The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

[a]Black and Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$
\begin{aligned}
t_2 &\equiv t_1 + \Delta t_1, \\
t_3 &\equiv t_2 + \Delta t_2.
\end{aligned}
$$

$$
\begin{array}{ccccc}
 & & & \nearrow & \\
 & & \ln r_{\mathrm{d}}(t_2) & & \\
 & \nearrow & & \searrow & \\
\ln r(t_1) & & & & \ln r_{\mathrm{du}}(t_3) = \ln r_{\mathrm{ud}}(t_3) \\
 & \searrow & & \nearrow & \\
 & & \ln r_{\mathrm{u}}(t_2) & & \\
 & & & \searrow &
\end{array}
$$

The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$\begin{aligned}\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\ \ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.\end{aligned}$$

- To make sure an up move followed by a down move coincides with a down move followed by an up move,

$$\begin{aligned}& \ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\ =\; & \ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.\end{aligned}$$

The Black-Karasinski Model: Discrete Time (continued)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{151}$$

- So from Δt_0, we can calculate the Δt_1 that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T$ (roughly).[a]

[a] As $\kappa(t), \theta(t), \sigma(t)$ are independent of r, the Δt_is will not depend on r.

The Black-Karasinski Model: Discrete Time (concluded)

- Unequal durations Δt_i are often necessary to ensure a combining tree.[a]

[a] Amin (1991); Chen (`R98922127`) (2011); Lok (`D99922028`) and Lyuu (2016).

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\,M(t)\,] = \infty$ for any finite t if they model the continuously compounded rate.[a]
- So periodically compounded rates should be modeled.[b]
- Another issue is computational.
- Lognormal models usually do not admit of analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

[a] Hogan and Weintraub (1993).
[b] Sandmann and Sondermann (1993).

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps.[a]
  - Use a fine time step up to the maturity of the short-dated derivative.
  - Use a coarse time step beyond the maturity.
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a] Hull and White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

- Like the Ho-Lee model, this is a normal model.
- The inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a]Hull and White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Many European-style securities can be evaluated analytically.
- Efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

[a]Hull and White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value

$$r_0 + j\Delta r$$

for some integer j.

[a] Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with

$$\begin{aligned} t_i &\equiv i\Delta t, \\ r_j &\equiv r_0 + j\Delta r, \end{aligned}$$

  as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\, t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

- Use

$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{152}$$

  to denote the drift rate[a] of the short rate as seen from node (i, j).

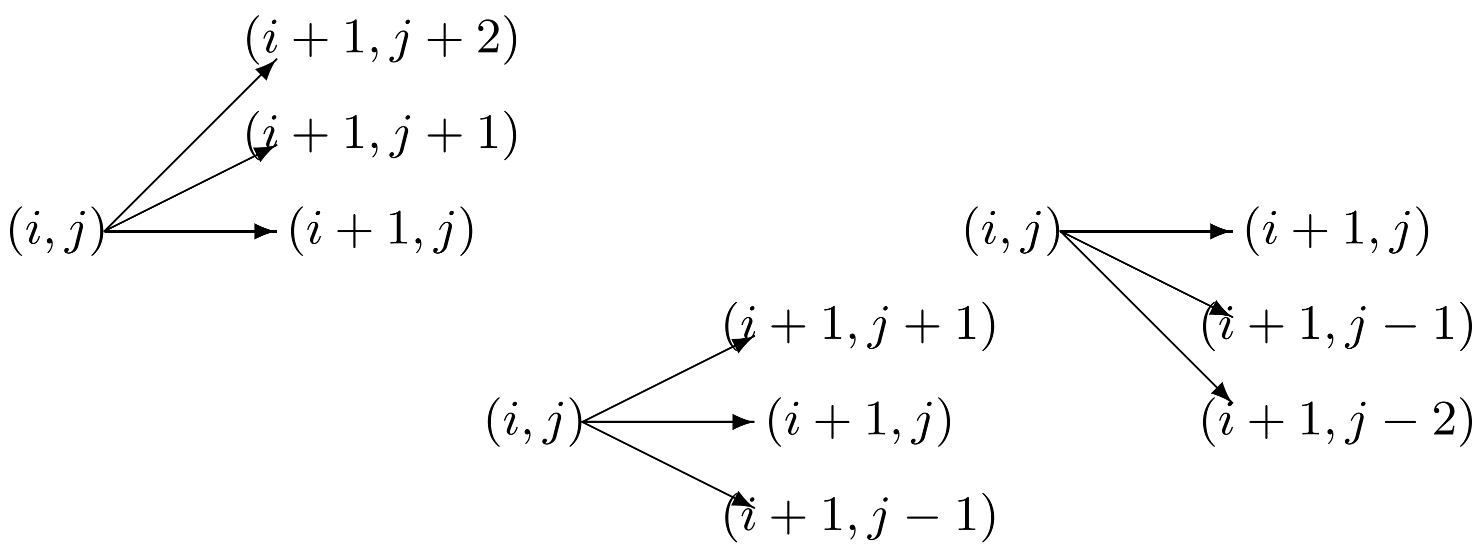
- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1109.

- The middle branch may be an increase of Δr, no change, or a decrease of Δr.

---

[a]Or, the annualized expected change.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j)\\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j)\\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of (the still unknown) $\theta(t_i)$.
  - Both describe the expected change from node (i,j).

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (152) on p. 1108.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.[a]

[a] Not $r(0, t_{i+1})$!

The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is done by letting the middle node be as closest to the current short rate r_j plus the drift $\mu_{i,j} \Delta t$.[b]

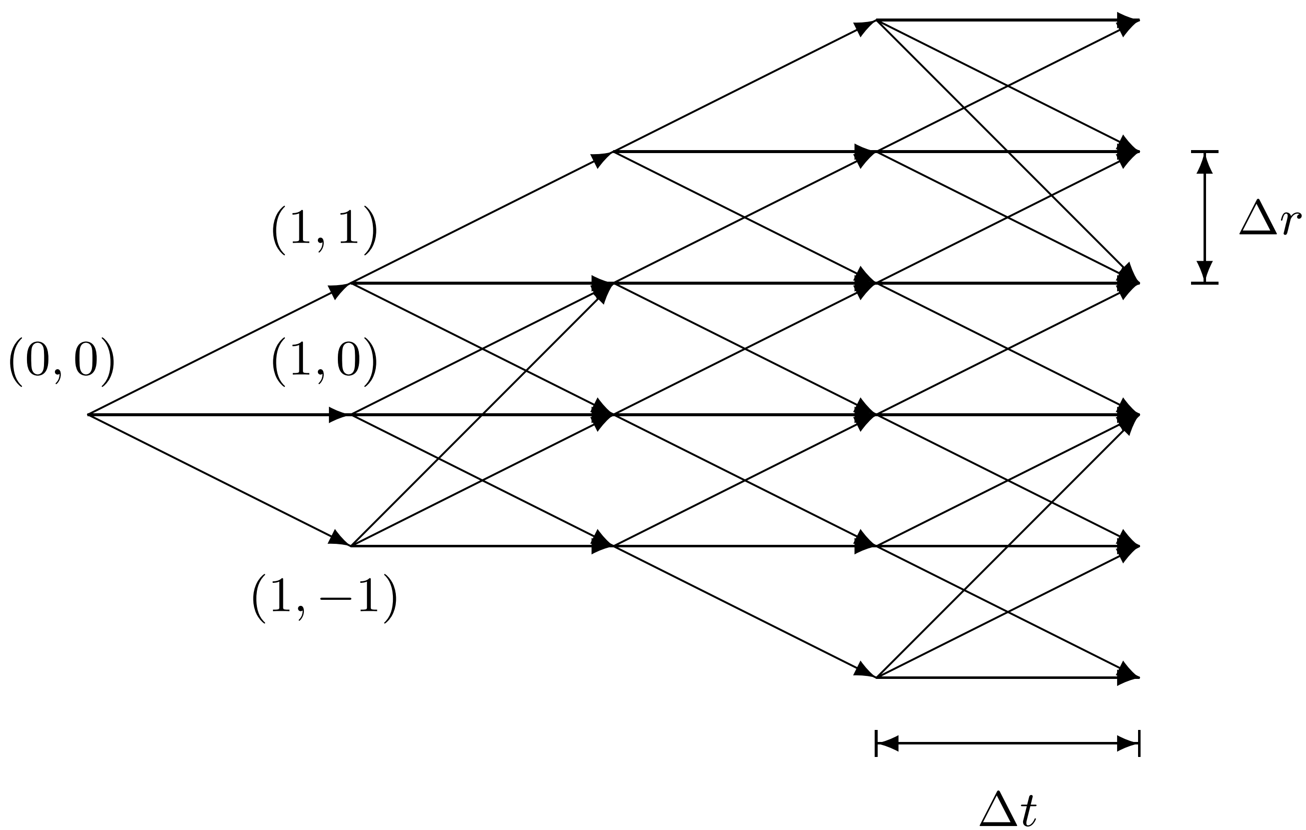
---

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

[b] A predecessor to Lyuu and Wu's (`R90723065`) (2003, 2005) mean-tracking idea, which is the precursor of the binomial-trinomial tree of Dai (`B82506025`, `R86526008`, `D8852600`) and Lyuu (2006, 2008, 2010).

The Hull-White Model: Calibration (continued)

- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to
$$r_j + \mu_{i,j} \Delta t.$$
  - But note that $\mu_{i,j}$ is still not computed yet.
- Then the three nodes following node (i, j) are nodes
$$(i+1, k+1), (i+1, k), (i+1, k-1).$$
- See p. 1114 for a possible geometry.
- The resulting tree combines because of the constant jump sizes to reach k from j.



$(1,1)$
$(0,0)$
$(1,0)$
$(1,-1)$
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{153}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{153$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{153$''$}$$

where

$$\eta \equiv \mu_{i,j} \Delta t + (j-k)\, \Delta r.$$

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise,[a] certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  – For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[b]

- Now it only remains to determine $\theta(t_i)$.

---

[a] Recall p. 761.

[b] Hull and White (1988).

The Hull-White Model: Calibration (continued)

- At this point at time t_i,

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_{i+1})$$

  have already been matched.

- Let $Q(i, j)$ be the state price at node (i, j).

- By construction, the state prices $Q(i, j)$ for all j are known by now.

- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= & \sum_j Q(i,j)\, e^{-r_j \Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \quad (154)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation in Eq. (154) can be approximated by

$$
\begin{aligned}
& E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\
\approx \quad & e^{-r_j \Delta t} \left(1 - \mu_{i,j} (\Delta t)^2 + \frac{\sigma^2 (\Delta t)^3}{2} \right). \qquad (155)
\end{aligned}
$$

  - This solves the chicken-egg problem!

- Substitute Eq. (155) into Eq. (154) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$
\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2 r_j \Delta t} \left(1 + a r_j (\Delta t)^2 + \sigma^2 (\Delta t)^3 / 2 \right) - e^{-r(0, t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2 r_j \Delta t}}.
$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (155) is actually known analytically by Eq. (25) on p. 162:

$$
\begin{aligned}
& E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\
= \quad & e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.
\end{aligned}
$$

- Therefore, alternatively,

$$
\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.
$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$,[a] the probabilities,[b] and finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i+1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^* (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

---

[a] See Eq. (152) on p. 1108.

[b] See Eqs. (153) on p. 1115.

Comments on the Hull-White Model

- One can try different values of a and σ for each option.
- Or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- But suppose the demand is for a single set of parameters that replicate *all* option prices.
- Then the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull and White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1114).
  - So it is harder to program (for nonprogrammers).
- The second shortcoming is again a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this chicken-egg dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (154) on p. 1118 that helps derive $\theta(t_i)$.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

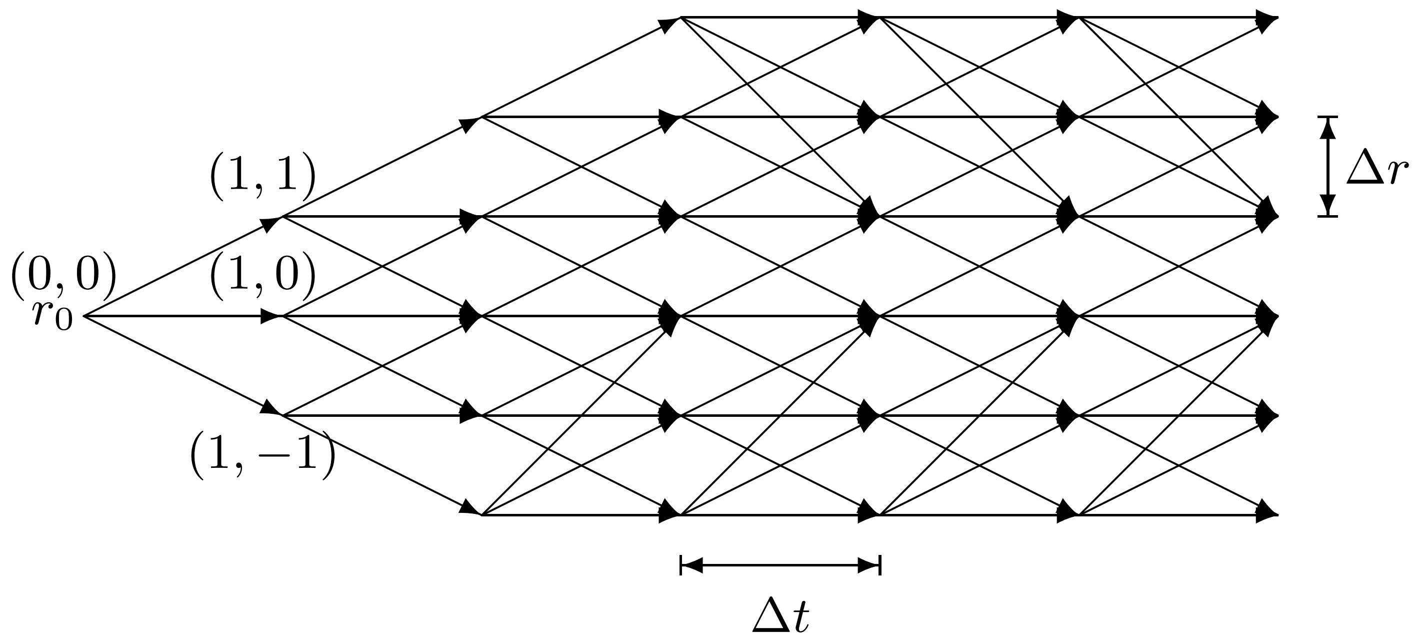
[a]Hull and White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (preview p. 1127).
  - The number of nodes above the r_0-line is $j_{\max}$, and that below the line is $j_{\min}$.
  - They will be picked so that the probabilities (153) on p. 1115 are positive for all nodes.



The short rate at node $(0, 0)$ equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- The tree's branches and probabilities are now in place.
- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to *add* to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.[a]

[a] Contrast this with the previous algorithm, where it was the spot rate $r(0, t_{i+2})$ that is matched!

The Hull-White Model: Calibration

- Set $\Delta r = \sigma\sqrt{3\Delta t}$ and assume that $a > 0$.
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j\Delta r$.
- Hence the probabilities in Eqs. (153) on p. 1115 use

$$\eta \equiv -aj\Delta r\Delta t + (j-k)\,\Delta r.$$

- Recall that k denotes the middle branch.

The Hull-White Model: Calibration (continued)

- The probabilities become

$$
\begin{aligned}
& p_1(i,j) \\
= \quad & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, \qquad (156) \\
& p_2(i,j) \\
= \quad & \frac{2}{3} - \left[a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \right], \qquad (157) \\
& p_3(i,j) \\
= \quad & \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. \qquad (158)
\end{aligned}
$$

- p_1: up move; p_2: flat move; p_3: down move.

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the same j.
- Furthermore, $p_1(i, j) = p_3(i, -j)$.

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{159}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As
$$\frac{3-\sqrt{6}}{3} \approx 0.184,$$
a good choice is
$$j_{\max} = \lceil 0.184/(a\Delta t) \rceil.$$

The Hull-White Model: Calibration (continued)

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0,0) = 1$.
- Inductively, suppose that spot rates

$$r(0,t_1), r(0,t_2), \ldots, r(0,t_i)$$

  have already been matched as of time t_i.
- By construction, the state prices $Q(i,j)$ for all j are known by now.

The Hull-White Model: Calibration (continued)

- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i,j)\, e^{-(\beta_i + r_j)\Delta t}$$

  by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0,t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i,j)\, e^{-r_j \Delta t}}{\Delta t}. \tag{160}$$

The Hull-White Model: Calibration (concluded)

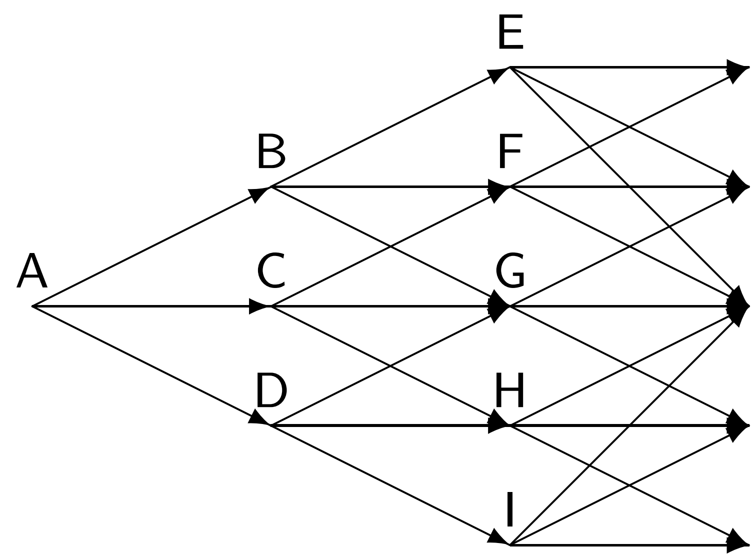
- The short rate at node (i, j) now equals $\beta_i + r_j$.
- The state prices at time t_{i+1},

$$Q(i+1, j), \quad -\min(i+1, j_{\max}) \leq j \leq \min(i+1, j_{\max}),$$

  can now be calculated as before.
- The total running time is $O(nj_{\max})$.
- The space requirement is $O(n)$.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\max} = 2$.
- The plot on p. 1138 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (156)–(158) on p. 1130 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	-1.73205	-3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve

$$0.08 - 0.05 \times e^{-0.18 \times t}.$$

- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now, by Eq. (160) on p. 1135,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2)\times 2 + \ln\left[\, Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \,\right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

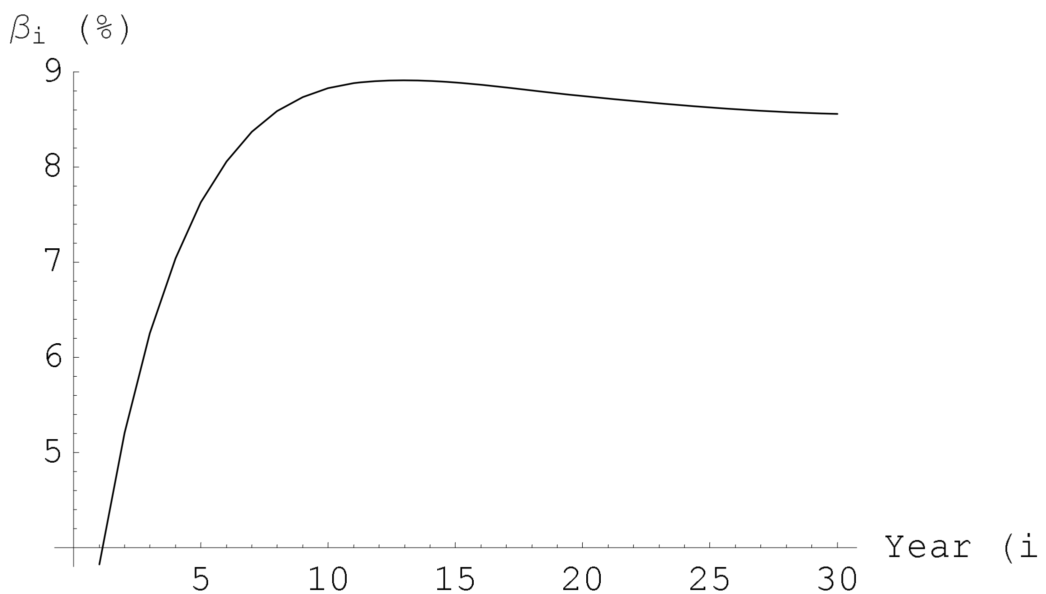
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) = 0.018209,\\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0)\\
&= 0.199799,\\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0)\\
&\quad + p_1(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1) = 0.473597,\\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1)\\
&= 0.203263,\\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\begin{aligned}\beta_2 &= r(0,3)\times 3+\ln\Big[\,Q(2,2)\,e^{-2\times\Delta r}+Q(2,1)\,e^{-\Delta r}+Q(2,0)\\ &\quad +Q(2,-1)\,e^{\Delta r}+Q(2,-2)\,e^{2\times\Delta r}\Big]=6.25359\%.\end{aligned}$$

- Hence the short rates at nodes E, F, G, H, and I equal β_2+r_j, where $j=2,1,0,-1,-2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1144 plots β_i for $i=0,1,\ldots,29$.


β_i (%)
9
8
7
6
5
5
10
15
20
25
30
Year (i)

The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.

The Heath-Jarrow-Morton (HJM) Model[a]

- This influential model is a forward rate model.
- The HJM model specifies the initial forward rate curve and the forward rate volatility structure.
  - The volatility structure describes the volatility of each forward rate for a given maturity date.
- Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

[a] Heath, Jarrow, and Morton (1992).

Introduction to Mortgage-Backed Securities

Anyone stupid enough to promise to be
responsible for a stranger's debts
deserves to have his own property
held to guarantee payment.
— Proverbs 27:13

Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- Suppose the borrower (the mortgagor) defaults, that is, fails to make the contractual payments.
- The lender (the mortgagee) can foreclose the loan by seizing the property.

Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.[a]
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.

[a]They can be traced to 1880s (Levy (2012)).

Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.[a]
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

[a]Merton (1994).

Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
  3. Stripped mortgage-backed security (SMBS).

Problems Investing in Mortgages

- The mortgage sector is one of the largest in the debt market (see p. 3 of the textbook).[a]
- Individual mortgages are unattractive for many investors.
- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.
- Most investors lack the resources and knowledge to assess the credit risk involved.

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[a]The outstanding balance was US$8.1 trillion as of 2012 vs. the US Treasury's US$10.9 trillion according to SIFMA.

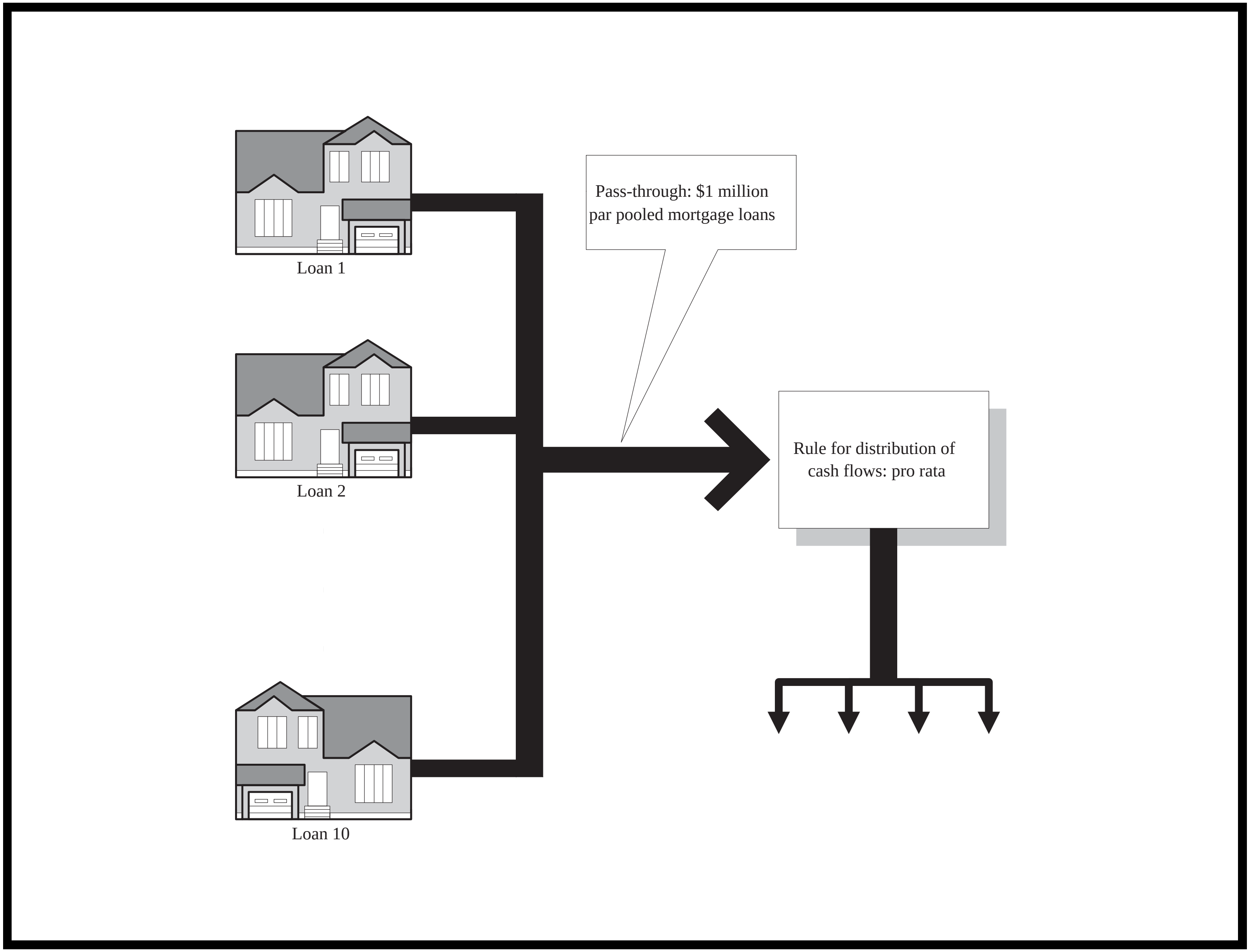
Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors.
- That makes the size and the timing of the cash flows unpredictable.

Mortgage Pass-Throughs[a]

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a \$25,000 certificate from a \$1 million pool is entitled to $2\,1/2\%$ (or 1/40th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.

[a]First issued by Ginnie Mae in 1970.


Pass-through: $1 million
par pooled mortgage loans
Loan 1
Loan 2
Loan 10
Rule for distribution of
cash flows: pro rata

Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.[a]
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

[a] In June 1983 by Freddie Mac with the help of First Boston, which was acquired by Credit Suisse in 1990.

Collateralized Mortgage Obligations (CMOs) (continued)

- CMOs are *multiple*-maturity, *multi*class debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.
- The total prepayment risk is now divided among classes of bonds called classes or tranches.[a]
- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

[a] *Tranche* is a French word for "slice."

Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors
  - The outstanding balance of agency CMOs was US$1.1 trillion as of the first quarter of 2015.[a]

[a]SIFMA (2015).

Sequential Tranche Paydown

- In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.
- Each tranche thus has a different effective maturity.
- Each tranche may even have a different coupon rate.

An Example

- Consider a two-tranche sequential-pay CMO backed by \$1,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The cash flow pattern for each tranche with zero prepayment and zero servicing fee is shown on p. 1162.
- The calculation can be carried out first for the Total columns, which make up the amortization schedule.
- Then the cash flow is allocated.
- Tranche A is retired after 4 months, and tranche B starts principal paydown at the end of month 4.

CMO Cash Flows without Prepayments

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Tota
							500,000	500,000	1,000,0
1	5,000	5,000	10,000	162,548	0	162,548	337,452	500,000	837,4
2	3,375	5,000	8,375	164,173	0	164,173	173,279	500,000	673,2
3	1,733	5,000	6,733	165,815	0	165,815	7,464	500,000	507,4
4	75	5,000	5,075	7,464	160,009	167,473	0	339,991	339,9
5	0	3,400	3,400	0	169,148	169,148	0	170,843	170,8
6	0	1,708	1,708	0	170,843	170,843	0	0	
Total	10,183	25,108	35,291	500,000	500,000	1,000,000			

The total monthly payment is \$172,548. Month-i numbers reflect the ith monthly payment.

Another Example

- When prepayments are present, the calculation is only slightly more complex.
- Suppose the single monthly mortality (SMM) per month is 5%.
- This means the prepayment amount is 5% of the *remaining* principal.
- The remaining principal at month i *after* prepayment then equals the scheduled remaining principal as computed by Eq. (7) on p. 46 times $(0.95)^i$.
- This done for all the months, the interest payment at any month is the remaining principal of the previous month times 1%.

Another Example (continued)

- The prepayment amount equals the remaining principal times $0.05/0.95$.
  - The division by 0.95 yields the remaining principal *before* prepayment.
- Page 1165 tabulates the cash flows of the same two-tranche CMO under 5% SMM.

Another Example (continued)

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Total
							500,000	500,000	1,000,0
1	5,000	5,000	10,000	204,421	0	204,421	295,579	500,000	795,5
2	2,956	5,000	7,956	187,946	0	187,946	107,633	500,000	607,6
3	1,076	5,000	6,076	107,633	64,915	172,548	0	435,085	435,0
4	0	4,351	4,351	0	158,163	158,163	0	276,922	276,9
5	0	2,769	2,769	0	144,730	144,730	0	132,192	132,1
6	0	1,322	1,322	0	132,192	132,192	0	0	
Total	9,032	23,442	32,474	500,000	500,000	1,000,000			

Month-i numbers reflect the ith monthly payment.

Another Example (continued)

- For instance, the total principal payment at month one, \$204,421, can be verified as follows.
- The *scheduled* remaining principal is \$837,452 from p. 1162.
- The remaining principal is hence

$$837452 \times 0.95 = 795579.$$

- That makes the total principal payment

$$1000000 - 795579 = 204421.$$

Another Example (concluded)

- As tranche A's remaining principal is \$500,000, all 204,421 dollars go to tranche A.
  - Incidentally, the prepayment is
$$837452 \times 5\% = 41873.$$
- Tranche A is retired after 3 months, and tranche B starts principal paydown at the end of month 3.

Stripped Mortgage-Backed Securities (SMBSs)[a]

- The principal and interest are divided between the PO strip and the IO strip.
- In the scenarios on p. 1161 and p. 1163:
  - The IO strip receives all the interest payments under the Interest/Total column.
  - The PO strip receives all the principal payments under the Principal/Total column.

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[a]They were created in February 1987 when Fannie Mae issued its Trust 1 stripped MBS.

Stripped Mortgage-Backed Securities (SMBSs) (concluded)

- These new instruments allow investors to better exploit anticipated changes in interest rates.[a]
- The collateral for an SMBS is a pass-through.
- CMOs and SMBSs are usually called derivative MBSs.

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[a]See p. 357 of the textbook.

