

## Set Things in Motion

- The abstract process is now in place.
- We need the annualized rates of return of the riskless bonds that make up the benchmark yield curve and their volatilities.
- In the U.S., for example, the on-the-run yield curve obtained by the most recently issued Treasury securities may be used as the benchmark curve.

## Set Things in Motion (concluded)

- The term structure of (yield) volatilities<sup>a</sup> can be estimated from:
  - Historical data (historical volatility).
  - Or interest rate option prices such as cap prices (implied volatility).
- The binomial tree should be found that is consistent with both term structures.
- Here we focus on the term structure of interest rates.

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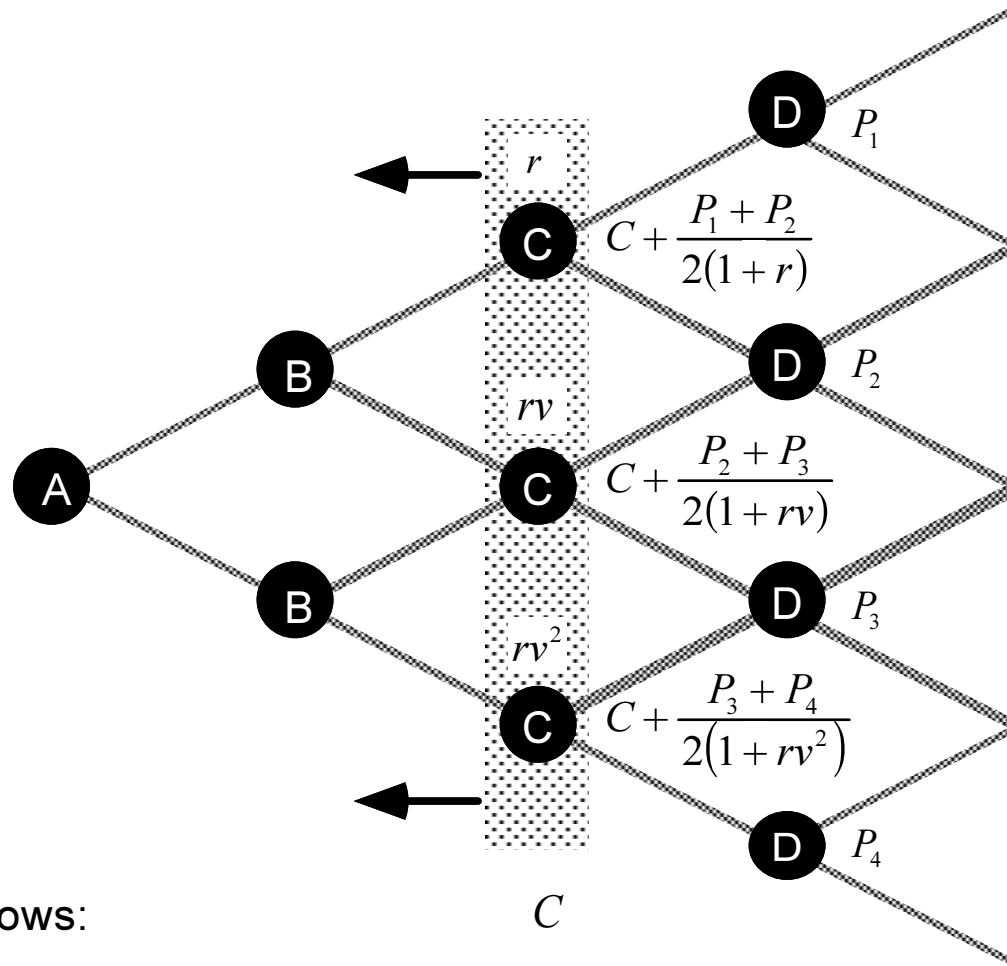
<sup>a</sup>Or simply the volatility (term) structure.

## Model Term Structures

- The model price is computed by backward induction.
- Refer back to the figure on p. 928.
- Given that the values at nodes B and C are  $P_B$  and  $P_C$ , respectively, the value at node A is then

$$\frac{P_B + P_C}{2(1 + r)} + \text{cash flow at node A.}$$

- We compute the values column by column without explicitly expanding the binomial interest rate tree (see next page).
- This takes  $O(n^2)$  time and  $O(n)$  space.



## Term Structure Dynamics

- An  $n$ -period zero-coupon bond's price can be computed by assigning \$1 to every node at period  $n$  and then applying backward induction.
- Repeating this step for  $n = 1, 2, \dots$ , one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics.
  - Taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.

## Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
  - This was called calibration (the reverse of pricing).
- Assume the short rate volatility is such that

$$v \equiv \frac{r_h}{r_\ell} = 1.5,$$

independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

## An Approximate Calibration Scheme

- Start with the implied one-period forward rates.
- Then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let  $f_j$  denote the forward rate in period  $j$ .
- This forward rate can be derived from the market discount function via

$$f_j = \frac{d(j)}{d(j+1)} - 1$$

(see Exercise 5.6.3 in text).

## An Approximate Calibration Scheme (continued)

- Since the  $i$ th short rate  $r_j v_j^{i-1}$ ,  $1 \leq i \leq j$ , occurs with probability  $2^{-(j-1)} \binom{j-1}{i-1}$ , this means

$$\sum_{i=1}^j 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left( \frac{2}{1 + v_j} \right)^{j-1} f_j. \quad (123)$$

- This binomial interest rate tree is trivial to set up, in  $O(n)$  time.



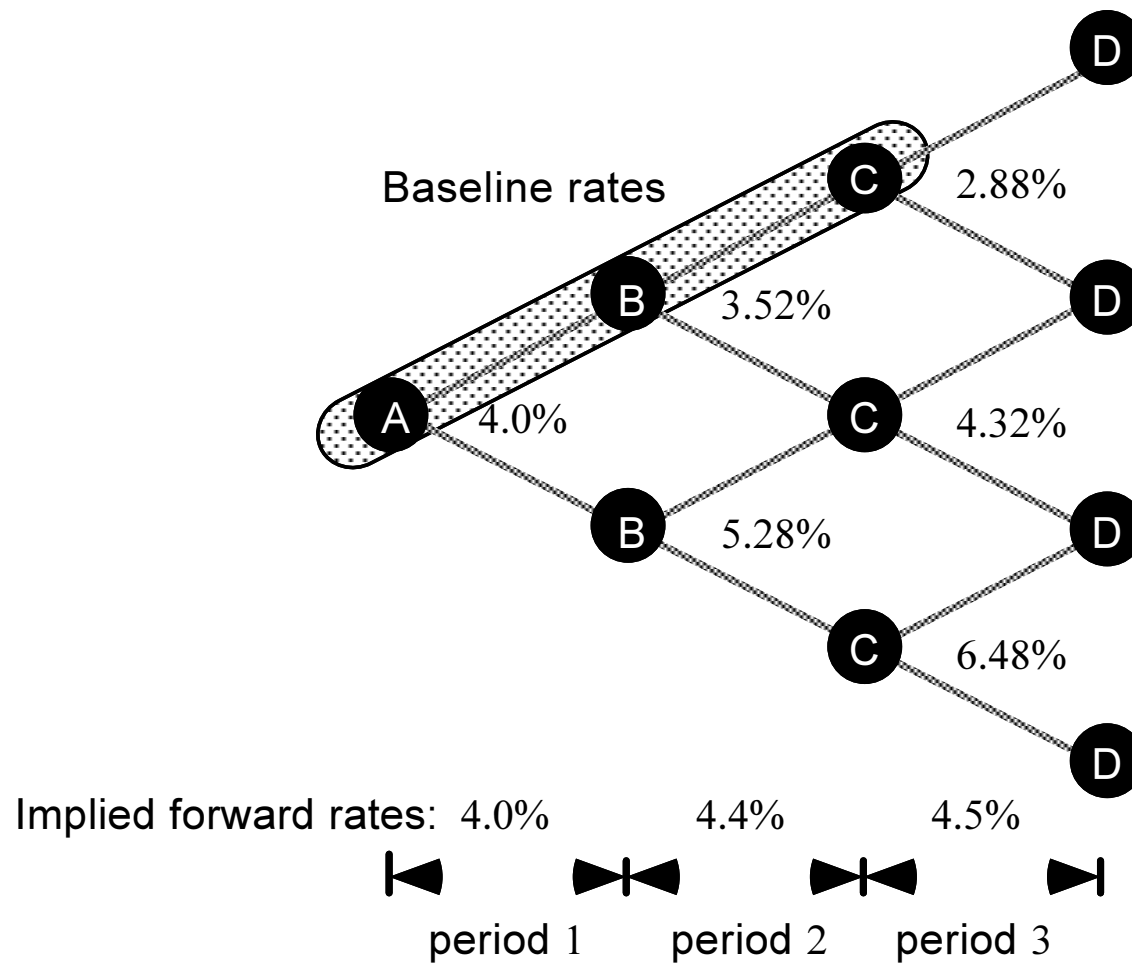
## An Approximate Calibration Scheme (continued)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

$$\frac{1}{4} \times \frac{1}{1.04} \times \left( \frac{1}{1.0352} \times \left( \frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left( \frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

or 0.88155, which exceeds discount factor 0.88135.

- The tree is thus *not* calibrated.



## An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree overprices the bonds.<sup>a</sup>
- Suppose we replace the baseline rates  $r_j$  by  $r_j v_j$ .
- Then the resulting tree underprices the bonds.<sup>b</sup>
- The true baseline rates are thus bounded between  $r_j$  and  $r_j v_j$ .

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<sup>a</sup>See Exercise 23.2.4 in text.

<sup>b</sup>Lyuu and Wang (F95922018) (2009, 2011).

## Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the  $m$ -period zero-coupon bond as computing some function  $f(r_m)$  of the unknown baseline rate  $r_m$  for period  $m$ .
- A root-finding method is applied to solve  $f(r_m) = P$  for  $r_m$  given the zero's price  $P$  and  $r_1, r_2, \dots, r_{m-1}$ .
- This procedure is carried out for  $m = 1, 2, \dots, n$ .
- It runs in  $O(n^3)$  time.

## Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in  $O(n^2)$  time by the use of forward induction.<sup>a</sup>
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price, the Arrow-Debreu price, or Green's function.
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 0 to time  $n$ .

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<sup>a</sup>Jamshidian (1991).

## Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time  $j$  and there are  $j + 1$  nodes.
  - The unknown baseline rate for period  $j$  is  $r \equiv r_j$ .
  - The multiplicative ratio is  $v \equiv v_j$ .
  - $P_1, P_2, \dots, P_j$  are the known state prices at *earlier* time  $j - 1$ , corresponding to rates  $r, rv, \dots, rv^{j-1}$  for period  $j$ .
- By definition,  $\sum_{i=1}^j P_i$  is the price of the  $(j - 1)$ -period zero-coupon bond.
- We want to find  $r$  based on  $P_1, P_2, \dots, P_j$  and the price of the  $j$ -period zero-coupon bond.

## Binomial Interest Rate Tree Calibration (continued)

- One dollar at time  $j$  has a known market value of  $1/[1 + S(j)]^j$ , where  $S(j)$  is the  $j$ -period spot rate.
- Alternatively, this dollar has a present value of

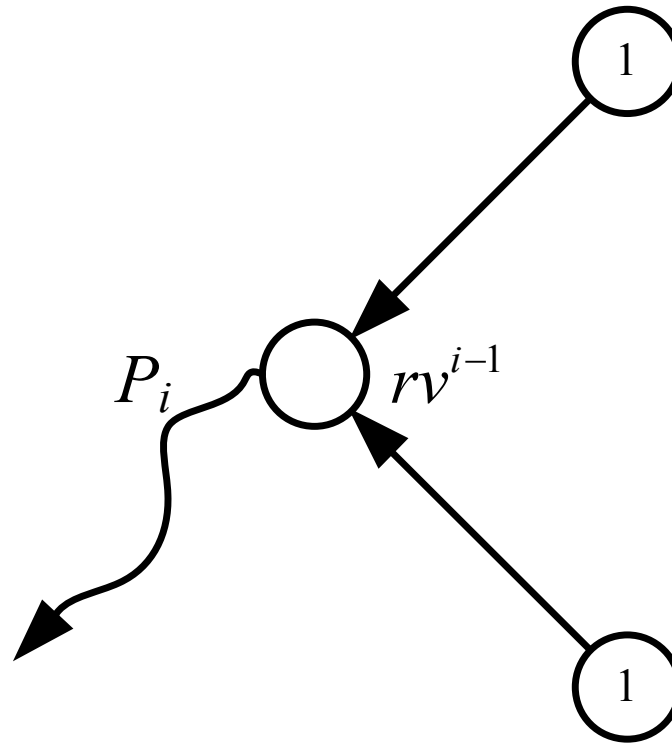
$$g(r) \equiv \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}$$

(see next plot).

- So we solve

$$g(r) = \frac{1}{[1 + S(j)]^j} \quad (124)$$

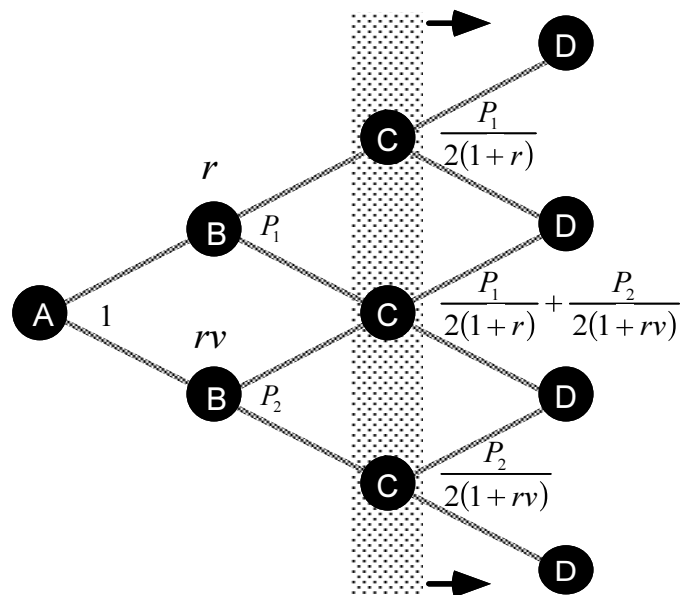
for  $r$ .



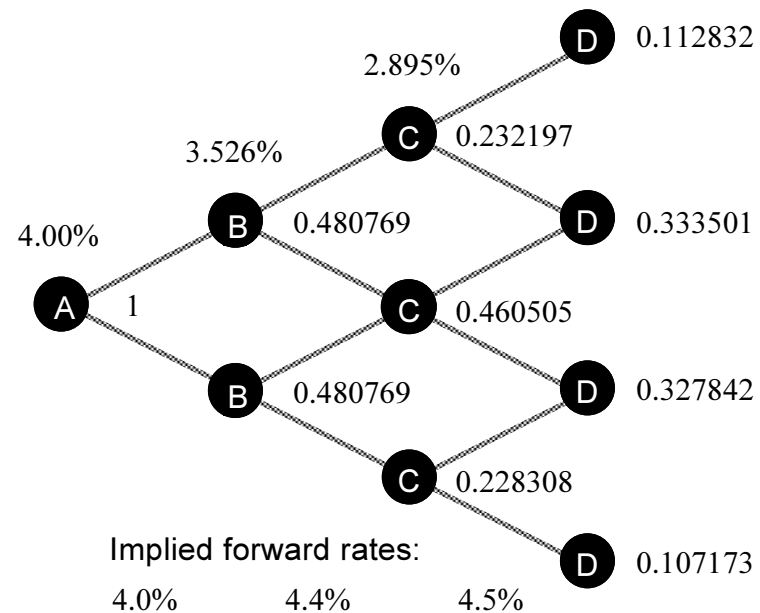


## Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for  $r$  is guaranteed.
- The state prices at time  $j$  can now be calculated (see panel (a) next page).
- We call a tree with these state prices a binomial state price tree (see panel (b) next page).
- The calibrated tree is depicted on p. 954.

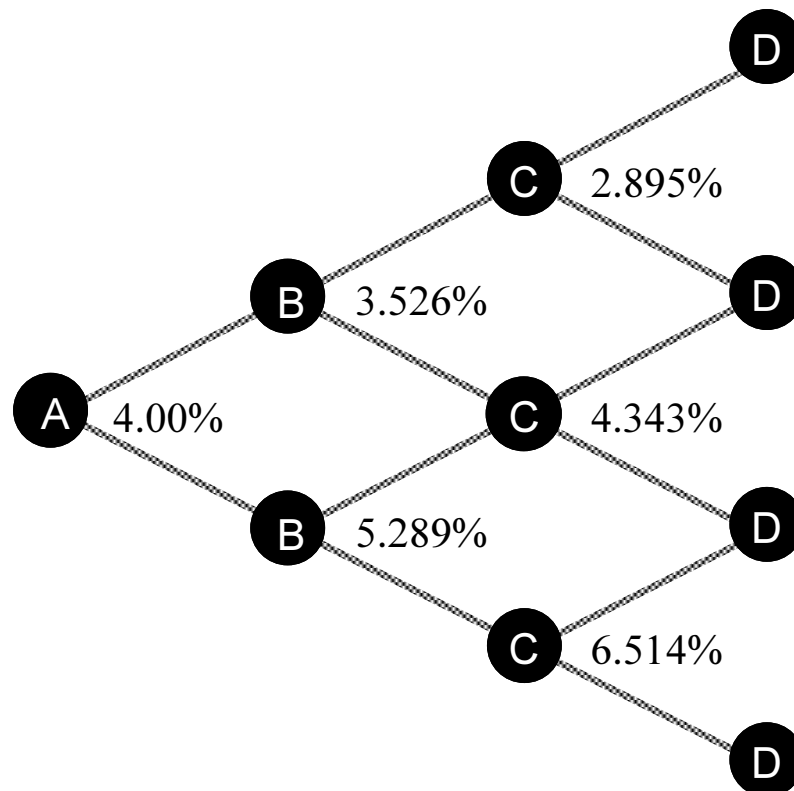


(a)



Implied forward rates:  
 4.0%      4.4%      4.5%  
 period 1   period 2   period 3

(b)



Implied forward rates: 4.0%      4.4%      4.5%

period 1      period 2      period 3

## Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the  $r$  in Eq. (124) on p. 950 as  $g'(r)$  is easy to evaluate.
- The monotonicity and the convexity of  $g(r)$  also facilitate root finding.
- The total running time is  $O(n^2)$ , as each root-finding routine consumes  $O(j)$  time.
- With a good initial guess,<sup>a</sup> the Newton-Raphson method converges in only a few steps.<sup>b</sup>

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<sup>a</sup>Such as the  $r_j = (\frac{2}{1+v_j})^{j-1} f_j$  on p. 943.

<sup>b</sup>Lyyu (1999).

## A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period,  $r_2$ , satisfies

$$\frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.$$

- The result is  $r_2 = 3.526\%$ .
- This is used to derive the next column of state prices shown in panel (b) on p. 953 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

## A Numerical Example (concluded)

- The baseline rate for the third period,  $r_3$ , satisfies

$$\frac{0.232197}{1 + r_3} + \frac{0.460505}{1 + 1.5 \times r_3} + \frac{0.228308}{1 + (1.5)^2 \times r_3} = 0.88135.$$

- The result is  $r_3 = 2.895\%$ .
- Now, redo the calculation on p. 944 using the new rates:

$$\frac{1}{4} \times \frac{1}{1.04} \times \left[ \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) + \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) \right],$$

which equals 0.88135, an exact match.

- The tree on p. 954 prices without bias the benchmark securities.

## Spread of Nonbenchmark Bonds

- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

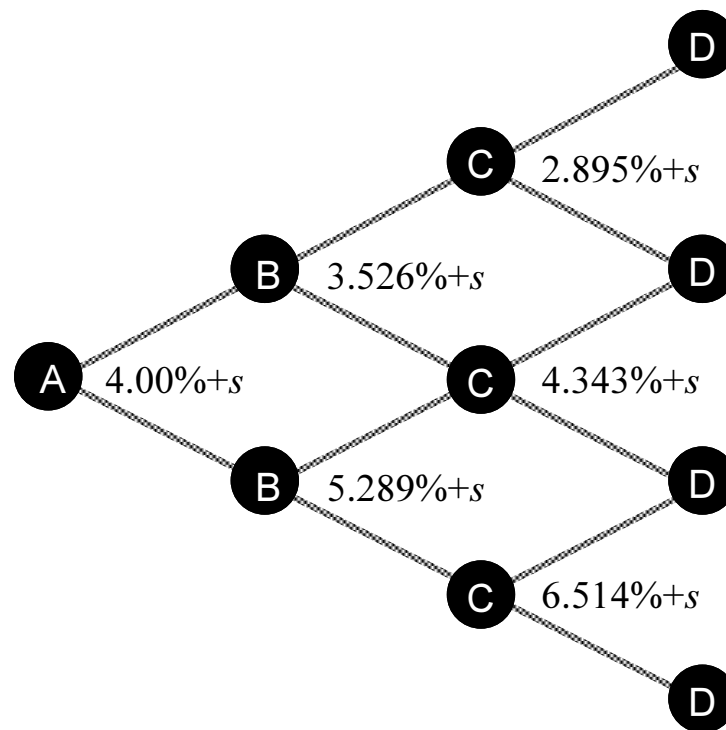
## Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 960.
- Consider a security with cash flow  $C_i$  at time  $i$  for  $i = 1, 2, 3$ .
- Its model price is  $p(s)$ , which is equal to

$$\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895 + s} + \frac{C_3}{1.04343 + s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343 + s} + \frac{C_3}{1.06514 + s} \right) \right) \right].$$

- Given a market price of  $P$ , the spread is the  $s$  that solves  $P = p(s)$ .





Implied forward rates: 4.0%      4.4%      4.5%

period 1      period 2      period 3

## Spread of Nonbenchmark Bonds (continued)

- The model price  $p(s)$  is a monotonically decreasing, convex function of  $s$ .
- We will employ the Newton-Raphson root-finding method to solve

$$p(s) - P = 0$$

for  $s$ .

- But a quick look at the equation for  $p(s)$  reveals that evaluating  $p'(s)$  directly is infeasible.
- Fortunately, the tree can be used to evaluate both  $p(s)$  and  $p'(s)$  during backward induction.

## Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node  $A$  in the tree associated with the short rate  $r$ .
- In the process of computing the model price  $p(s)$ , a price  $p_A(s)$  is computed at  $A$ .
- Prices computed at  $A$ 's two successor nodes  $B$  and  $C$  are discounted by  $r + s$  to obtain  $p_A(s)$  as follows,

$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)},$$

where  $c$  denotes the cash flow at  $A$ .

## Spread of Nonbenchmark Bonds (continued)

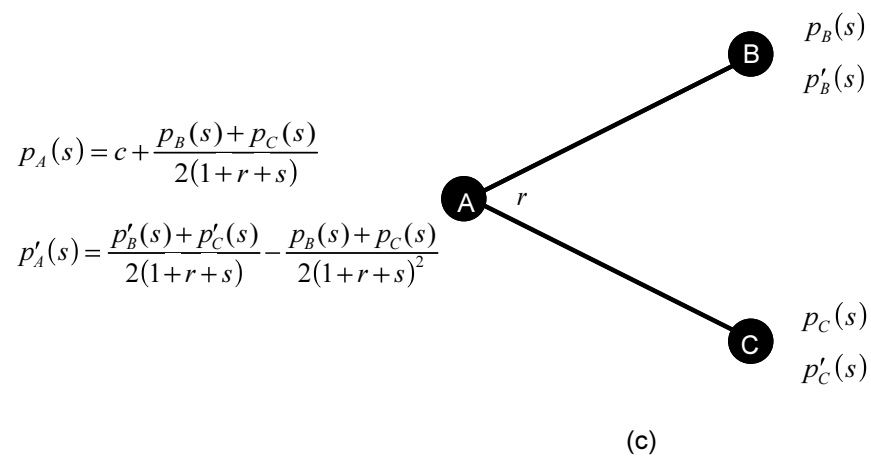
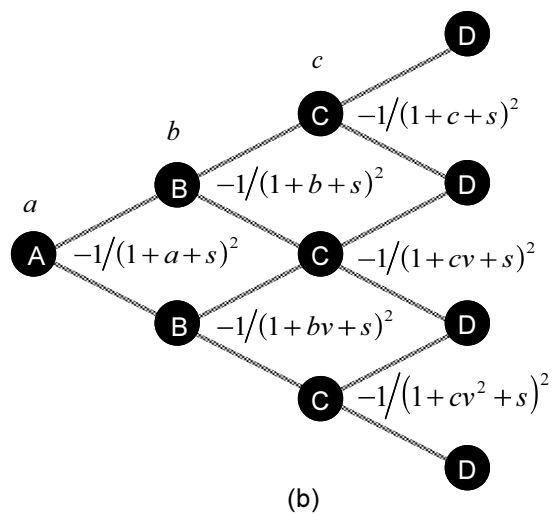
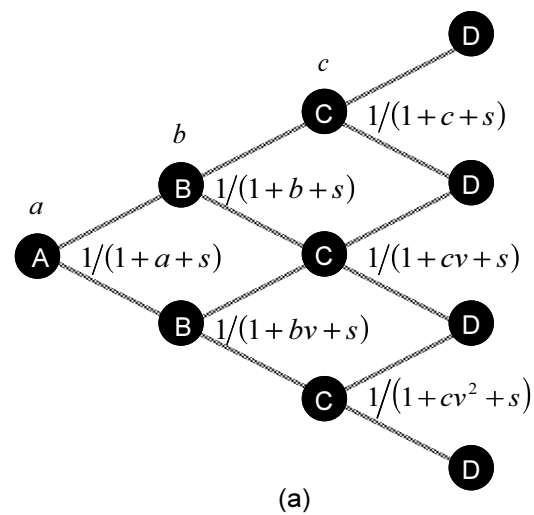
- To compute  $p'_A(s)$  as well, node A calculates

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}. \quad (125)$$

- This is easy if  $p'_B(s)$  and  $p'_C(s)$  are also computed at nodes B and C.
- When A is a terminal node, simply use the payoff function for  $p_A(s)$ .<sup>a</sup>

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<sup>a</sup>Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.



$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1+r+s)}$$

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1+r+s)} - \frac{p_B(s) + p_C(s)}{2(1+r+s)^2}$$

## Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield  $p(s)$  and  $p'(s)$  at the root (p. 964).
- This is called the differential tree method.<sup>a</sup>
  - Similar ideas can be found in automatic differentiation (AD)<sup>b</sup> and backpropagation<sup>c</sup> in artificial neural networks.
- The total running time is  $O(n^2)$ .
- The memory requirement is  $O(n)$ .

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<sup>a</sup>Lyu (1999).

<sup>b</sup>Rall (1981).

<sup>c</sup>Rumelhart, Hinton, and Williams (1986).

## Spread of Nonbenchmark Bonds (continued)

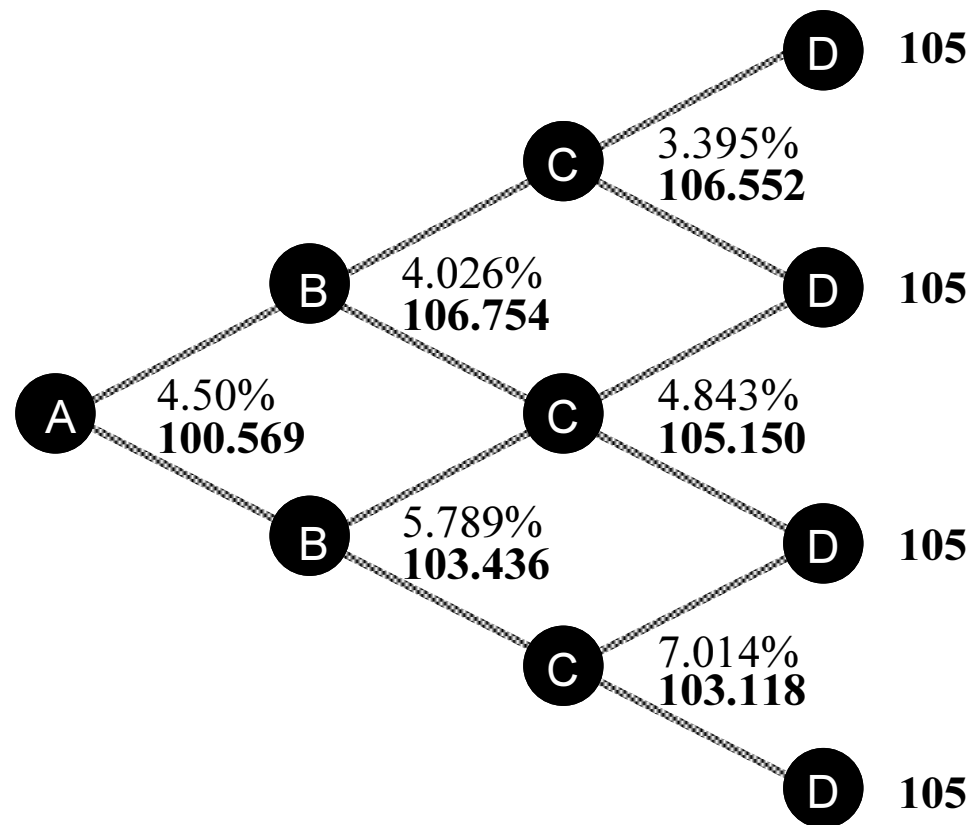
Number of partitions $n$	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	.....	.....	.....

75MHz Sun SPARCstation 20.

## Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 968).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 121) and static spread (p. 122) of the nonbenchmark bond over an otherwise identical benchmark bond.





Cash flows:                      5                      5                      105

## More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)<sup>a</sup>

American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

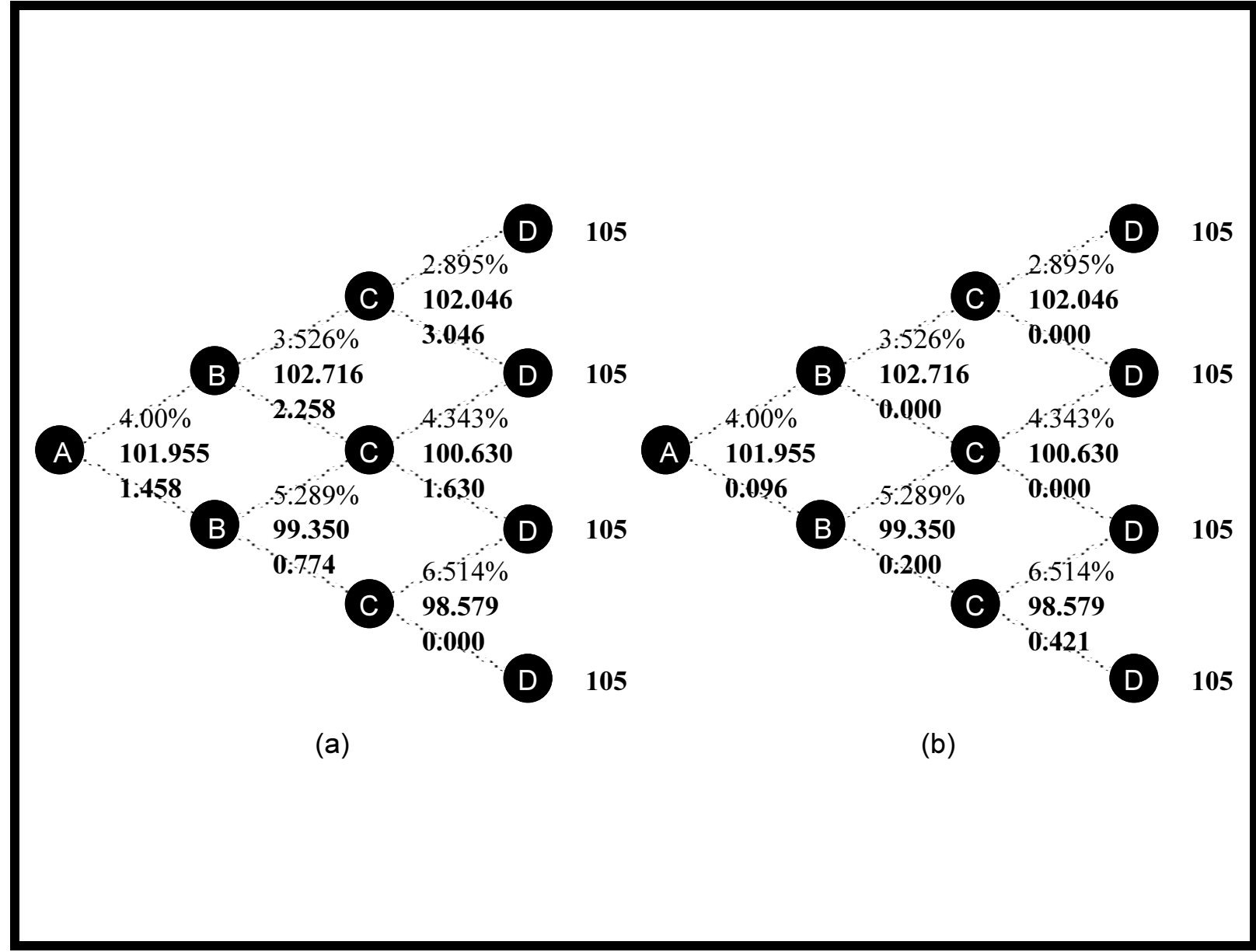
Intel 166MHz Pentium, running on Microsoft Windows 95.

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<sup>a</sup>Lyuu (1999).

## Fixed-Income Options

- Consider a 2-year 99 European call on the 3-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 971 the 3-year Treasury's price minus the \$5 interest at year 2 could be \$102.046, \$100.630, or \$98.579 two years from now.
  - The accrued interest is not included as it belongs to the original bondholder.
- Now compare the strike price against the bond prices.
- The call is in the money in the first two scenarios out of the money in the third.



## Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 971(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 971(b).

## Fixed-Income Options (concluded)

- The present value of the strike price is  
 $PV(X) = 99 \times 0.92101 = 91.18$ .
- The Treasury is worth  $B = 101.955$ .
- The present value of the interest payments during the life of the options is

$$PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth  $C = 1.458$  and  $P = 0.096$ , respectively.
- Hence the put-call parity is preserved:

$$C = P + B - PV(I) - PV(X).$$

## Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_h - O_\ell}{P_h - P_\ell}.$$

- In the above  $P_h$  and  $P_\ell$  denote the bond prices if the short rate moves up and down, respectively.
- Similarly,  $O_h$  and  $O_\ell$  denote the option values if the short rate moves up and down, respectively.

## Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 971 as examples.
- Their deltas are

$$\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441,$$

$$\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,$$

respectively.



## Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an  $n$ -period zero-coupon bond.
- First find its yield to maturity  $y_h$  ( $y_\ell$ , respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

$$\frac{1}{2} \ln \left( \frac{y_h}{y_\ell} \right). \quad (126)$$

## Volatility Term Structures (continued)

- For example, based on the tree on p. 954, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.$$

## Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is  $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$ .
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

## Volatility Term Structures (continued)

- Thus its yield is  $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$ .
- The yield volatility is hence

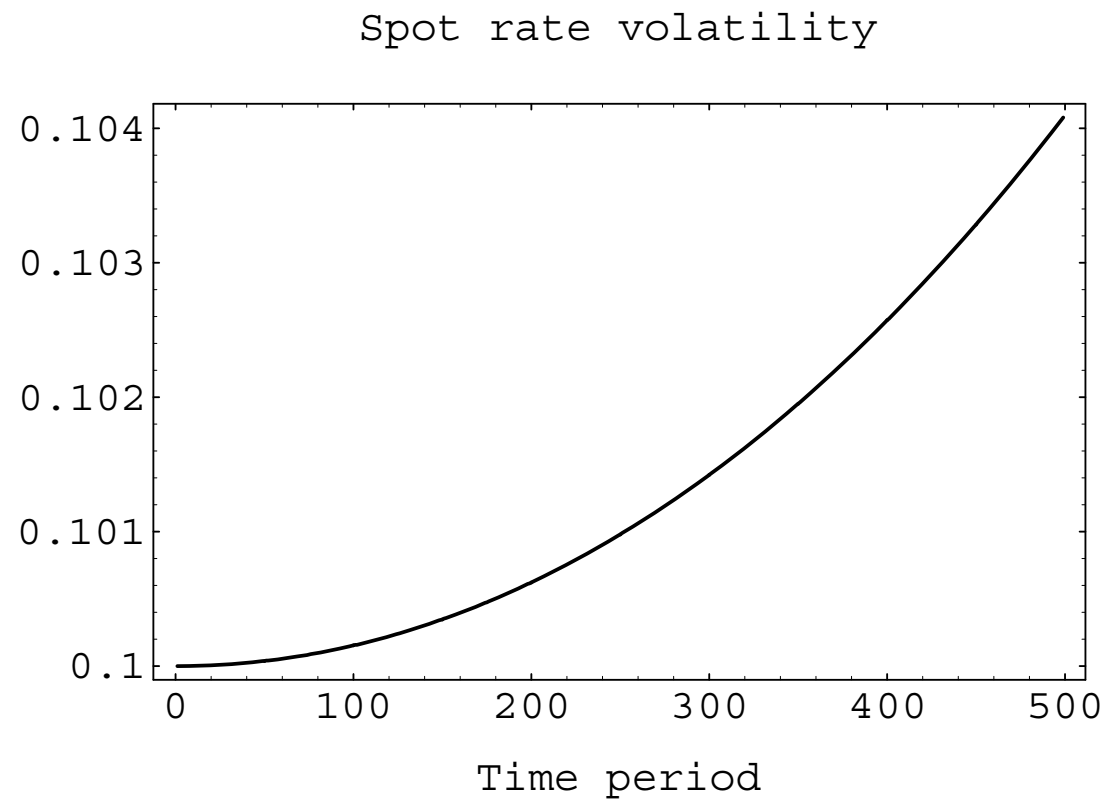
$$\frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\%,$$

slightly less than the one-year yield volatility.

- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.<sup>a</sup>
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

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<sup>a</sup>The relation is reversed for price volatilities (duration).



Short rate volatility given flat %10 volatility term structure.

## Volatility Term Structures (concluded)

- We started with  $v_i$  and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The  $v_i$ —hence the short rate volatilities via Eq. (121) on p. 932—and the  $r_i$  are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.<sup>a</sup>

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<sup>a</sup>Black, Derman, and Toy (1990).

# *Foundations of Term Structure Modeling*

[Meriwether] scoring especially high marks  
in mathematics — an indispensable subject  
for a bond trader.  
— Roger Lowenstein,  
*When Genius Failed* (2000)



[The] fixed-income traders I knew  
seemed smarter than the equity trader [...]  
there's no competitive edge to  
being smart in the equities business[.]  
— Emanuel Derman,  
*My Life as a Quant* (2004)

Bond market terminology was designed less  
to convey meaning than to bewilder outsiders.  
— Michael Lewis, *The Big Short* (2011)

## Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time  $t$ , the next time instant refers to time  $t + dt$  in the continuous-time model and time  $t + 1$  in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

## Standard Notations

The following notation will be used throughout.

$t$ : a point in time.

$r(t)$ : the one-period riskless rate prevailing at time  $t$  for repayment one period later.<sup>a</sup>

$P(t, T)$ : the present value at time  $t$  of one dollar at time  $T$ .

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<sup>a</sup>Alternatively, the instantaneous spot rate, or short rate, at time  $t$ .

## Standard Notations (continued)

$r(t, T)$ : the  $(T - t)$ -period interest rate prevailing at time  $t$  stated on a per-period basis and compounded once per period.<sup>a</sup>

$F(t, T, M)$ : the forward price at time  $t$  of a forward contract that delivers at time  $T$  a zero-coupon bond maturing at time  $M \geq T$ .

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<sup>a</sup>In other words, the  $(T - t)$ -period spot rate at time  $t$ .

## Standard Notations (concluded)

$f(t, T, L)$ : the  $L$ -period forward rate at time  $T$  implied at time  $t$  stated on a per-period basis and compounded once per period.

$f(t, T)$ : the one-period or instantaneous forward rate at time  $T$  as seen at time  $t$  stated on a per period basis and compounded once per period.

- It is  $f(t, T, 1)$  in the discrete-time model and  $f(t, T, dt)$  in the continuous-time model.
- Note that  $f(t, t)$  equals the short rate  $r(t)$ .

## Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t, T) = \begin{cases} (1 + r(t, T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t, T)(T-t)}, & \text{in continuous time.} \end{cases}$$

- $r(t, T)$  as a function of  $T$  defines the spot rate curve at time  $t$ .
- By definition,

$$f(t, t) = \begin{cases} r(t, t + 1), & \text{in discrete time,} \\ r(t, t), & \text{in continuous time.} \end{cases}$$

## Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \quad (127)$$

- The forward price equals the future value at time  $T$  of the underlying asset.<sup>a</sup>
- Equation (127) holds whether the model is discrete-time or continuous-time.

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<sup>a</sup>See Exercise 24.2.1 of the textbook for proof.

## Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \quad (128)$$

in discrete time.

- The analog to Eq. (128) under simple compounding is

$$f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right).$$



## Fundamental Relations (continued)

- In continuous time,

$$f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L} \quad (129)$$

by Eq. (127) on p. 990.

- Furthermore,

$$\begin{aligned} f(t, T, \Delta t) &= \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t, T)}{\partial T} \\ &= -\frac{\partial P(t, T)/\partial T}{P(t, T)}. \end{aligned}$$

## Fundamental Relations (continued)

- So

$$f(t, T) \equiv \lim_{\Delta t \rightarrow 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \quad (130)$$

- Because Eq. (130) is equivalent to

$$P(t, T) = e^{-\int_t^T f(t, s) ds}, \quad (131)$$

the spot rate curve is

$$r(t, T) = \frac{\int_t^T f(t, s) ds}{T - t}.$$

## Fundamental Relations (concluded)

- The discrete analog to Eq. (131) is

$$P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}.$$

- The short rate and the market discount function are related by

$$r(t) = - \left. \frac{\partial P(t, T)}{\partial T} \right|_{T=t}.$$

## Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all  $t + 1 < T$ ,

$$\frac{E_t[P(t + 1, T)]}{P(t, T)} = 1 + r(t). \quad (132)$$

- Relation (132) in fact follows from the risk-neutral valuation principle.<sup>a</sup>

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<sup>a</sup>Theorem 17 on p. 514.

## Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability  $\pi$ .
- Rewrite Eq. (132) as

$$\frac{E_t^\pi [P(t+1, T)]}{1 + r(t)} = P(t, T).$$

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

## Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned} & P(t, T) \\ = & E_t^\pi \left[ \frac{P(t+1, T)}{1+r(t)} \right] \\ = & E_t^\pi \left[ \frac{E_{t+1}^\pi [P(t+2, T)]}{(1+r(t))(1+r(t+1))} \right] = \dots \\ = & E_t^\pi \left[ \frac{1}{(1+r(t))(1+r(t+1)) \cdots (1+r(T-1))} \right]. \quad (133) \end{aligned}$$

## Risk-Neutral Pricing (concluded)

- Equation (132) on p. 995 can also be expressed as

$$E_t[ P(t + 1, T) ] = F(t, t + 1, T).$$

- Verify that with, e.g., Eq. (127) on p. 990.
- Hence the forward price for the next period is an unbiased estimator of the expected bond price.<sup>a</sup>

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<sup>a</sup>But the forward rate is not an unbiased estimator of the expected future short rate (p. 946).

## Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t, T) = E_t \left[ e^{-\int_t^T r(s) ds} \right], \quad t < T. \quad (134)$$

- Note that  $e^{\int_t^T r(s) ds}$  is the bank account process, which denotes the rolled-over money market account.



## Interest Rate Swaps

- Consider an interest rate swap made at time  $t$  (now) with payments to be exchanged at times  $t_1, t_2, \dots, t_n$ .
- The fixed rate is  $c$  per annum.
- The floating-rate payments are based on the future annual rates  $f_0, f_1, \dots, f_{n-1}$  at times  $t_0, t_1, \dots, t_{n-1}$ .
- For simplicity, assume  $t_{i+1} - t_i$  is a fixed constant  $\Delta t$  for all  $i$ , and the notional principal is one dollar.
- If  $t < t_0$ , we have a forward interest rate swap.
- The ordinary swap corresponds to  $t = t_0$ .

## Interest Rate Swaps (continued)

- The amount to be paid out at time  $t_{i+1}$  is  $(f_i - c) \Delta t$  for the *floating-rate payer*.
- Simple rates are adopted here.
- Hence  $f_i$  satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

## Interest Rate Swaps (continued)

- The value of the swap at time  $t$  is thus

$$\begin{aligned} & \sum_{i=1}^n E_t^\pi \left[ e^{-\int_t^{t_i} r(s) ds} (f_{i-1} - c) \Delta t \right] \\ &= \sum_{i=1}^n E_t^\pi \left[ e^{-\int_t^{t_i} r(s) ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\ &= \sum_{i=1}^n [P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)] \\ &= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^n P(t, t_i). \end{aligned}$$

## Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

## Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^n P(t, t_i) \Delta t}. \quad (135)$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap,  $P(t, t_0) = 1$ .

## The Term Structure Equation<sup>a</sup>

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price  $P(r, t, T)$  follow

$$\frac{dP}{P} = \mu_p dt + \sigma_p dW.$$

- At time  $t$ , short one unit of a bond maturing at time  $s_1$  and buy  $\alpha$  units of a bond maturing at time  $s_2$ .

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<sup>a</sup>Vasicek (1977).

## The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r, t, s_1) + \alpha dP(r, t, s_2) \\ = & (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) dt \\ & + (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) dW. \end{aligned}$$

- Pick

$$\alpha \equiv \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}.$$

## The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)}{-P(r, t, s_1) + \alpha P(r, t, s_2)} = r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r.$$

- This becomes

$$\frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)}$$

after rearrangement.



## The Term Structure Equation (continued)

- Since the above equality holds for any  $s_1$  and  $s_2$ ,

$$\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \equiv \lambda(r, t) \quad (136)$$

for some  $\lambda$  independent of the bond maturity  $s$ .

- As  $\mu_p = r + \lambda\sigma_p$ , all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.
- The term  $\lambda(r, t)$  is called the market price of risk.
- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

## The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r, t) dt + \sigma(r, t) dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right) / P, \quad (137)$$

$$\sigma_p = \left( \sigma(r, t) \frac{\partial P}{\partial r} \right) / P, \quad (137')$$

subject to  $P(\cdot, T, T) = 1$ .

## The Term Structure Equation (concluded)

- Substitute  $\mu_p$  and  $\sigma_p$  into Eq. (136) on p. 1008 to obtain

$$-\frac{\partial P}{\partial T} + [\mu(r, t) - \lambda(r, t) \sigma(r, t)] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP. \quad (138)$$

- This is called the term structure equation.
- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.
- Once  $P$  is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$