Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- The main difficulty with pricing discrete barrier options lies in matching the monitored times.
- Here is why.

Pricing Discrete Barrier Options (continued)

• Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \dots, t_\ell = T.$$

- It is unlikely that all monitored times coincide with the end of a period on the tree, or Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t₁, followed by one from time t₁ to time t₂, and so on until time t_ℓ.



Pricing Discrete Barrier Options (concluded)

• This procedure works even if each t_i is associated with a distinct barrier or if each window $[t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.^a
 - 1. The one we saw earlier (p. 305) models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
 - One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

^aFrishling (2002).

- Realistic models assume:
 - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
 - The dividend is part cash and part yield (i.e., $\alpha(t)S_0 + \beta(t)S_t$), for practitioners.^a
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But they result in binomial trees that grow exponentially (recall p. 304).
- The binomial-trinomial tree can avoid this problem in most cases.

^aHenry-Labordère (2009).

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \stackrel{\Delta}{=} t/\Delta t$ periods between time 0 and the ex-dividend date.^a
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \ge D.$

- Or,

$$\Delta t \ge \left[\frac{t\sigma}{\ln(S/D)}\right]^2.$$

^aOr simply assume *m* is an integer input and $\Delta t \stackrel{\Delta}{=} t/m$.

- Build a CRR tree from time 0 to time t as before.
- Subtract *D* from all the stock prices on the tree at time *t* to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
 - As usual, its two successor nodes will have prices S'u and $S'u^{-1}$.
- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \dots,$$

same as the CRR tree.



- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 703.
 - And even with the displacements $\pm 2\sigma\sqrt{\Delta t}$ (as on p. 731).

- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.^a

^aDai (B82506025, R86526008, D8852600) & Lyuu (2004); Dai (B82506025, R86526008, D8852600) (2009).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.^a
- Option pricing with stochastic volatilities.^b
- Efficient Parisian option pricing.^c
- Option pricing with time-varying volatilities and time-varying barriers.^d
- Defaultable bond pricing.^e

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<sup>a</sup>H. Wu (R96723058) (2009).
<sup>b</sup>C. Huang (R97922073) (2010).
<sup>c</sup>Y. Huang (R97922081) (2010).
<sup>d</sup>C. Chou (R97944012) (2010); C. Chen (R98922127) (2011).
<sup>e</sup>Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2009, 2010, 2014).
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General Properties of Trees^a

• Consider the Ito process,

 $dX = a(X, t) dt + \sigma dW,$

where a(X, t) = O(1) and σ is a constant.

- The mean and volatility of the next move's size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.
- Note that $\sqrt{\Delta t} \gg \Delta t$.
- The tree spacing must be in the order of $\sigma\sqrt{\Delta t}$ if the variance is to be matched.^b

^aChiu (R98723059) (2012); C. H. Wu (R99922149) (2012). ^bLyuu & C. Wang (F95922018) (2009, 2011); Lyuu & Wen (D94922003) (2012).

Merton's Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.
- Stochastic volatility and jump processes have been proposed to address this problem.
- Merton's (1976) jump-diffusion model is our focus.

- This model superimposes a jump component on a diffusion component.
- The diffusion component is the familiar geometric Brownian motion.
- The jump component is composed of lognormal jumps driven by a Poisson process.
 - It models the sudden changes in the stock price because of the arrival of important new information.

- Let S_t be the stock price at time t.
- The risk-neutral jump-diffusion process for the stock price follows

$$\frac{dS_t}{S_t} = (r - \lambda \bar{k}) dt + \sigma dW_t + k dq_t.$$
(96)

• Above, σ denotes the volatility of the diffusion component.

• The jump event is governed by a compound Poisson process q_t with intensity λ , where k denotes the magnitude of the *random* jump.

- The distribution of k obeys

 $\ln(1+k) \sim N\left(\gamma, \delta^2\right)$

with mean $\bar{k} \stackrel{\Delta}{=} E(k) = e^{\gamma + \delta^2/2} - 1.$

• The model with $\lambda = 0$ reduces to the Black-Scholes model.

• The solution to Eq. (96) on p. 738 is

$$S_t = S_0 e^{(r - \lambda \bar{k} - \sigma^2/2)t + \sigma W_t} U(n(t)), \qquad (97)$$

where

$$U(n(t)) = \prod_{i=0}^{n(t)} (1+k_i).$$

-
$$k_i$$
 is the magnitude of the *i*th jump with
 $\ln(1+k_i) \sim N(\gamma, \delta^2).$
- $k_0 = 0.$

-n(t) is a Poisson process with intensity λ .

- Recall that n(t) denotes the number of jumps that occur up to time t.
- As k > -1, stock prices will stay positive.
- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.

Tree for Merton's Jump-Diffusion $\mathsf{Model}^{\mathrm{a}}$

- Define the S-logarithmic return of the stock price S' as $\ln(S'/S)$.
- Define the logarithmic distance between stock prices S'and S as

$$|\ln(S') - \ln(S)| = |\ln(S'/S)|.$$

^aDai (B82506025, R86526008, D8852600), C. Wang (F95922018), Lyuu, & Y. Liu (2010).

• Take the logarithm of Eq. (97) on p. 740:

$$M_t \stackrel{\Delta}{=} \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t,\tag{98}$$

where

$$X_{t} \stackrel{\Delta}{=} \left(r - \lambda \bar{k} - \frac{\sigma^{2}}{2}\right) t + \sigma W_{t}, \qquad (99)$$
$$Y_{t} \stackrel{\Delta}{=} \sum_{i=0}^{n(t)} \ln\left(1 + k_{i}\right). \qquad (100)$$

• It decomposes the S_0 -logarithmic return of S_t into the diffusion component X_t and the jump component Y_t .

- Motivated by decomposition (98) on p. 743, the tree construction divides each period into a diffusion phase followed by a jump phase.
- In the diffusion phase, X_t is approximated by the BOPM.
- So X_t makes an up move to $X_t + \sigma \sqrt{\Delta t}$ with probability p_u or a down move to $X_t - \sigma \sqrt{\Delta t}$ with probability p_d .

• According to BOPM,

$$p_u = \frac{e^{\mu\Delta t} - d}{u - d},$$

$$p_d = 1 - p_u,$$

except that $\mu = r - \lambda \bar{k}$ here.

- The diffusion component gives rise to diffusion nodes.
- They are spaced at $2\sigma\sqrt{\Delta t}$ apart such as the white nodes A, B, C, D, E, F, and G on p. 746.



White nodes are diffusion nodes. Gray nodes are jump nodes. In the diffusion phase, the solid black lines denote the binomial structure of BOPM; the dashed lines denote the trinomial structure. Here m is set to one for simplicity. Only the double-circled nodes will remain after the construction. Note that a and b are diffusion nodes because no jump occurs in the jump phase.

Tree for Merton's Jump-Diffusion Model (concluded)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from *each* diffusion node to 2m jump nodes that match the first 2m moments of the lognormal jump.
- The *m* jump nodes above the diffusion node are spaced at $h \stackrel{\Delta}{=} \sqrt{\gamma^2 + \delta^2}$ apart.
- The same holds for the *m* jump nodes below the diffusion node.
- The gray nodes at time $\ell \Delta t$ on p. 746 are jump nodes.
- The size of the tree is $O(n^{2.5})$.

Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff

$$\max\left(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0\right),\,$$

where α_i is the percentage of asset *i*.

- Basket options are essentially options on a portfolio of stocks; they are index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Multivariate Contingent Claims (concluded) a

Name	Payoff	
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$	
Better-off option	$\max(S_1(\tau),\ldots,S_k(\tau),0)$	
Worst-off option	$\min(S_1(\tau),\ldots,S_k(\tau),0)$	
Binary maximum option	$I\{\max(S_1(\tau),\ldots,S_k(\tau))>X\}$	
Maximum option	$\max(\max(S_1(\tau),\ldots,S_k(\tau))-X,0)$	
Minimum option	$\max(\min(S_1(\tau),\ldots,S_k(\tau))-X,0)$	
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$	
Basket average option	$\max((S_1(\tau) + \dots + S_k(\tau))/k - X, 0)$	
Multi-strike option	$\max(S_1(\tau) - X_1, \dots, S_k(\tau) - X_k, 0)$	
Pyramid rainbow option	$\max(S_1(\tau) - X_1 + \dots + S_k(\tau) - X_k - X$	0)
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \dots + (S_k(\tau) - X_k)^2})$	-X, 0)
^a Lyuu & Teng (R91723054) (2011).		

Correlated Trinomial Model $^{\rm a}$

• Two risky assets S_1 and S_2 follow

$$\frac{dS_i}{S_i} = r \, dt + \sigma_i \, dW_i$$

in a risk-neutral economy, i = 1, 2.

• Let

$$M_i \stackrel{\Delta}{=} e^{r\Delta t},$$
$$V_i \stackrel{\Delta}{=} M_i^2 (e^{\sigma_i^2 \Delta t} - 1).$$

 $-S_iM_i$ is the mean of S_i at time Δt .

 $-S_i^2 V_i$ the variance of S_i at time Δt .

^aBoyle, Evnine, & Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of S_1S_2 at time Δt has a joint lognormal distribution with mean $S_1S_2M_1M_2e^{\rho\sigma_1\sigma_2\Delta t}$, where ρ is the correlation between dW_1 and dW_2 .
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are 5 distinct outcomes.

Correlated Trinomial Model (continued)

• The five-point probability distribution of the asset prices is

Probability	Asset 1	Asset 2
p_1	S_1u_1	$S_2 u_2$
p_2	S_1u_1	$S_2 d_2$
p_3	S_1d_1	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

• As usual, impose $u_i d_i = 1$.

Correlated Trinomial Model (continued)

• The probabilities must sum to one, and the means must be matched:

$$1 = p_1 + p_2 + p_3 + p_4 + p_5,$$

$$S_1 M_1 = (p_1 + p_2) S_1 u_1 + p_5 S_1 + (p_3 + p_4) S_1 d_1,$$

$$S_2 M_2 = (p_1 + p_4) S_2 u_2 + p_5 S_2 + (p_2 + p_3) S_2 d_2.$$

Correlated Trinomial Model (concluded)

- Let $R \stackrel{\Delta}{=} M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$.
- Match the variances and covariance:

$$S_{1}^{2}V_{1} = (p_{1} + p_{2})((S_{1}u_{1})^{2} - (S_{1}M_{1})^{2}) + p_{5}(S_{1}^{2} - (S_{1}M_{1})^{2}) + (p_{3} + p_{4})((S_{1}d_{1})^{2} - (S_{1}M_{1})^{2}),$$

$$S_{2}^{2}V_{2} = (p_{1} + p_{4})((S_{2}u_{2})^{2} - (S_{2}M_{2})^{2}) + p_{5}(S_{2}^{2} - (S_{2}M_{2})^{2}) + (p_{2} + p_{3})((S_{2}d_{2})^{2} - (S_{2}M_{2})^{2}),$$

$$S_{1}S_{2}R = (p_{1}u_{1}u_{2} + p_{2}u_{1}d_{2} + p_{3}d_{1}d_{2} + p_{4}d_{1}u_{2} + p_{5})S_{1}S_{2}.$$

• The solutions appear on p. 246 of the textbook.



^aMadan, Milne, & Shefrin (1989).

Correlated Trinomial Model Simplified (continued)

• All of the probabilities lie between 0 and 1 if and only if

$$-1 + \lambda \sqrt{\Delta t} \left| \frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2} \right| \leq \rho \leq 1 - \lambda \sqrt{\Delta t} \left| \frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2} \right| (101)$$

$$1 \leq \lambda \qquad (102)$$

• We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as

$$-1+O(\sqrt{\Delta t})<\rho<1-O(\sqrt{\Delta t}),$$

such as the above one.^a

^aW. Kao (**R98922093**) (2011); W. Kao (**R98922093**), Lyuu, & Wen (**D94922003**) (2014).

Correlated Trinomial Model Simplified (concluded)

- But this model cannot price 2-asset 2-barrier options accurately.^a
- Few multivariate trees are both optimal and able to handle multiple barriers.^b
- An alternative is to use orthogonalization.^c

^aSee Y. Chang (B89704039, R93922034), Hsu (R7526001, D89922012), & Lyuu (2006); W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for solutions.

^bSee W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for one.
^cHull & White (1990); Dai (B82506025, R86526008, D8852600), Lyuu,
& C. Wang (F95922018) (2012).

Extrapolation

- It is a method to speed up numerical convergence.
- Say f(n) converges to an unknown limit f at rate of 1/n:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{103}$$

• Assume c is an unknown constant independent of n.

- Convergence is basically monotonic and smooth.
Extrapolation (concluded)

• From two approximations $f(n_1)$ and $f(n_2)$ and ignoring the smaller terms,

$$f(n_1) = f + \frac{c}{n_1},$$

$$f(n_2) = f + \frac{c}{n_2}.$$

• A better approximation to the desired f is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}.$$
 (104)

- This estimate should converge faster than 1/n.^a
- The Richardson extrapolation uses $n_2 = 2n_1$.

^aIt is identical to the forward rate formula (21) on p. 140!

Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by f(n).
- It is known that BOPM convergences at the rate of 1/n, consistent with Eq. (103) on p. 758.
- But the plots on p. 288 (redrawn on next page) demonstrate that convergence to the true option value oscillates with *n*.
- Extrapolation is inapplicable at this stage.



Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 761.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 763).^a
- Apply extrapolation (104) on p. 759 with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 763.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

^aThis can be proved (L. Chang & Palmer, 2007).



Numerical Methods

All science is dominated by the idea of approximation. — Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 767).
- Solve the equation numerically by introducing difference equations in place of derivatives.



Example: Poisson's Equation

- It is $\partial^2 \theta / \partial x^2 + \partial^2 \theta / \partial y^2 = -\rho(x, y)$, which describes the electrostatic field.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite difference form is

$$-\rho(x_i, y_j) = \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \stackrel{\Delta}{=} x_i x_{i-1}$ and $\Delta y \stackrel{\Delta}{=} y_j y_{j-1}$ for $i, j = 1, 2, \dots$
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$-h^{2}\rho(x_{i}, y_{j}) = \theta(x_{i+1}, y_{j}) + \theta(x_{i-1}, y_{j}) + \theta(x_{i}, y_{j+1}) + \theta(x_{i}, y_{j-1}) - 4\theta(x_{i}, y_{j}).$$

- Given boundary values, we can solve for the x_i s and the y_j s within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation $D(\partial^2 \theta / \partial x^2) - (\partial \theta / \partial t) = 0, D > 0.$
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt , where $\Delta x \stackrel{\Delta}{=} x_{i+1} x_i$ and $\Delta t \stackrel{\Delta}{=} t_{j+1} t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\frac{\partial \theta(x,t)}{\partial t}\Big|_{t=t_{j}} = \frac{\theta(x,t_{j+1}) - \theta(x,t_{j})}{\Delta t} + \cdots, \qquad (105)$$

$$\frac{\partial^2 \theta(x,t)}{\partial x^2}\Big|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots (106)$$

Explicit Methods (continued)

- Next, assemble Eqs. (105) and (106) into a single equation at (x_i, t_j) .
- But we need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (106), we might as well use x_i for x in Eq. (105).
- Two choices are possible for t in Eq. (106).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}.$$
(107)

Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- Rearrange Eq. (107) on p. 771 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2} \,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right) \theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2} \,\theta_{i-1,j}.$$

• We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see exhibit (a) on next page).



Explicit Methods (concluded)

• Starting from the initial conditions at t_0 , that is, $\theta_{i,0} = \theta(x_i, t_0), i = 1, 2, \dots$, we calculate

$$\theta_{i,1}, \quad i=1,2,\ldots$$

• And then

$$\theta_{i,2}, \quad i=1,2,\ldots$$

• And so on.

Stability

• The explicit method is numerically unstable unless

 $\Delta t \le (\Delta x)^2 / (2D).$

- A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.
- The stability condition may lead to high running times and memory requirements.
- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time 8 times as much.

Explicit Method and Trinomial Tree

• Recall that

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2} \,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right) \theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2} \,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!

Explicit Method and Trinomial Tree (concluded)

- The freedom in choosing Δx corresponds to similar freedom in the construction of trinomial trees.
- The explicit finite-difference equation is also identical to backward induction on a binomial tree.^a
 - Let the binomial tree take 2 steps each of length $\Delta t/2.$
 - It is now a trinomial tree.

^aHilliard (2014).

Implicit Methods

- Suppose we use $t = t_{j+1}$ in Eq. (106) on p. 770 instead.
- The finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}.$$
(108)

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is implicit:
 - The value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known.
 - See exhibit (b) on p. 773.

Implicit Methods (continued)

• Equation (108) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma) \,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma \theta_{i,j},$$

where $\gamma \stackrel{\Delta}{=} (\Delta x)^2 / (D\Delta t)$.

- This equation is unconditionally stable.
- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.
- After $\theta_{i,j}$ has been calculated for i = 1, 2, ..., N, the values of $\theta_{i,j+1}$ at time t_{j+1} can be computed as the solution to the following tridiagonal linear system,



Implicit Methods (concluded)

• Tridiagonal systems can be solved in O(N) time and O(N) space.

- Never invert a matrix to solve a tridiagonal system.

- The matrix above is nonsingular when $\gamma \geq 0$.
 - A square matrix is nonsingular if its inverse exists.

Crank-Nicolson Method

• Take the average of explicit method (107) on p. 771 and implicit method (108) on p. 778:

$$= \frac{\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t}}{\left(D \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2} + D \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}\right)$$

• After rearrangement,

$$\gamma \theta_{i,j+1} - \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{2} = \gamma \theta_{i,j} + \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{2}.$$

• This is an unconditionally stable implicit method with excellent rates of convergence.



Numerically Solving the Black-Scholes PDE (82) on p. 629

- See text.
- Brennan and Schwartz (1978) analyze the stability of the implicit method.

Monte Carlo Simulation $^{\rm a}$

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.
- When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo may very well be the only strategy that succeeds consistently.

^aA top 10 algorithm (Dongarra & Sullivan, 2000).

The Big Idea

- Assume X_1, X_2, \ldots, X_n have a joint distribution.
- $\theta \stackrel{\Delta}{=} E[g(X_1, X_2, \dots, X_n)]$ for some function g is desired.
- We generate

$$\left(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}\right), \quad 1 \le i \le N$$

independently with the same joint distribution as (X_1, X_2, \ldots, X_n) .

• Set

$$Y_i \stackrel{\Delta}{=} g\left(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}\right).$$

The Big Idea (concluded)

- Y_1, Y_2, \ldots, Y_N are independent and identically distributed random variables.
- Each Y_i has the same distribution as

$$Y \stackrel{\Delta}{=} g(X_1, X_2, \dots, X_n).$$

- Since the average of these N random variables, \overline{Y} , satisfies $E[\overline{Y}] = \theta$, it can be used to estimate θ .
- The strong law of large numbers says that this procedure converges almost surely.
- The number of replications (or independent trials), N, is called the sample size.

Accuracy

- The Monte Carlo estimate and true value may differ owing to two reasons:
 - 1. Sampling variation.
 - 2. The discreteness of the sample paths.^a
- The first can be controlled by the number of replications.
- The second can be controlled by the number of observations along the sample path.

^aThis may not be an issue if the financial derivative only requires discrete sampling along the time dimension, such as the discrete barrier option.

Accuracy and Number of Replications

- The statistical error of the sample mean \overline{Y} of the random variable Y grows as $1/\sqrt{N}$.
 - Because $\operatorname{Var}[\overline{Y}] = \operatorname{Var}[Y]/N$.
- In fact, this convergence rate is asymptotically optimal.^a
- So the variance of the estimator \overline{Y} can be reduced by a factor of 1/N by doing N times as much work.
- This is amazing because the same order of convergence holds independently of the dimension n.

^aThe Berry-Esseen theorem.

Accuracy and Number of Replications (concluded)

- In contrast, classic numerical integration schemes have an error bound of O(N^{-c/n}) for some constant c > 0.
 - n is the dimension.
- The required number of evaluations thus grows exponentially in n to achieve a given level of accuracy.
 The curse of dimensionality.
- The Monte Carlo method is more efficient than alternative procedures for multivariate derivatives when *n* is large.

Monte Carlo Option Pricing

- For the pricing of European options on a dividend-paying stock, we may proceed as follows.
- Assume

$$\frac{dS}{S} = \mu \, dt + \sigma \, dW.$$

• Stock prices S_1, S_2, S_3, \ldots at times $\Delta t, 2\Delta t, 3\Delta t, \ldots$ can be generated via

$$S_{i+1} = S_i e^{(\mu - \sigma^2/2) \Delta t + \sigma \sqrt{\Delta t} \xi}, \quad \xi \sim N(0, 1).$$
(109)

Monte Carlo Option Pricing (continued)

• If we discretize $dS/S = \mu dt + \sigma dW$ directly, we will obtain

$$S_{i+1} = S_i + S_i \mu \,\Delta t + S_i \sigma \sqrt{\Delta t} \,\xi.$$

- But this is locally normally distributed, not lognormally, hence biased.^a
- In practice, this is not expected to be a major problem as long as Δt is sufficiently small.

^aContributed by Mr. Tai, Hui-Chin (R97723028) on April 22, 2009.

Monte Carlo Option Pricing (continued)

• Non-dividend-paying stock prices in a risk-neutral economy can be generated by setting $\mu = r$ and $\Delta t = T$.

1: C := 0; {Accumulated terminal option value.} 2: for i = 1, 2, 3, ..., N do 3: $P := S \times e^{(r - \sigma^2/2)T + \sigma\sqrt{T} \xi}, \xi \sim N(0, 1);$ 4: $C := C + \max(P - X, 0);$ 5: end for 6: return $Ce^{-rT}/N;$


How about American Options?

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise (why?).
 - Given a sample path S_0, S_1, \ldots, S_n , how to decide which S_i is an early-exercise point?
 - What is the option price at each S_i if the option is not exercised?
- It is difficult to determine the early-exercise point based on one single path.
- But Monte Carlo simulation can be modified to price American options with small biases (pp. 847ff).^a

^aLongstaff & Schwartz (2001).