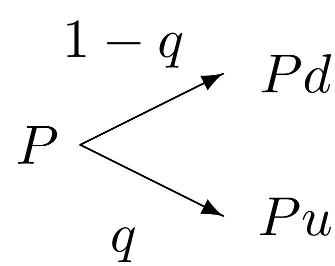
The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:



The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{140}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{141}$$

The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate r.
- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- As in the continuous-time case, it can be shown that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \qquad (142)$$

which is independent of bond maturity and q.

  - Recall the BOPM.

- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

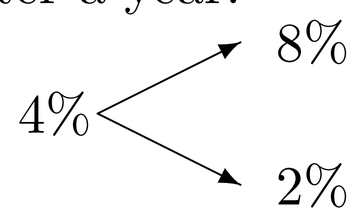
- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



Numerical Examples (continued)

- *No* real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 1017.

Numerical Examples (continued)

$$90.703 \nearrow 92.593\ (= 100/1.08), \quad \searrow 98.039\ (= 100/1.02)$$

$$96.154 \nearrow 100, \quad \searrow 100$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

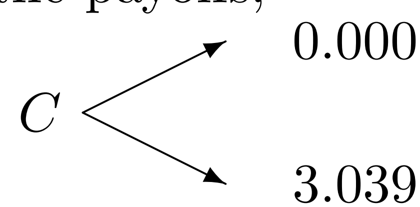
where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,



- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

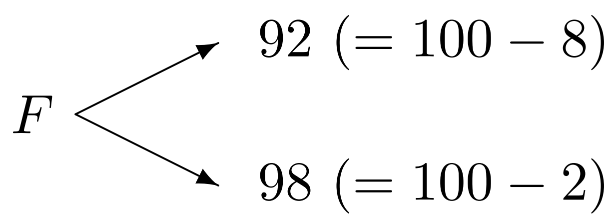
- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:



- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

[a]See Exercise 13.2.11 of the textbook or p. 515.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

---

[a]By Eq. (128) on p. 990.

[b]Recall p. 459.

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- We now survey equilibrium models.
- Recall that the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}$$

by Eq. (127) on p. 989.

- Hence the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\,r(T)\,|\,r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$

from Eq. (76) on p. 576.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{143}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

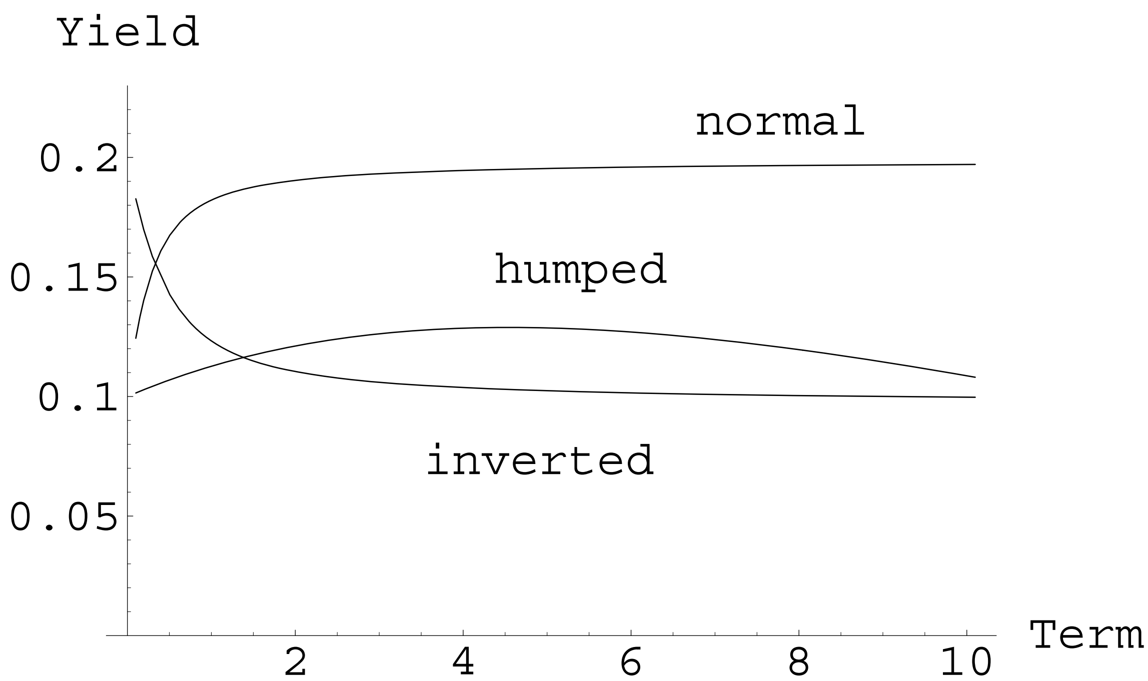
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve

$$(\partial r(t, T)/\partial r)\, \sigma = \sigma B(t, T)/(T - t).$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Indeed, higher β leads to greater attenuation of volatility with maturity.


Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
Term
2
4
6
8
10

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a]Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[\,0, T\,]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.$$

[a]Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases} .$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{144}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

---

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- By Ito's lemma,

$$dx = m(x)\,dt + dW,$$

where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a constant volatility, and its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation

$$r = f(x) \equiv \frac{x^2 \sigma^2}{4}$$

(see p. 1042).

- When $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

[a] Nawalkha and Beliaeva (2007).

$$
\begin{array}{ccccccccc}
 & & & & x + 2\sqrt{\Delta t} & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
x & & & & x \qquad f(x) & & & & f(x) \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t} & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{145}$$

  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

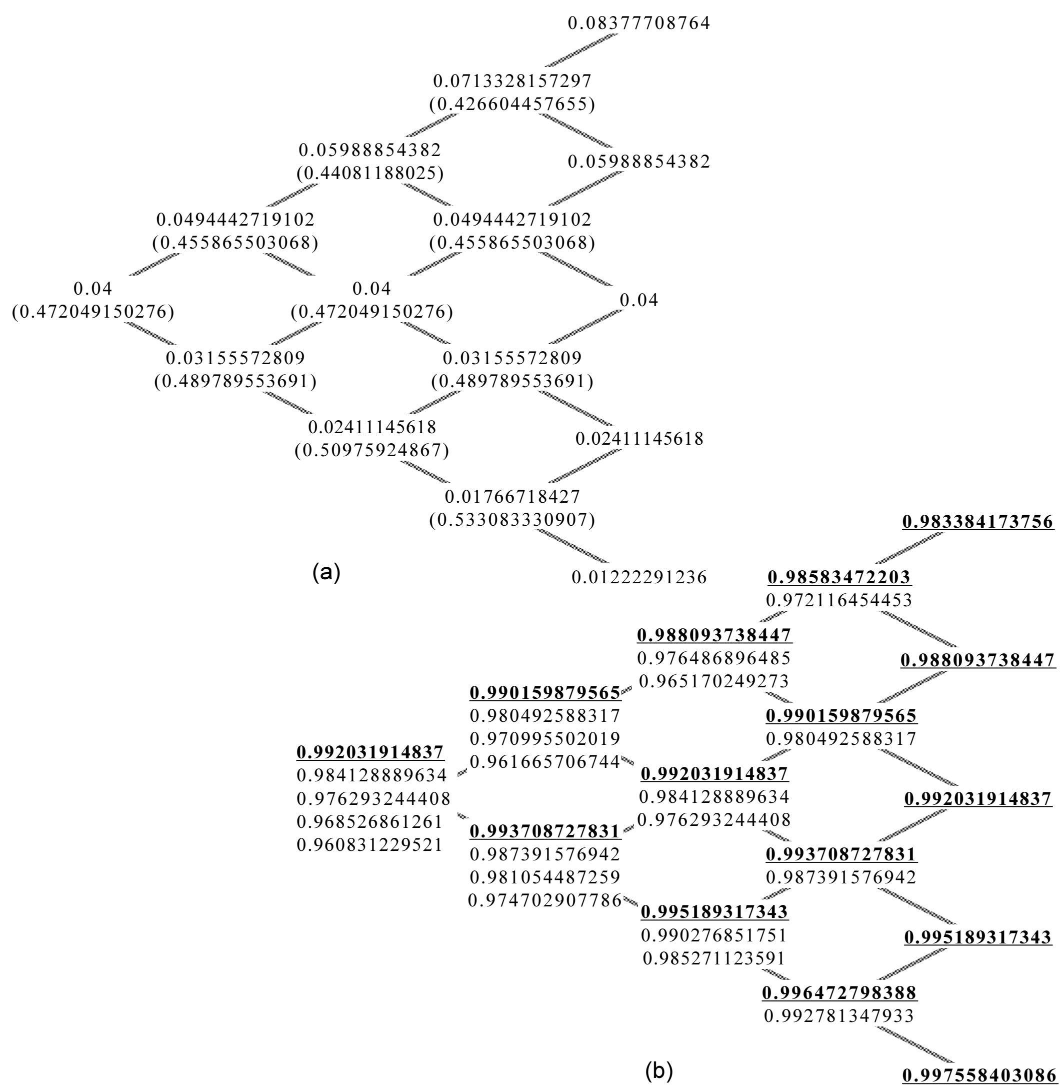
Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 1045(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.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0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
  - I suspect that
$$p(r) = A\sqrt{\frac{\Delta t}{r}} + B - C\sqrt{r\Delta t}$$
    for some $A, B, C > 0$.[a]
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see p. 369 of the textbook).

[a]Thanks to a lively class discussion on May 28, 2014.

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.

[a]Nelson and Ramaswamy (1990).

A General Method (continued)

- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t+\Delta t)\sqrt{\Delta t} \\ \neq \; & -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t+\Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.
- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows

$$dx = m(y,t)\, dt + dW$$

for some $m(y,t)$.[a]

- The diffusion term is now a constant, and the binomial tree for x combines.

[a] See Exercise 25.2.13 of the textbook.

A General Method (concluded)

- The transformation is unique.[a]
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.

- Note that

$$\begin{aligned} y_{\mathrm{u}}(x,t) &\equiv y(x+\sqrt{\Delta t}, t+\Delta t), \\ y_{\mathrm{d}}(x,t) &\equiv y(x-\sqrt{\Delta t}, t+\Delta t). \end{aligned}$$

[a] Chiu (R98723059) (2012).

Examples

- The transformation is
$$\int^r (\sigma\sqrt{z})^{-1}\, dz = \frac{2\sqrt{r}}{\sigma}$$
for the CIR model.
- The transformation is
$$\int^S (\sigma z)^{-1}\, dz = \frac{\ln S}{\sigma}$$
for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.

Options on Coupon Bonds[a]

- Assume the market discount function P is a monotonically decreasing function of the short rate r.
  - Such as a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max \left\{ \left[\sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - X, 0 \right\}.$$

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.

Options on Coupon Bonds (continued)

- This r^* can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for r.

- Let

$$X_i \equiv P(r^*, T, t_i),$$

the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

- Note that $P(r, T, t_i) \geq X_i$ if and only if $r \leq r^*$.

Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\begin{aligned} & \max\left\{\left[\sum_{i=1}^{n} c_i P(r(T), T, t_i)\right] - \left[\sum_i c_i X_i\right], 0\right\} \\ = & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0). \end{aligned}$$

- Thus the call is a package of n options on the underlying zero-coupon bond.

- Why can't we do the same thing for Asian options?[a]

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

[a]Ho and Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

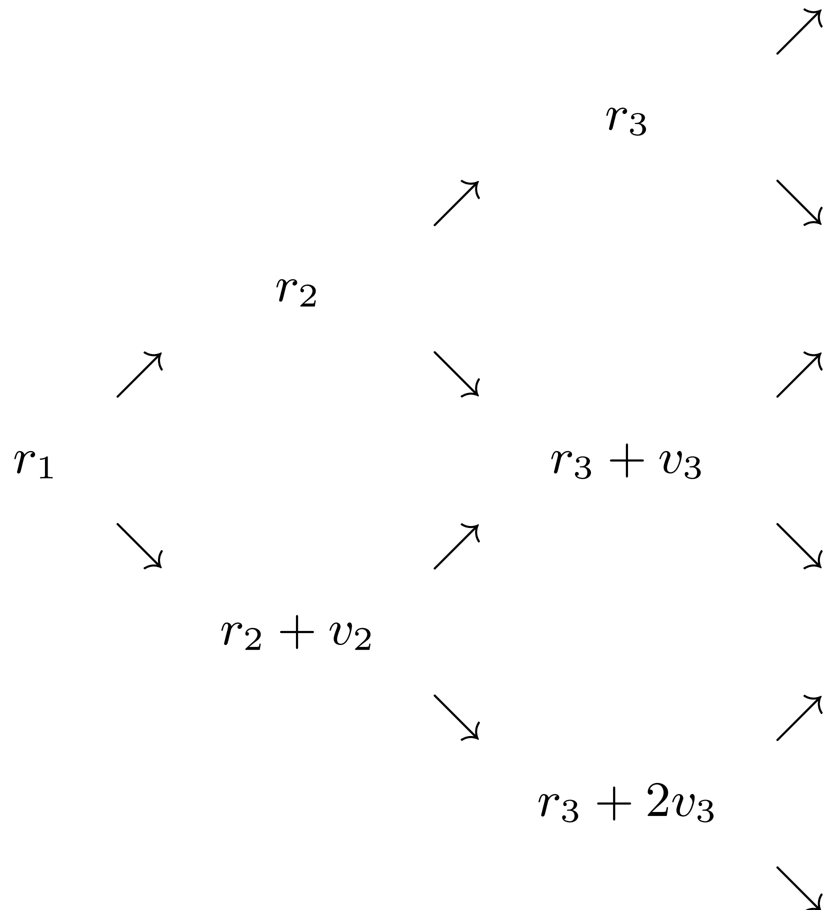
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] Ho and Lee (1986).


r_3
r_2
r_1
$r_3 + v_3$
$r_2 + v_2$
$r_3 + 2v_3$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\rm d}(t+1, t+2), P_{\rm d}(t+1, t+3), \ldots & \text{if short rate moves down} \\ P_{\rm u}(t+1, t+2), P_{\rm u}(t+1, t+3), \ldots & \text{if short rate moves up} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$P_{\rm u}(t+1, t+n) = P_{\rm d}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{146}$$

(see p. 376 of the textbook).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$
\begin{aligned}
y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1, t+n)}{n-1} \\
y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1, t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1}
\end{aligned}
$$

- The volatility of the yield to maturity for this bond is therefore

$$
\begin{aligned}
\kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\
&= \sqrt{p(1-p)}\ (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\
&= \sqrt{p(1-p)}\ \frac{v_2 + \cdots + v_n}{n-1}.
\end{aligned}
$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{147}$$

- The variance of the short rate therefore equals

$$p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2,$$

where r_{u} and r_{d} are the two successor rates.[a]

---

[a]Contrast this with the lognormal model (120) on p. 930.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

$$\kappa_2, \kappa_3, \ldots .$$

  - It is independent of

$$r_2, r_3, \ldots .$$

- It is easy to compute the v_is from the volatility structure, and vice versa (review p. 1068).
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t+n) = [\, p P_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n) \,]\, P(t, t+1).$$

- Combine the above with Eq. (146) on p. 1067 and assume $p = 1/2$ to obtain[a]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[\, v_2 + \cdots + v_n \,]}{1 + \exp[\, v_2 + \cdots + v_n \,]}, \tag{148}$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[\, v_2 + \cdots + v_n \,]}. \tag{148'}$$

[a]In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.[a]
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$\begin{aligned} P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\ P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}. \end{aligned}$$

- Short rate volatility $\sigma = v/2$ by Eq. (147) on p. 1069.
- Price derivatives by taking expectations under the risk-neutral probability.

---

[a]See Exercise 26.2.3 of the textbook.

Calibration

- The Ho-Lee model can be calibrated in $O(n^2)$ time using state prices.
- But it can actually be calibrated in $O(n)$ time.
- Derive the v_i's in linear time.
- Derive the r_i's in linear time.[a]

[a] See Programming Assignment 26.2.6 of the textbook.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \equiv \ln \left(\frac{P(t+1, t+n)}{P(t, t+n)} \right).$$

- Its two possible value are

$$\ln \frac{P_{\mathrm{d}}(t+1, t+n)}{P(t, t+n)} \quad \text{and} \quad \ln \frac{P_{\mathrm{u}}(t+1, t+n)}{P(t, t+n)}.$$

- Thus the variance of return is

$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2 \sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is[a]

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is one.
- Strong correlation between rates is inherent in all one-factor Markovian models.

[a] See Exercise 26.2.7 of the textbook.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\, dt + \sigma(t)\, dW.$$

- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

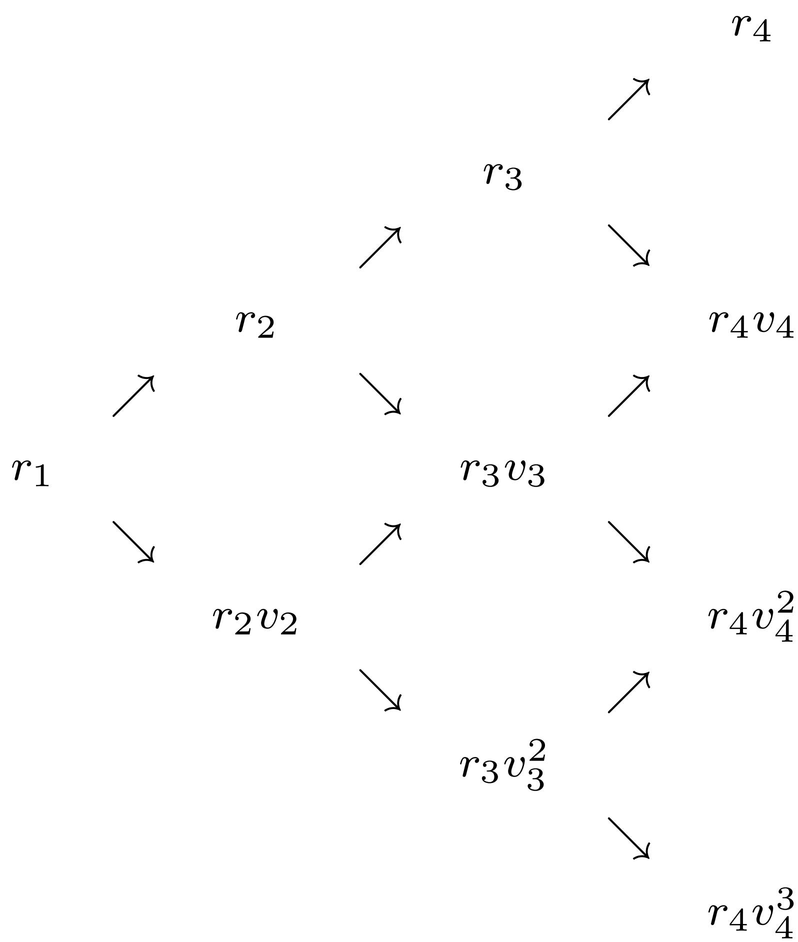
- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 926ff.[b]
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

[a] Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

[b] Repeated on next page.


r_4
r_3
r_2
$r_4 v_4$
r_1
$r_3 v_3$
$r_2 v_2$
$r_4 v_4^2$
$r_3 v_3^2$
$r_4 v_4^3$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.
- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$y_{\mathrm{u}} \equiv P_{\mathrm{u}}^{-1/(i-1)} - 1,$$
$$y_{\mathrm{d}} \equiv P_{\mathrm{d}}^{-1/(i-1)} - 1.$$

- The yield volatility is defined as

$$\kappa_i \equiv \frac{\ln(y_{\mathrm{u}}/y_{\mathrm{d}})}{2}$$

(recall Eq. (126) on p. 976).

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to $P_{\rm u}$ or $P_{\rm d}$ one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\rm u}+P_{\rm d}}{2(1+r_1)}=\frac{1}{(1+y)^i}. \tag{149}$$

- Obviously, $P_{\rm u}$ and $P_{\rm d}$ are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively (p. 1084).
- With κ_i^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{150}$$

- Solving Eqs. (149)–(150) for r and v with backward induction takes $O(i^2)$ time.
  - That leads to a cubic-time algorithm.

The BDT Model: Calibration (continued)

- We next employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving *forward* in time, how much $1 at a node contributes to the price.[b]
- This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

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[a] Chen (R84526007) and Lyuu (1997); Lyuu (1999).

[b] Review p. 953(a).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be

$$P_1, P_2, \dots, P_i.$$

- They correspond to rates

$$r, rv, \dots, rv^{i-1}$$

for period i, respectively.

- One dollar at time i has a present value of

$$f(r, v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- By Eq. (150) on p. 1087, the yield volatility is

$$g(r,v) \equiv \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right).$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 1081).
- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 1081).

The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices: $P_i, P_{\text{u},i}, P_{\text{d},i}$.
- Now solve

$$\begin{aligned} f(r, v) &= \frac{1}{(1+y)^i}, \\ g(r, v) &= \kappa_i, \end{aligned}$$

for $r = r_i$ and $v = v_i$.

- This $O(n^2)$-time algorithm appears on p. 382 of the textbook.

Calibrating the BDT Model with the Differential Tree (in seconds)[a]

Number of years	Running time	Number of years	Running time	Number of years	Running time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

[a]Lyuu (1999).

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t)\, dW.$$

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant volatility will not attain mean reversion.