+s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1+r+s)^2}. \tag{127}$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- When A is a terminal node, simply use the payoff function for $p_{\mathrm{A}}(s)$.[a]

---

[a]Contributed by Mr. Chou, Ming-Hsin (`R02723073`) on May 28, 2014.


a
b
c
A
B
B
C
C
C
D
D
D
D
1/(1+a+s)
1/(1+b+s)
1/(1+bv+s)
1/(1+c+s)
1/(1+cv+s)
1/(1+cv^2+s)
(a)
−1/(1+a+s)^2
−1/(1+b+s)^2
−1/(1+bv+s)^2
−1/(1+c+s)^2
−1/(1+cv+s)^2
−1/(1+cv^2+s)^2
(b)
p_A(s) = c + (p_B(s) + p_C(s)) / (2(1+r+s))
p'_A(s) = (p'_B(s) + p'_C(s)) / (2(1+r+s)) − (p_B(s) + p_C(s)) / (2(1+r+s)^2)
r
p_B(s)
p'_B(s)
p_C(s)
p'_C(s)
(c)

Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 980).
- This is called the differential tree method.[a]
  - Similar ideas can be found in automatic differentiation (AD)[b] and backpropagation[c] in artificial neural networks.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n)$.

---

[a] Lyuu (1999).
[b] Rall (1981).
[c] Werbos (1974); Rumelhart, Hinton, & Williams (1986).

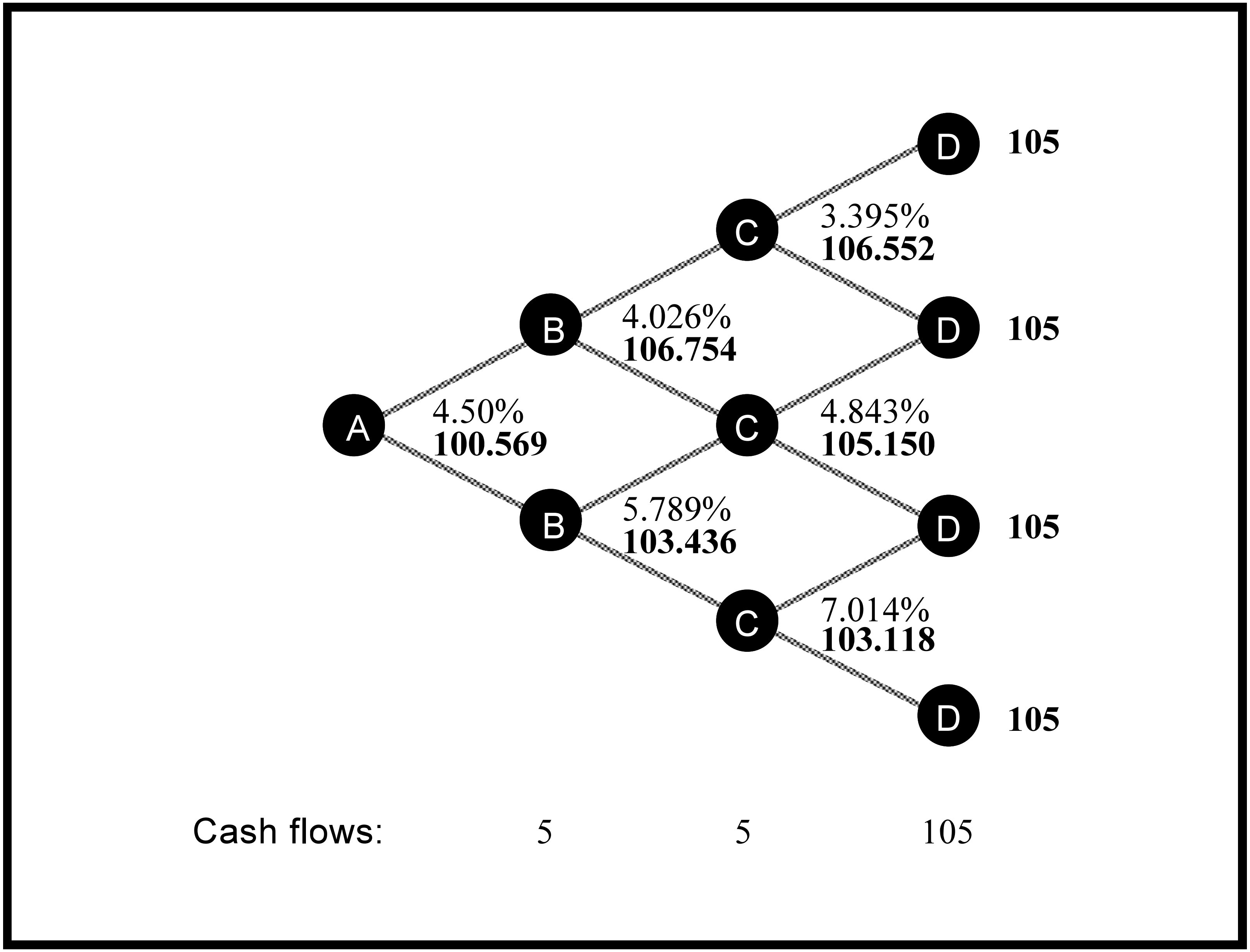
Spread of Nonbenchmark Bonds (continued)

Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 984).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 124) and static spread (p. 125) of the nonbenchmark bond over an otherwise identical benchmark bond.


D
105
C
3.395%
106.552
B
4.026%
106.754
D
105
A
4.50%
100.569
C
4.843%
105.150
B
5.789%
103.436
D
105
C
7.014%
103.118
D
105
Cash flows:
5
5
105

More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)[a]

American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

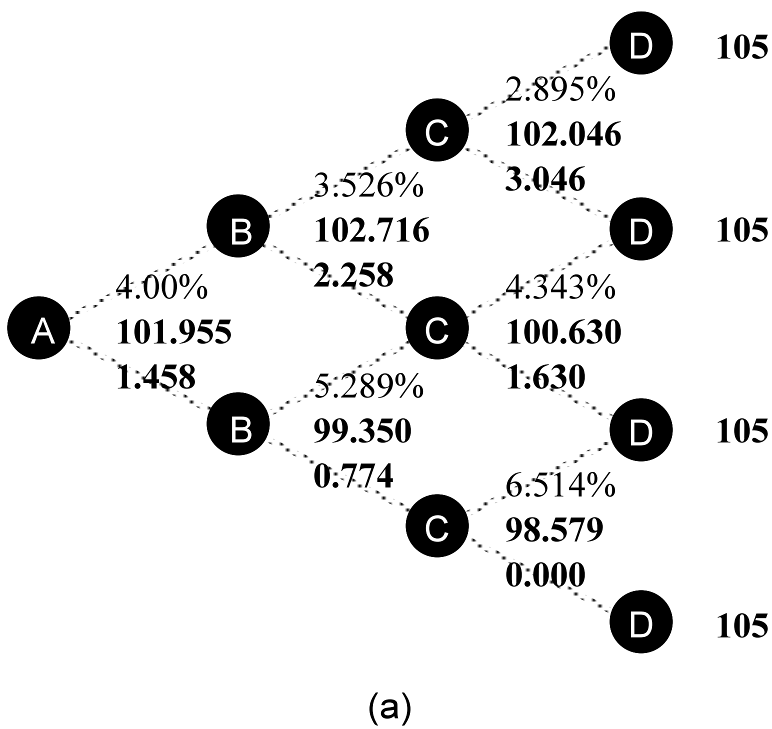
Intel 166MHz Pentium, running on Microsoft Windows 95.

---

[a]Lyuu (1999).

Fixed-Income Options

- Consider a 2-year 99 European call on the 3-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 987 the 3-year Treasury's price minus the \$5 interest at year 2 could be \$102.046, \$100.630, or \$98.579 two years from now.
  - The accrued interest is *not* included as it belongs to the original bondholder.
- Now compare the strike price against the bond prices.
- The call is in the money in the first two scenarios out of the money in the third.



A
4.00%
101.955
1.458
B
3.526%
102.716
2.258
B
5.289%
99.350
0.774
C
2.895%
102.046
3.046
C
4.343%
100.630
1.630
C
6.514%
98.579
0.000
D
105
D
105
D
105
D
105
(a)
A
4.00%
101.955
0.096
B
3.526%
102.716
0.000
B
5.289%
99.350
0.200
C
2.895%
102.046
0.000
C
4.343%
100.630
0.000
C
6.514%
98.579
0.421
D
105
D
105
D
105
D
105
(b)

Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 987(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 987(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\text{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\text{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \text{PV}(I) - \text{PV}(X).$$

Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above P_{h} and P_{ℓ} denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_{ℓ} denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 987 as examples.
- Their deltas are

$$\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441,$$
$$\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_{ℓ}, respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

$$\frac{1}{2} \ln \left(\frac{y_{\mathrm{h}}}{y_{\ell}} \right). \tag{128}$$

Volatility Term Structures (continued)

- For example, based on the tree on p. 970, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2} \ln \left(\frac{0.05289}{0.03526} \right) = 20.273\%.$$

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

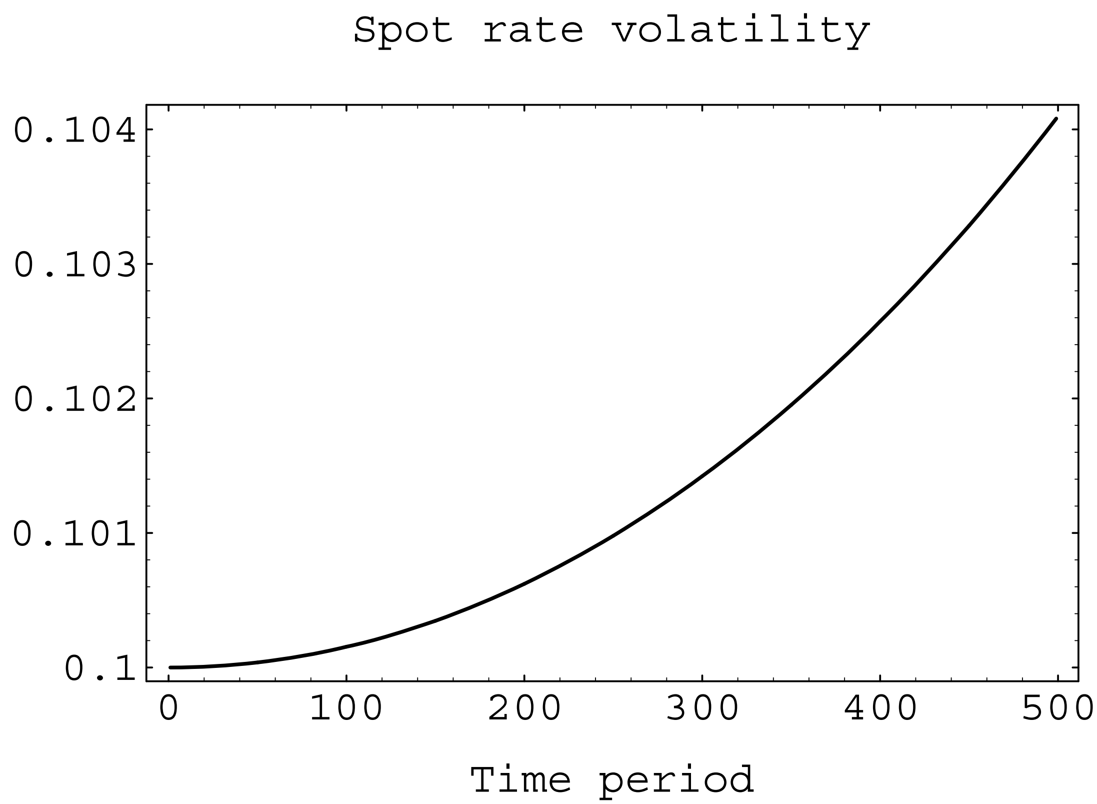
Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

$$\frac{1}{2} \ln \left(\frac{0.053531}{0.0357} \right) = 20.256\%,$$

  slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.[a]
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

[a]The relation is reversed for *price* volatilities (duration).



Short rate volatility given flat %10 volatility term structure.

Volatility Term Structures (concluded)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (123) on p. 948—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.[a]

[a] Black, Derman, & Toy (1990).

Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.
— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader [$\cdots$]
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later.[a]

$P(t, T)$**:** the present value at time t of one dollar at time T.

---

[a] Alternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

$r(t, T)$**:** the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.[a]

$F(t, T, M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

---

[a] In other words, the $(T - t)$-period spot rate at time t.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases} \tag{129}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{130}$$

  - The forward price equals the future value at time T of the underlying asset.[a]

- Equation (130) holds whether the model is discrete-time or continuous-time.

[a] See Exercise 24.2.1 of the textbook for proof.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)} \right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)} \right)^{1/L} - 1 \tag{131}$$

  in discrete time.

- The analog to Eq. (131) under simple compounding is

$$f(t,T,L) = \frac{1}{L} \left(\frac{P(t,T)}{P(t,T+L)} - 1 \right).$$

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{132}$$

by Eq. (130) on p. 1006.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \stackrel{\Delta}{=} \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{133}$$

- Because Eq. (133) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{134}$$

the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s)\,ds}{T-t}.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (134) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{135}$$

  - Relation (135) in fact follows from the risk-neutral valuation principle.[a]

[a]Theorem 18 on p. 521.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (135) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t, T) \\
= \; & E_t^{\pi}\left[\frac{P(t+1, T)}{1+r(t)}\right] \\
= \; & E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\,P(t+2, T)\,]}{(1+r(t))(1+r(t+1))}\right]=\cdots \\
= \; & E_t^{\pi}\left[\frac{1}{(1+r(t))(1+r(t+1)) \cdots(1+r(T-1))}\right]. \qquad (136)
\end{aligned}
$$

Risk-Neutral Pricing (concluded)

- Equation (135) on p. 1011 can also be expressed as

$$E_t[\, P(t+1,T)\,] = F(t,t+1,T).$$

  - Verify that with, e.g., Eq. (130) on p. 1006.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.[a]

[a]But the forward rate is not an unbiased estimator of the expected future short rate (p. 962).

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t, T) = E_t \left[e^{-\int_t^T r(s)\, ds} \right], \quad t < T. \tag{137}$$

- Note that $e^{\int_t^T r(s)\, ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the *floating-rate payer*.
- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(e^{\int_{t_{i-1}}^{t_i} r(s)\,ds} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} \left[P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right] \\
= & P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \stackrel{\Delta}{=} \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{138}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Term Structure Equation[a]

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

[a] Vasicek (1977).

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\ = \quad & (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\ & + (-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW. \end{aligned}$$

- Pick

$$\alpha \stackrel{\Delta}{=} \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \stackrel{\Delta}{=} \lambda(r,t) \tag{139}$$

  for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right)/P, \tag{140}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r}\right)/P, \tag{140'}$$

subject to $P(\,\cdot\,,T,T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (139) on p. 1024 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{141}$$

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once P is available, the spot rate curve emerges via

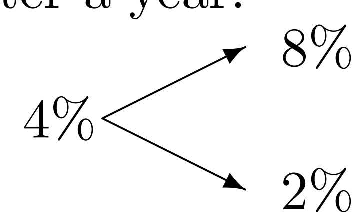
$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



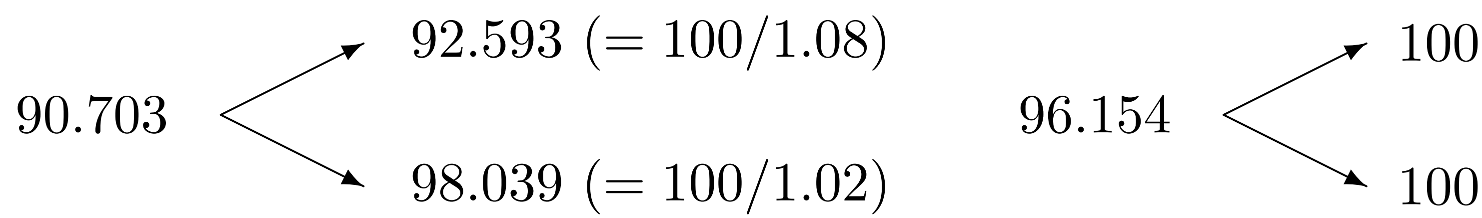
Numerical Examples (continued)

- *No* real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 1029.

Numerical Examples (continued)



The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$C \begin{cases} \nearrow & 0.000 \\ \searrow & 3.039 \end{cases}$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$
\begin{aligned}
x \times 100 + y \times 92.593 &= 0.000, \\
x \times 100 + y \times 98.039 &= 3.039.
\end{aligned}
$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

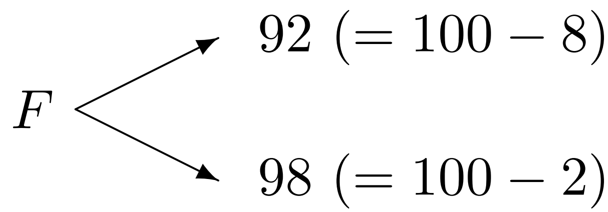
- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:



- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

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[a] See Exercise 13.2.11 of the textbook or p. 522.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

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[a]By Eq. (130) on p. 1006.

[b]Recall p. 466.