# Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is[a]

$$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F \left( \frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F.$$

- Can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k,$$

  where $F_i$ is the futures price at time $i$.

- This equation holds under stochastic interest rates, too.[b]

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[a] Recall p. 476.

[b] See Exercise 13.2.11 of the textbook.

## Martingale Pricing and Numeraire[a]

- The martingale pricing formula (67) on p. 518 uses the money market account as numeraire.[b]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset $S$'s value is positive at all times.

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[a] John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value."

[b] Leon Walras (1834–1910).

## Martingale Pricing and Numeraire (concluded)

- Choose $S$ as numeraire.
- Martingale pricing says there exists a risk-neutral probability $\pi$ under which the relative price of any asset $C$ is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of $S$ at time $j$.
- So the discount process remains a martingale.[a]

[a] This result is related to Girsanov's theorem (1960).

## Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from $S$ to $S_1$ or $S_2$.
- In a period, asset two's price can go from $P$ to $P_1$ or $P_2$.
- Both assets must move up or down at the same time.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2} \tag{68}$$

to rule out arbitrage opportunities.

## Example (continued)

- For any derivative security, let $C_1$ be its price at time one if asset one's price moves to $S_1$.
- Let $C_2$ be its price at time one if asset one's price moves to $S_2$.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using $\alpha$ units of asset one and $\beta$ units of asset two.

## Example (continued)

- By Eqs. (68) on p. 524, $\alpha$ and $\beta$ have unique solutions.
- In fact,

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

# Example (continued)

- It is easy to verify that

$$\frac{C}{P} = p\,\frac{C_1}{P_1} + (1-p)\,\frac{C_2}{P_2}.$$

  - Above,

$$p \triangleq \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

  - By Eqs. (68) on p. 524, $0 < p < 1$.

- $C$'s price using asset two as numeraire (i.e., $C/P$) is a martingale under the risk-neutral probability $p$.

- The expected returns of the two assets are *irrelevant*.

# Example (concluded)

- In the BOPM, $S$ is the stock and $P$ is the bond.
- Furthermore, $p$ assumes the bond is the numeraire.
- In the binomial option pricing formula (p. 255), the $S \sum b(j; n, pu/R)$ term uses the stock as the numeraire.
  - It results in a different probability measure $pu/R$.
- In the limit, $SN(x)$ for the call and $SN(-x)$ for the put in the Black-Scholes formula (p. 285) use the stock as the numeraire.[a]

[a] See Exercise 13.2.12 of the textbook.

## Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  **1.** $X(0) = 0$, unless stated otherwise.

  **2.** for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  **3.** for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where $\mu$ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

# Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a $(\mu, \sigma)$ Brownian motion with drift $\mu$ and variance $\sigma^2$.
- Although Brownian motion is a continuous function of $t$ with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.
- If condition 3 is replaced by "$X(t) - X(s)$ depends only on $t - s$," we have the more general Levy process.[b]

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[a] Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

[b] Paul Levy (1886–1971).

## Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then

$$X(t) - X(s) \sim N(0, t - s).$$

- A $(\mu, \sigma)$ Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{69}$$

- Note that

$$Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s).$$

## Brownian Motion as Limit of Random Walk

**Claim 1** *A* $(\mu, \sigma)$ *Brownian motion is the limiting case of random walk.*

- A particle moves $\Delta x$ to the right with probability $p$ after $\Delta t$ time.
- It moves $\Delta x$ to the left with probability $1 - p$.
- Define

$$X_i \stackrel{\Delta}{=} \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  – $X_i$ are independent with

$$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,].$$

## Brownian Motion as Limit of Random Walk (continued)

- Assume $n \stackrel{\Delta}{=} t/\Delta t$ is an integer.
- Its position at time $t$ is

$$Y(t) \stackrel{\Delta}{=} \Delta x \left( X_1 + X_2 + \cdots + X_n \right).$$

- Recall

$$\begin{aligned} E[\, X_i \,] &= 2p - 1, \\ \mathrm{Var}[\, X_i \,] &= 1 - (2p - 1)^2. \end{aligned}$$

## Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$

$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \stackrel{\Delta}{=} \sigma\sqrt{\Delta t}$ and $p \stackrel{\Delta}{=} [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,[a]

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

---

[a] Identical to Eq. (38) on p. 278!

## Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a $(\mu, \sigma)$ Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Similarity to the the BOPM: The $p$ is identical to the probability in Eq. (38) on p. 278 and $\Delta x = \ln u$.
- Note that

$$
\begin{aligned}
&\mathrm{Var}[\, Y(t + \Delta t) - Y(t) \,] \\
=&\mathrm{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \mathrm{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t.
\end{aligned}
$$

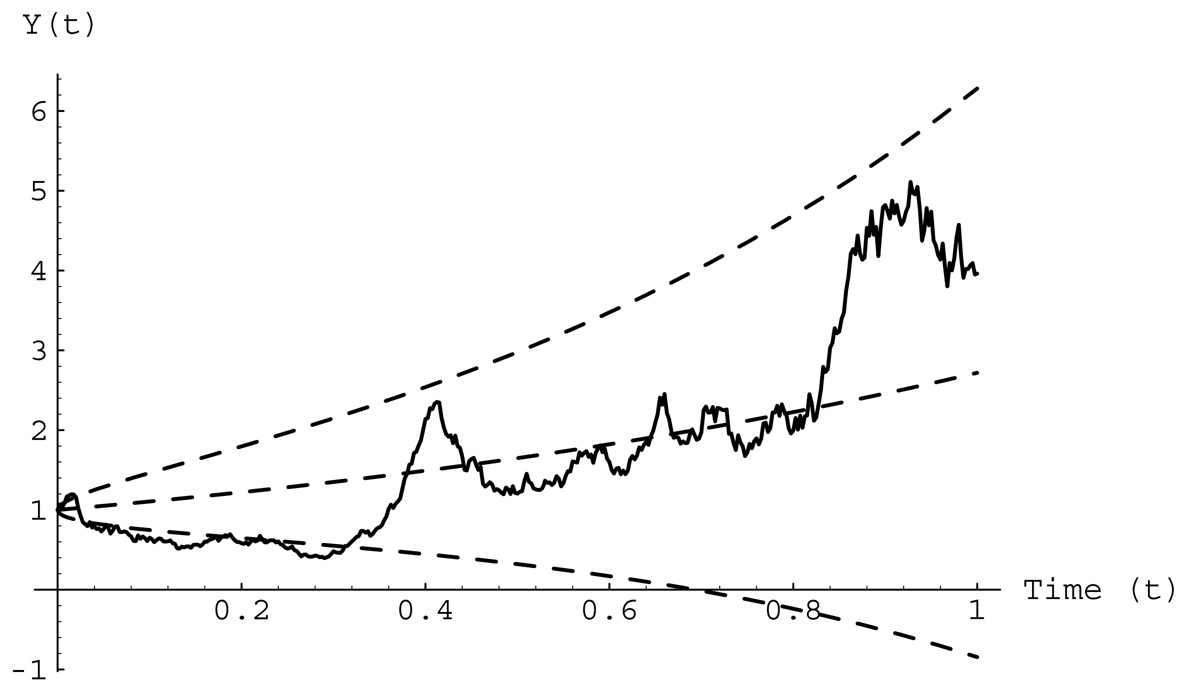
# Geometric Brownian Motion

- Let $X \stackrel{\Delta}{=} \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process
$$\{ Y(t) \stackrel{\Delta}{=} e^{X(t)}, t \geq 0 \},$$
is called geometric Brownian motion.
- Suppose further that $X$ is a $(\mu, \sigma)$ Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function
$$E\left[ e^{sX(t)} \right] = E\left[ Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$
from Eq. (25) on p 158.

# Geometric Brownian Motion (concluded)

- In particular,

$$
\begin{aligned}
E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)},\\
\mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2\\
&= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t}-1\right).
\end{aligned}
$$


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

## A Case for Long-Term Investment[a]

- Suppose the stock follows the geometric Brownian motion

$$S(t) = S(0)\, e^{N(\mu t, \sigma^2 t)} = S(0)\, e^{t N(\mu, \sigma^2 / t)}, \quad t \geq 0,$$

where $\mu > 0$.

- The annual rate of return has a normal distribution:

$$N\left(\mu, \frac{\sigma^2}{t}\right).$$

- The larger the $t$, the likelier the return is positive.
- The smaller the $t$, the likelier the return is negative.

[a] Contributed by Prof. King, Gow-Hsing on April 9, 2015. See `http://www.cb.idv.tw/phpbb3/viewtopic.php?f=7&t=1025`

*Continuous-Time Financial Mathematics*

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
*Science and the Modern World*

## Stochastic Integrals

- Use $W \stackrel{\Delta}{=} \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of $X$ from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \stackrel{\Delta}{=} \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of $X$ with respect to $W$.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a] Kiyoshi Ito (1915–2008).

## Stochastic Integrals (concluded)

- Typical requirements for $X$ in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time $t$ includes the history of $X$ and $W$ up to that point in time.
  - But it contains nothing about the evolution of $X$ or $W$ after $t$ (nonanticipating, so to speak).
  - The future cannot influence the present.

## Ito Integral

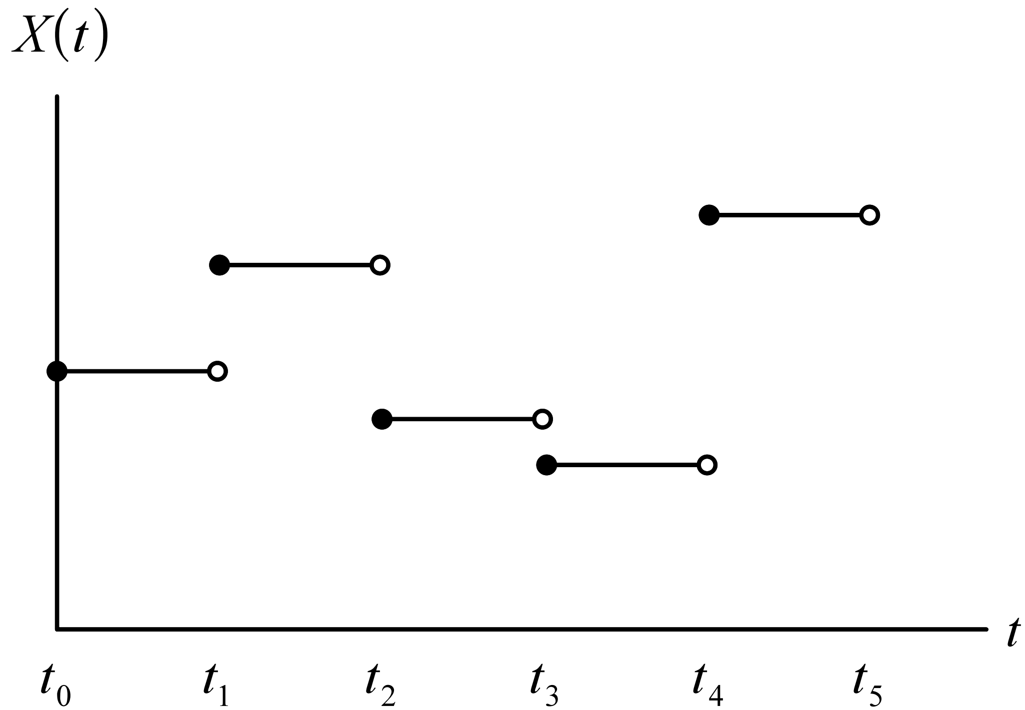
- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist

$$0 = t_0 < t_1 < \cdots$$

such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots$$

for any realization (see figure on next page).



X(t)
t
t0
t1
t2
t3
t4
t5

# Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \triangleq \sum_{k=0}^{n-1} X(t_k)[\, W(t_{k+1}) - W(t_k) \,], \tag{70}$$

  where $t_n = t$.

  – The integrand $X$ is evaluated at $t_k$, not $t_{k+1}$.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

## Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that $X_n$ converges in probability to $X$.
- If $X$ is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as

$$\delta_n \stackrel{\Delta}{=} \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

# Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.[a]

**Theorem 19** *The Ito integral* $\int X\, dW$ *is a martingale.*

- A corollary is the mean value formula

$$E\left[\int_a^b X\, dW\right] = 0.$$

[a]See Exercise 14.1.1 for simple stochastic processes.

## Discrete Approximation

- Recall Eq. (70) on p. 546.
- The following *simple* stochastic process $\{\,\widehat{X}(t)\,\}$ can be used in place of $X$ to approximate $\int_0^t X\,dW$,

$$\widehat{X}(s) \stackrel{\Delta}{=} X(t_{k-1}) \text{ for } s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time $s$,

$$\{\,\widehat{X}(t), W(t), 0 \le t \le s\,\},$$

    cannot determine the future evolution of $X$ or $W$.

## Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

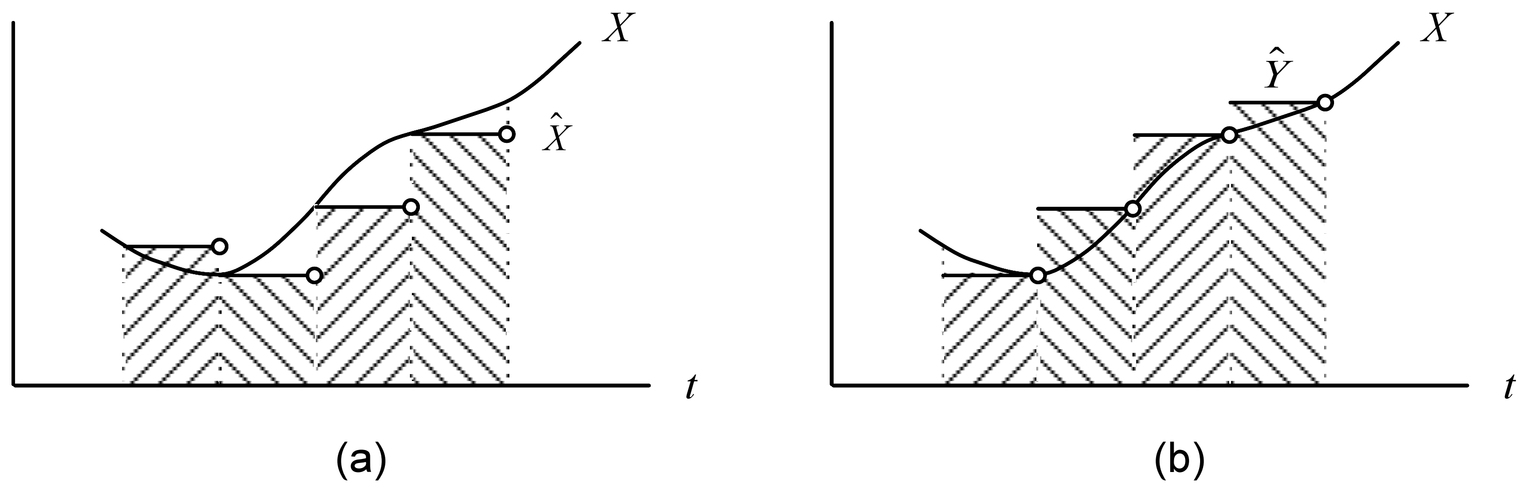
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \stackrel{\Delta}{=} X(t_k) \quad \text{for } \; s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of $X$.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.



X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

## Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  - $X_0$ is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.
  - $a(X_t, t)$: the drift.
  - $b(X_t, t)$: the diffusion.

# Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential $dX_t$,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{71}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an *instantaneous* drift $a_t$ and an *instantaneous* variance $b_t^2$.

- $X$ is a martingale if $a_t = 0$ (Theorem 19 on p. 548).

[a]Paul Langevin (1872–1946) in 1904.

## Ito Process (concluded)

- $dW$ is normally distributed with mean zero and variance $dt$.
- An equivalent form of Eq. (71) is
$$dX_t = a_t\,dt + b_t\sqrt{dt}\;\xi, \tag{72}$$
where $\xi \sim N(0,1)$.

# Euler Approximation

- Define $t_n \stackrel{\Delta}{=} n\Delta t$.
- The following approximation follows from Eq. (72),

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n).\end{aligned} \tag{73}$$

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as

$$W(t_{n+1}) - W(t_n),$$

not $W(t_n) - W(t_{n-1})$!

## Euler Approximation (concluded)

- With the Euler method, one can obtain a sample path

$$\widehat{X}(t_1), \widehat{X}(t_2), \widehat{X}(t_3), \ldots$$

  from a sample path

$$W(t_0), W(t_1), W(t_2), \ldots .$$

- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

## More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (73) on p. 555 can be replaced by

$$
\begin{aligned}
&\widehat{X}(t_{n+1}) \\
=&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).
\end{aligned}
$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

# More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$
\begin{aligned}
&\widehat{X}(t_{n+1})\\
=&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,\xi.
\end{aligned}
$$

  - $\text{Prob}[\,\xi = 1\,] = \text{Prob}[\,\xi = -1\,] = 1/2$.
  - Note that $E[\,\xi\,] = 0$ and $\text{Var}[\,\xi\,] = 1$.

- This is a binomial model.

- As $\Delta t$ goes to zero, $\widehat{X}$ converges to $X$.[a]

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[a] He (1990).

## Trading and the Ito Integral

- Consider an Ito process

$$d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t.$$

  - $\boldsymbol{S}_t$ is the vector of security prices at time $t$.

- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time $t$.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time $t$.

- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \stackrel{\Delta}{=} \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time $t$.

## Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \stackrel{\Delta}{=} \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

# Ito's Lemma[a]

A smooth function of an Ito process is itself an Ito process.

**Theorem 20** *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

*for* $t \geq 0$.

---

[a] Ito (1944).

# Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\,a\,dt + f'(X)\,b\,dW + \frac{1}{2}\,f''(X)\,b^2\,dt. \tag{74}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.

- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\,dX + \frac{1}{2}\,f''(X)(dX)^2.$$

# Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

| $\times$ | $dW$ | $dt$ |
|---|---|---|
| $dW$ | $dt$ | $0$ |
| $dt$ | $0$ | $0$ |

  - The $(dW)^2 = dt$ entry is justified by a known result.
- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$.
- This form is easy to remember because of its similarity to the Taylor expansion.

## Ito's Lemma (continued)

**Theorem 21 (Higher-Dimensional Ito's Lemma)** *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \triangleq (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* $X_i$ *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

*where* $f_i \triangleq \partial f / \partial X_i$ *and* $f_{ik} \triangleq \partial^2 f / \partial X_i \partial X_k$*.*

## Ito's Lemma (continued)

- The multiplication table for Theorem 21 is

| $\times$ | $dW_i$ | $dt$ |
|---|---|---|
| $dW_k$ | $\delta_{ik}\, dt$ | $0$ |
| $dt$ | $0$ | $0$ |

in which

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

## Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say $X_1$, is time $t$ and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all $j$ and $a_1 = 1$.
- As an example, let

$$dX_t = a_t \, dt + b_t \, dW_t.$$

- Consider the process $f(X_t, t)$.

## Ito's Lemma (continued)

- Then

$$
\begin{aligned}
df &= \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(dX_t)^2 \\
&= \frac{\partial f}{\partial X_t}\,(a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(a_t\, dt + b_t\, dW_t)^2 \\
&= \left( \frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, b_t^2 \right) dt \\
&\quad + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \qquad (75)
\end{aligned}
$$

# Ito's Lemma (continued)

**Theorem 22 (Alternative Ito's Lemma)** *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \stackrel{\Delta}{=} (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* $X_i$ *is an Ito process with* $dX_i = a_i\,dt + b_i\,dW_i$. *Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\,dX_i + \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{m} f_{ik}(X)\,dX_i\,dX_k.$$

# Ito's Lemma (concluded)

- The multiplication table for Theorem 22 is

| $\times$ | $dW_i$ | $dt$ |
|---|---|---|
| $dW_k$ | $\rho_{ik}\,dt$ | 0 |
| $dt$ | 0 | 0 |

- Above, $\rho_{ik}$ denotes the correlation between $dW_i$ and $dW_k$.

# Geometric Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \stackrel{\Delta}{=} e^{X(t)}.$$

  - $X(t)$ is a $(\mu, \sigma)$ Brownian motion.
  - By Eq. (69) on p. 531,

$$dX = \mu\, dt + \sigma\, dW.$$

- Note that

$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

## Geometric Brownian Motion (continued)

- Ito's formula (74) on p. 562 implies

$$\begin{aligned} dY &= Y\,dX + (1/2)\,Y\,(dX)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\,(\mu\,dt + \sigma\,dW)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\sigma^2\,dt. \end{aligned}$$

- Hence

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\,dW. \tag{76}$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not $\mu$).[a]

---

[a] Consistent with Lemma 11 (p. 283).

## Geometric Brownian Motion (concluded)

- Similarly, suppose

$$\frac{dY}{Y} = \mu\, dt + \sigma\, dW.$$

- Then $X(t) \stackrel{\Delta}{=} \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma\, dW.$$

## Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned}\frac{dY}{Y} &= a\,dt + b\,dW_Y,\\ \frac{dZ}{Z} &= f\,dt + g\,dW_Z.\end{aligned}$$

- Assume $dW_Y$ and $dW_Z$ have correlation $\rho$.
- Consider the Ito process

$$U \stackrel{\Delta}{=} YZ.$$

# Product of Geometric Brownian Motion Processes (continued)

- Apply Ito's lemma (Theorem 22 on p. 568):

$$
\begin{aligned}
dU &= Z\,dY + Y\,dZ + dY\,dZ \\
&= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\
&\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\
&= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z.
\end{aligned}
$$

- The product of correlated geometric Brownian motion processes thus remains geometric Brownian motion.

## Product of Geometric Brownian Motion Processes (continued)

- Note that

$$
\begin{aligned}
Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\
Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\
U &= \exp\left[\,\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\,\right].
\end{aligned}
$$

  – There is no $bg\rho$ term in $U$!

## Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if $Y$ and $Z$ are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation $\rho$.

## Quotients of Geometric Brownian Motion Processes

- Suppose $Y$ and $Z$ are drawn from p. 573.
- Let
$$U \triangleq Y/Z.$$
- We now show that[a]
$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{77}$$
- Keep in mind that $dW_Y$ and $dW_Z$ have correlation $\rho$.

[a]Exercise 14.3.6 of the textbook is erroneous.

## Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 22 on p. 568) can be employed to show that

$$
\begin{aligned}
& dU \\
= \;& (1/Z)\,dY - (Y/Z^2)\,dZ - (1/Z^2)\,dY\,dZ + (Y/Z^3)\,(dZ)^2 \\
= \;& (1/Z)(aY\,dt + bY\,dW_Y) - (Y/Z^2)(fZ\,dt + gZ\,dW_Z) \\
& -(1/Z^2)(bgYZ\rho\,dt) + (Y/Z^3)(g^2 Z^2\,dt) \\
= \;& U(a\,dt + b\,dW_Y) - U(f\,dt + g\,dW_Z) \\
& -U(bg\rho\,dt) + U(g^2\,dt) \\
= \;& U(a - f + g^2 - bg\rho)\,dt + Ub\,dW_Y - Ug\,dW_Z.
\end{aligned}
$$

## Forward Price

- Suppose $S$ follows
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$
- Consider $F(S,t) \stackrel{\Delta}{=} Se^{y(T-t)}$ for some constants $y$ and $T$.
- As $F$ is a function of two variables, we need the various partial derivatives of $F(S,t)$ with respect to $S$ and $t$.
- Note that in partial differentiation with respect to one variable, other variables are held constant.[a]

[a]Contributed by Mr. Sun, Ao (`R05922147`) on April 26, 2017.

# Forward Prices (continued)

- Now,

$$\begin{aligned}\frac{\partial F}{\partial S} &= e^{y(T-t)},\\ \frac{\partial^2 F}{\partial S^2} &= 0,\\ \frac{\partial F}{\partial t} &= -ySe^{y(T-t)}.\end{aligned}$$

- Then

$$\begin{aligned} dF &= e^{y(T-t)}\, dS - ySe^{y(T-t)}\, dt\\ &= Se^{y(T-t)}\,(\mu\, dt + \sigma\, dW) - ySe^{y(T-t)}\, dt\\ &= F(\mu - y)\, dt + F\sigma\, dW.\end{aligned}$$

## Forward Prices (concluded)

- One can also prove it by Eq. (75) on p. 567.
- Thus $F$ follows
$$\frac{dF}{F} = (\mu - y)\,dt + \sigma\,dW.$$
- This result has applications in forward and futures contracts.
- In Eq. (52) on p. 446, $\mu = r = y$.
- So
$$\frac{dF}{F} = \sigma\,dW,$$
a martingale.[a]

[a] It is also consistent with p. 521.

## Ornstein-Uhlenbeck (OU) Process

- The OU process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- For $t_0 \leq s \leq t$ and $X(t_0) = x_0$, it is known that

$$\begin{aligned}
E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\
\mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\
\mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[1 - e^{-2\kappa(s-t_0)}\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,].
\end{aligned}$$

## Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if $x_0$ is a constant or normally distributed.
- $X$ is said to be a normal process.
- $E[\,x_0\,] = x_0$ and $\text{Var}[\,x_0\,] = 0$ if $x_0$ is a constant.
- The OU process has the following mean reversion property.
  - When $X > 0$, $X$ is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

## Ornstein-Uhlenbeck Process (continued)

- A generalized version:

$$dX = \kappa(\mu - X)\,dt + \sigma\,dW,$$

where $\kappa, \sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\,X(t)\,] &= \mu + (x_0 - \mu)\,e^{-\kappa(t-t_0)}, \qquad (78) \\ \mathrm{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right], \end{aligned}$$

for $t_0 \leq t$.

# Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly $\mu$ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large $t$, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - $X$ tends to move toward $\mu$.
  - Useful for modeling term structure, stock price volatility, and stock price return.

## Square-Root Process

- Suppose $X$ is an OU process.
- Consider
$$V \stackrel{\Delta}{=} X^2.$$
- Ito's lemma says $V$ has the differential,
$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right)dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

  a square-root process.

## Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

$$dX = \kappa(\mu - X)\, dt + \sigma\sqrt{X}\, dW,$$

where $\kappa, \sigma \geq 0$ and $X(0)$ is a nonnegative constant.

- Like the OU process, it possesses mean reversion: $X$ tends to move toward $\mu$, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

## Square-Root Process (continued)

- When $X$ hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rates.[a]
- The OU process, in contrast, allows negative interest rates.[b]
- The two processes are related (see p. 586).

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[a] Cox, Ingersoll, & Ross (1985).

[b] But some rates have gone negative in Europe in 2015!

## Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \triangleq (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\,\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\,\left(1 - e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a]William Feller (1906–1970) in 1951.