Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Calculate the IRR of the risky bond.
- Calculate the IRR of a riskless bond with comparable maturity.
- Yield spread is their difference.

Static Spread

- Were the risky bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

- But as risk must be compensated, in reality $P < P^*$.
- The static spread is the amount s by which the spot rate curve has to shift *in parallel* to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a] See a counterexample in the text.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1 + S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[\,1 + S(i)\,]^i[\,1 + S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j - i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

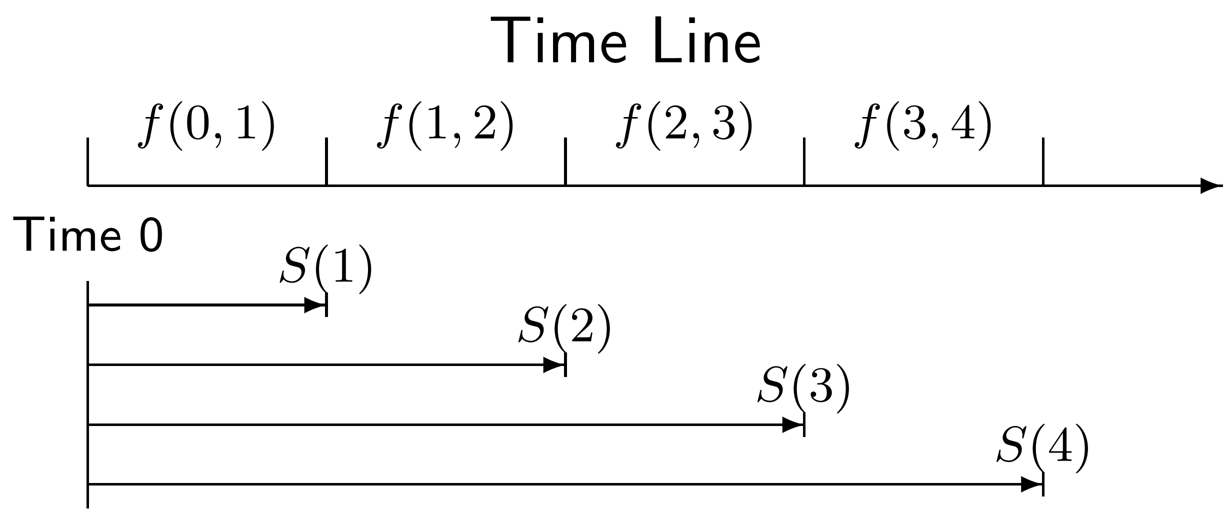
- When $S(i,j)$ equals

$$f(i,j) \equiv \left[\frac{(1+S(j))^j}{(1+S(i))^i}\right]^{1/(j-i)} - 1, \tag{18}$$

  we will end up with $[\,1+S(j)\,]^j$ dollars again.

- By definition, $f(0,j) = S(j)$.

- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate i periods from now.


Time Line
$f(0,1)$
$f(1,2)$
$f(2,3)$
$f(3,4)$
Time 0
$S(1)$
$S(2)$
$S(3)$
$S(4)$

Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.

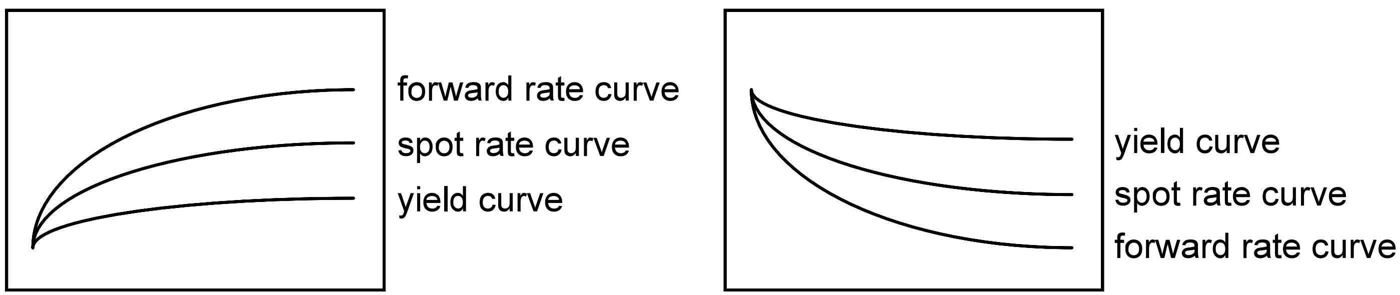
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
yield curve
spot rate curve
forward rate curve

(a) (b)

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive

$$[\,1 + S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,].$$

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[\,1 + S(1)\,][\,1 + f(1,2)\,] \cdots [\,1 + f(n-1,n)\,]\}^{1/n} - 1. \tag{19}$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

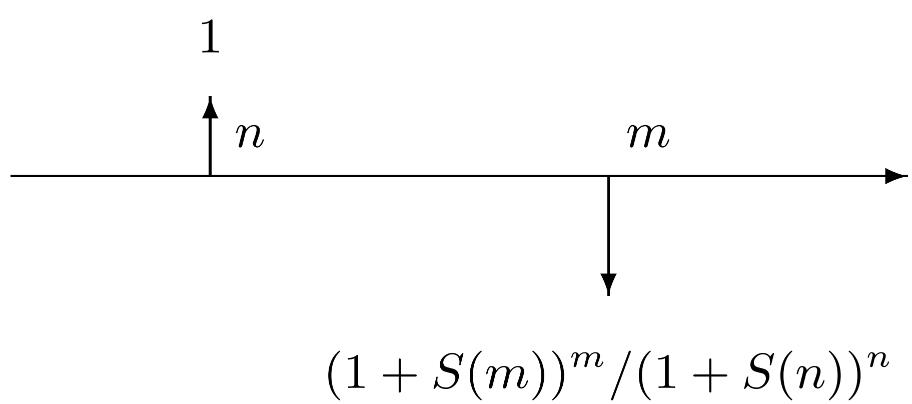
$$f(T, T+1) = \frac{d(T)}{d(T+1)} - 1. \tag{20}$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.

Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the rate $f(n, m)$ between times n and m.



Forward Contracts

- We had generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to
  - Borrow money at time n in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the forward rate
$$f(n, m).$$
- Can the spot rate curve be an arbitrary curve?[a]

[a] Contributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

---

[a] Compare it with Eq. (19) on p. 135.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}. \tag{21}$$

  – Compare the above formula with Eq. (18) on p. 129.

- The one-period forward rate:

$$f(j, j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

  – Compare the above formula with Eq. (20) on p. 135.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now,

$$\begin{aligned} f(T) &\equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) \\ &= S(T) + T\, \frac{\partial S}{\partial T}. \end{aligned}$$

- So $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S/\partial T)$, then $f(T) < 0$.[a]

[a] Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\, S(a, b)\,]. \tag{22}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on average.

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (18) on p. 129.
  - So $E[\, S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A "Bad" Expectations Theory

- The expected returns[a] on all possible riskless bond strategies are equal for *all* holding periods.
- So
$$(1+S(2))^2 = (1+S(1))\,E[\,1+S(1,2)\,] \qquad (23)$$
because of the equivalency between buying a two-period bond and rolling over one-period bonds.
- After rearrangement,
$$\frac{1}{E[\,1+S(1,2)\,]} = \frac{1+S(1)}{(1+S(2))^2}.$$

[a] More precisely, the one-plus returns.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1+S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is
$$E[\,(1+S(1,2))^{-1}\,]/(1+S(2))^{-2}.$$
  - Strategy two buys a one-period bond with a return of $1+S(1)$.

A "Bad" Expectations Theory (concluded)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (23) on p. 144 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use $g(x) \equiv (1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \ \text{ for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration in Practice

- To handle more general types of spot rate curve changes, define a vector $[\, c_1, c_2, \ldots, c_n\,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\, 1, 1, \ldots, 1\,]$.
  - Twist: $[\, 1, 1, \ldots, 1, -1, \ldots, -1\,]$, $[\, 1.8, 1.6, 1.4, 1, 0, -1, -1.4, \ldots]$, etc.
  - $\ldots$.
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let

$$P(y) \equiv \sum_i C_i/(1+S(i)+yc_i)^i$$

be the price associated with the cash flow $C_1, C_2, \ldots$.

- Define duration as

$$-\left.\frac{\partial P(y)/P(0)}{\partial y}\right|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y)-P(-\Delta y)}{2P(0)\Delta y}.$$

- Modified duration equals the above when

$$[\,c_1, c_2, \ldots, c_n\,] = [\,1, 1, \ldots, 1\,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \equiv E\,[\,(X - \mu_X)(Y - \mu_Y)\,]\,,$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,
$$\sigma_X \equiv \sqrt{\mathrm{Var}[\,X\,]}\,.$$
- The correlation (or correlation coefficient) between X and Y is
$$\rho_{X,Y} \equiv \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$
provided both have nonzero standard deviations.[a]

[a]Paul Wilmott (2009), "the correlations between financial quantities are notoriously unstable." It may even break down "at high-frequency time intervals" (Budish, Cramton, and Shim, 2015).

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\, X_i, X_j \,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\, X_i \,]$$

when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation $E[\, X \mid I\,]$ is the expected value of X conditional on I; it is a random variable.
- The law of iterated conditional expectations:

$$E[\, X\,] = E[\, E[\, X \mid I\,]\,].$$

- If I_2 contains at least as much information as I_1, then

$$E[\, X \mid I_1\,] = E[\, E[\, X \mid I_2\,] \mid I_1\,]. \tag{24}$$

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is defined as
$$\theta_X(t) \equiv E[\, e^{tX}\,].$$
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{25}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\, X_i, X_j\,]\right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$.[a]

- X_i are said to have a multivariate normal distribution.

[a] Corrected by Mr. Huang, Guo-Hua (`R98922107`) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1]$ so that

$$\text{Prob}[\, X \leq x \,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \equiv \sqrt{-2(\ln \omega)/\omega}\,.$$

[a] As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i \right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- The general formula is

$$\frac{(\sum_{i=1}^{n} \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

[a] Contributed by Mr. Chen, Shih-Hang (R02723031) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated as follows.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$\begin{aligned} U &\equiv aX_1, \\ V &\equiv \rho U + \sqrt{1-\rho^2}\, aX_2. \end{aligned}$$

Generation of Bivariate Normal Distributions (concluded)

- U and V are the desired random variables with

$$\begin{aligned} \mathrm{Var}[\,U\,] &= \mathrm{Var}[\,V\,] = a^2, \\ \mathrm{Cov}[\,U, V\,] &= \rho a^2. \end{aligned}$$

- Note that the mapping between (X_1, X_2) and (U, V) is one-to-one.

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu+\sigma^2/2} \quad \text{and} \quad \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \tag{26}$$

  respectively.
  - They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.

The Lognormal Distribution (continued)

- Conversely, suppose Y is lognormally distributed with mean μ and variance σ^2.
- Then

$$\begin{aligned} E[\ln Y] &= \ln(\mu/\sqrt{1+(\sigma/\mu)^2}), \\ \mathrm{Var}[\ln Y] &= \ln(1+(\sigma/\mu)^2). \end{aligned}$$

- If X and Y are joint-lognormally distributed, then

$$\begin{aligned} E[XY] &= E[X]\,E[Y]\,e^{\mathrm{Cov}[\ln X, \ln Y]}, \\ \mathrm{Cov}[X,Y] &= E[X]\,E[Y]\left(e^{\mathrm{Cov}[\ln X, \ln Y]}-1\right). \end{aligned}$$

The Lognormal Distribution (concluded)

- Let Y be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then

$$\int_a^\infty y f(y)\, dy = e^{\mu+\sigma^2/2}\, N\left(\frac{\mu - \ln a}{\sigma} + \sigma\right).$$

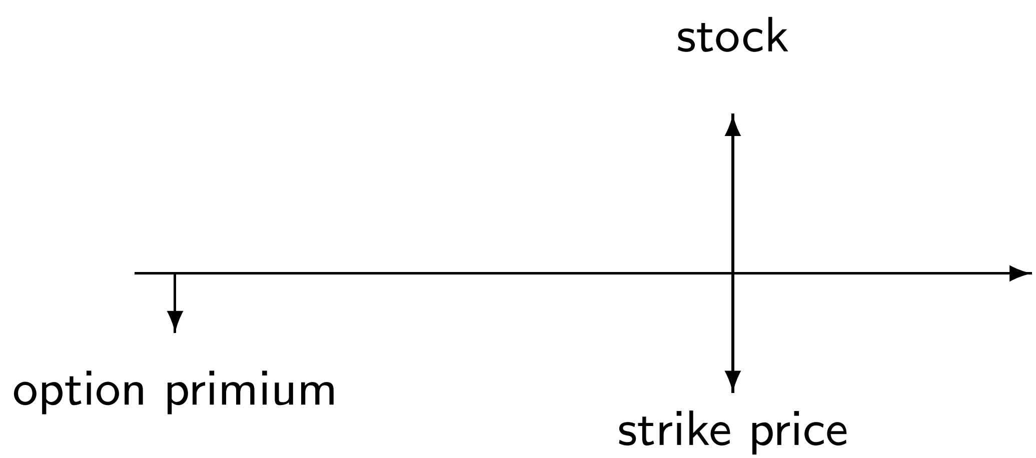
Option Basics

The shift toward options as
the center of gravity of finance [...]
— Merton H. Miller (1923–2000)

Too many potential physicists and engineers
spend their careers shifting money around
in the financial sector,
instead of applying their talents to
innovating in the real economy.
— Barack Obama (2016)

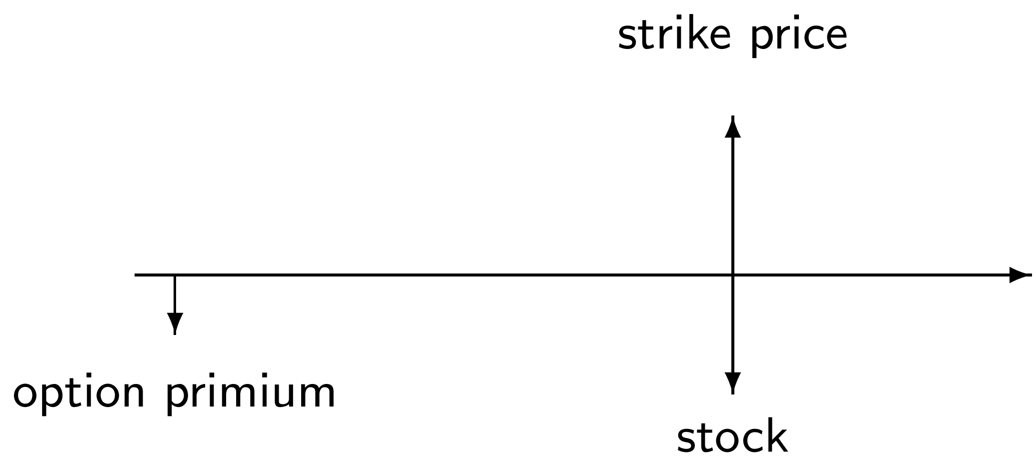
Calls and Puts

- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.



Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
  - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

Payoff, Mathematically Speaking

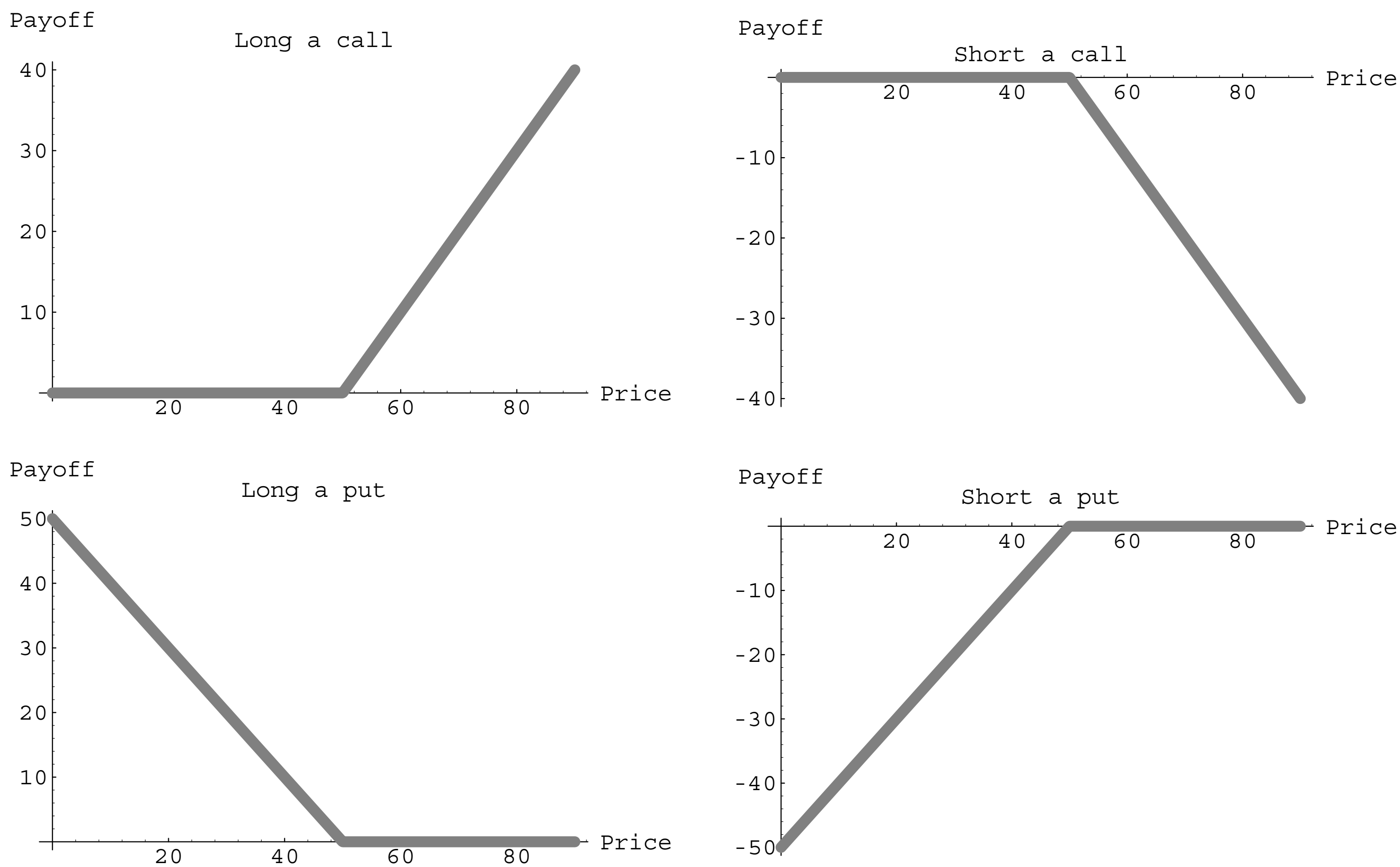
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

the intrinsic value of a call.

- At any time t before the expiration date, we call

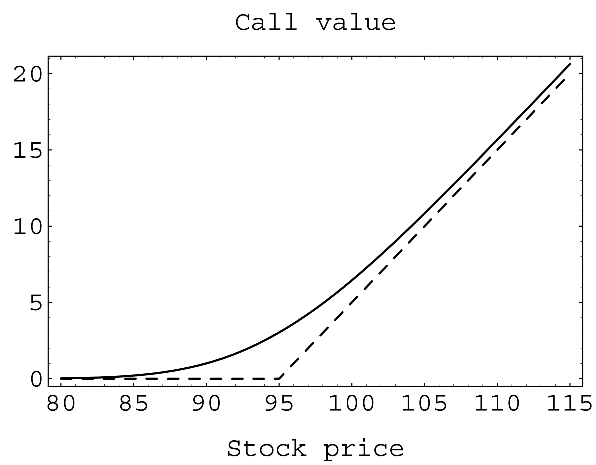
$$\max(0, X - S_t)$$

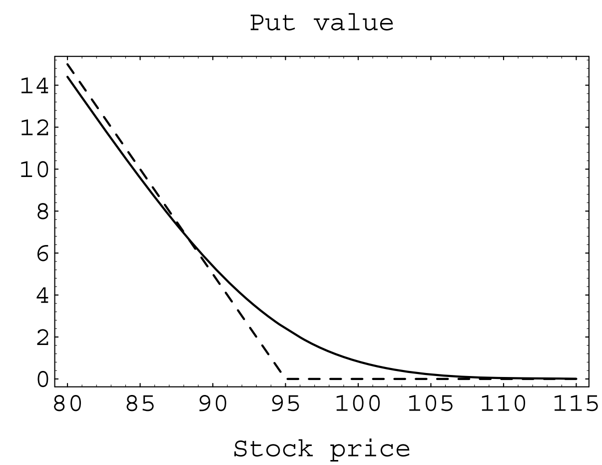
the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time *before* expiration is a major intellectual breakthrough.

[a] 11% of option holders let in-the-money options expire worthless.


Call value
0
5
10
15
20
80 85 90 95 100 105 110 115
Stock price


Put value
0
2
4
6
8
10
12
14
80 85 90 95 100 105 110 115
Stock price