The Barrier-Too-Close Problem (p. 664) Revisited

- Our idea solves it.
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 671ff).
  - Unlike an earlier solution using combinatorics (p. 655), now the choice of n is not that restricted.
- The handling of single-barrier options is similar.

The Barrier-Too-Close Problem Revisited (continued)

- We can build the tree treating S as if it were a second barrier.
- So both H and S are matched.
- Alternatively, we can pick $\Delta\tau \equiv T/m$ as our length of a period Δt without any adjustment.
- Then build the tree from the price H down.
- So H is matched.
- As usual, the initial price S will be matched by the trinomial structure.

The Barrier-Too-Close Problem Revisited (concluded)

- The earlier trinomial tree is impractical as it needs a very large n when the barrier H is very close to S.[a]
  - It needs at least one *up* move to connect S to H as its middle branch is flat.
  - But when $S \approx H$, that up move must take a very small step, necessitating a small Δt.
- Now the trinomial structure's middle branch is not necessarily flat.
- So S can be connected to H via the middle branch, and the need of a very large n no longer exists!

---

[a]Recall the table on p. 665.

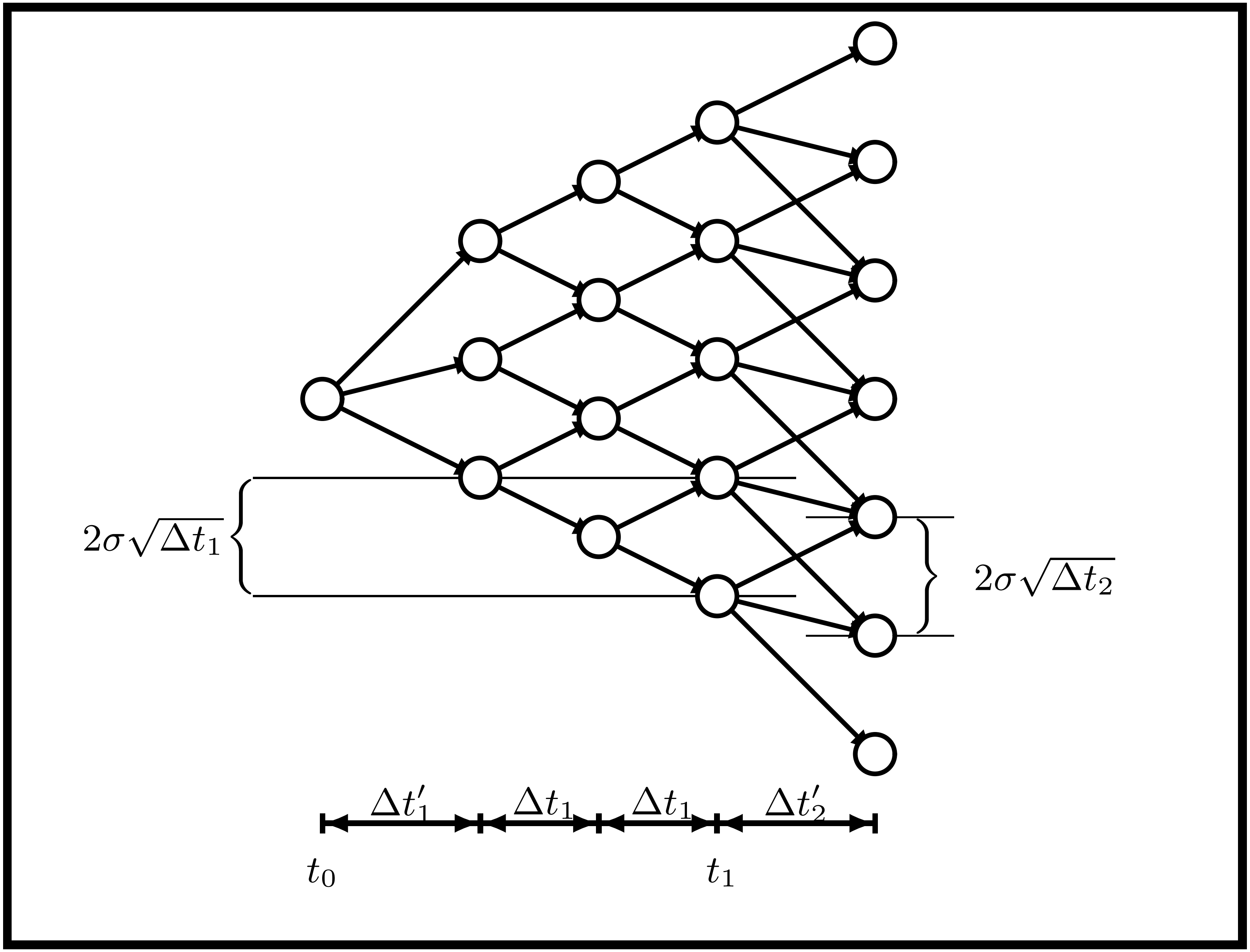
Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- The main difficulty with pricing discrete barrier options lies in matching the monitored times.
- Here is why.
- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \dots, t_\ell = T.$$

Pricing Discrete Barrier Options (continued)

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, meaning Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.
- See p. 711.



$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t_1'$
Δt_1
Δt_1
$\Delta t_2'$
t_0
t_1

Pricing Discrete Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[\,t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]
  1. The one we saw earlier (p. 303) models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

---

[a]Frishling (2002).

Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., $\alpha(t) S_0 + \beta(t) S_t$), for practitioners.[a]
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But they result in binomial trees that grow exponentially (recall p. 302).
- The binomial-trinomial tree can often avoid this problem.

---

[a] Henry-Labordère (2009).

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \equiv t/\Delta t$ periods between time 0 and the ex-dividend date.[a]
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D$.
  - Equivalently,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)} \right]^2.$$

[a] Assume m is an integer or simply an input.

Options on a Stock That Pays Known Dividends (continued)

- Build a binomial tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \dots ,$$

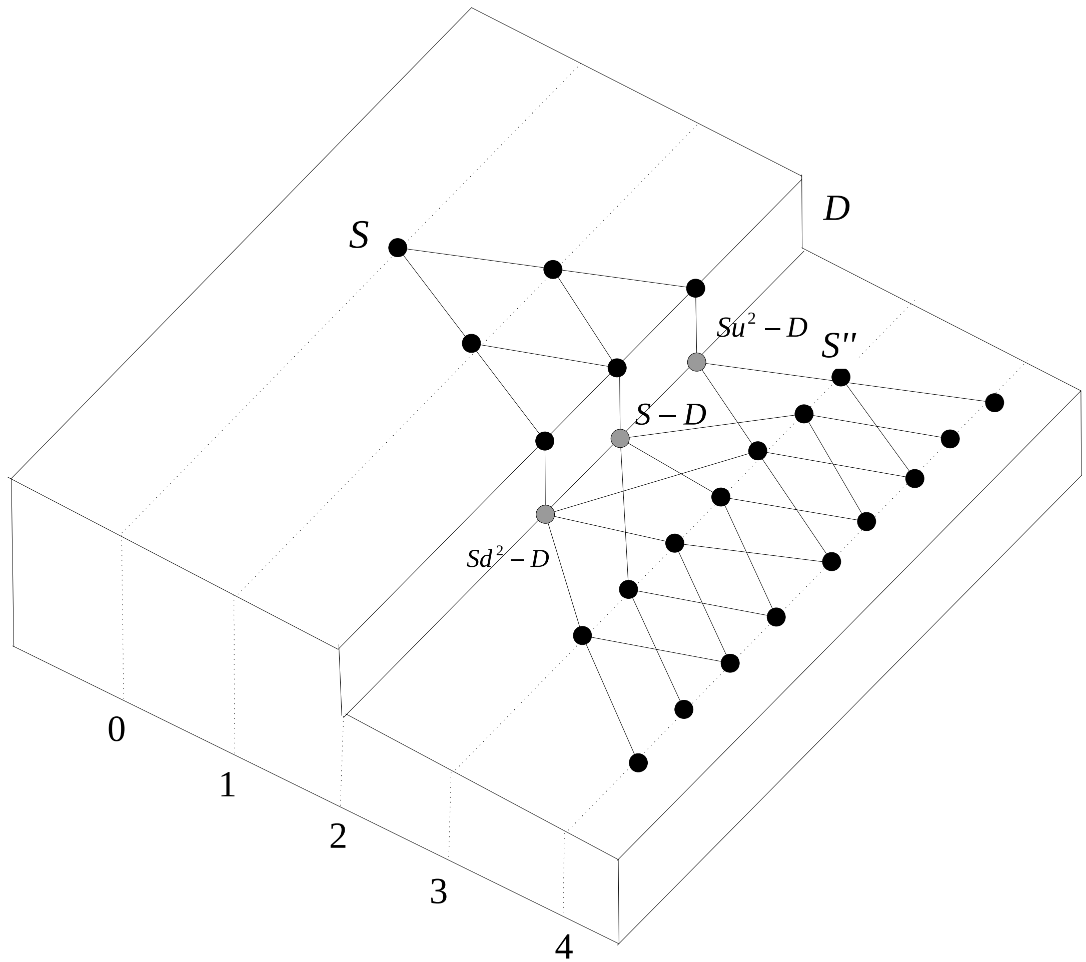
same as the binomial tree.

Options on a Stock That Pays Known Dividends (concluded)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 689.
- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

[a] Dai (B82506025, R86526008, D8852600) and Lyuu (2004); Dai (B82506025, R86526008) (2009).

A Stair Tree



Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Option pricing with time-varying volatilities and time-varying barriers.[d]
- Defaultable bond pricing.[e]

---

[a] Wu (R96723058) (2009).
[b] Huang (R97922073) (2010).
[c] Huang (R97922081) (2010).
[d] Chou (R97944012) (2010) and Chen (R98922127) (2011).
[e] Dai (B82506025, R86526008, D8852600), Lyuu, and Wang (F95922018) (2009, 2010, 2014).

General Properties of Trees[a]

- Consider the Ito process,

$$dX = a(X, t)\, dt + \sigma\, dW,$$

where $a(X, t) = O(1)$ and σ is a constant.

- The mean and volatility of the next move's size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.
- Note that $\sqrt{\Delta t} \gg \Delta t$.
- The tree spacing must be in the order of $\sigma\sqrt{\Delta t}$ if the variance is to be matched.[b]

---

[a] Chiu (R98723059) (2012) and Wu (R99922149) (2012).

[b] Lyuu and Wang (F95922018) (2009, 2011) and Lyuu and Wen (D94922003) (2012).

Merton's Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.
- Stochastic volatility and jump processes have been proposed to address this problem.
- Merton's jump-diffusion model is our focus.[a]

---

[a]Merton (1976).

Merton's Jump-Diffusion Model (continued)

- This model superimposes a jump component on a diffusion component.
- The diffusion component is the familiar geometric Brownian motion.
- The jump component is composed of lognormal jumps driven by a Poisson process.
  - It models the sudden changes in the stock price because of the arrival of important new information.

Merton's Jump-Diffusion Model (continued)

- Let S_t be the stock price at time t.
- The risk-neutral jump-diffusion process for the stock price follows

$$\frac{dS_t}{S_t} = (r - \lambda\bar{k})\, dt + \sigma\, dW_t + k\, dq_t. \tag{94}$$

- Above, σ denotes the volatility of the diffusion component.

Merton's Jump-Diffusion Model (continued)

- The jump event is governed by a compound Poisson process q_t with intensity λ, where k denotes the magnitude of the *random* jump.
  - The distribution of k obeys
    $$\ln(1+k) \sim N\left(\gamma, \delta^2\right)$$
    with mean $\bar{k} \equiv E(k) = e^{\gamma+\delta^2/2} - 1$.
- The model with $\lambda = 0$ reduces to the Black-Scholes model.

Merton's Jump-Diffusion Model (continued)

- The solution to Eq. (94) on p. 723 is

$$S_t = S_0 e^{(r - \lambda \bar{k} - \sigma^2/2)\, t + \sigma W_t} U(n(t)), \tag{95}$$

where

$$U(n(t)) = \prod_{i=0}^{n(t)} (1 + k_i)\,.$$

  - k_i is the magnitude of the ith jump with $\ln(1 + k_i) \sim N(\gamma, \delta^2)$.
  - $k_0 = 0$.
  - $n(t)$ is a Poisson process with intensity λ.

Merton's Jump-Diffusion Model (concluded)

- Recall that $n(t)$ denotes the number of jumps that occur up to time t.
- As $k > -1$, stock prices will stay positive.
- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.

Tree for Merton's Jump-Diffusion Model[a]

- Define the S-logarithmic return of the stock price S' as
$$\ln(S'/S).$$
- Define the logarithmic distance between stock prices S' and S as
$$|\ln(S') - \ln(S)| = |\ln(S'/S)|.$$

[a]Dai (B82506025, R86526008, D8852600), Wang (F95922018), Lyuu, and Liu (2010).

Tree for Merton's Jump-Diffusion Model (continued)

- Take the logarithm of Eq. (95) on p. 725:

$$M_t \equiv \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t, \tag{96}$$

where

$$X_t \equiv \left(r - \lambda\bar{k} - \sigma^2/2\right) t + \sigma W_t, \tag{97}$$

$$Y_t \equiv \sum_{i=0}^{n(t)} \ln\left(1 + k_i\right). \tag{98}$$

- It decomposes the S_0-logarithmic return of S_t into the diffusion component X_t and the jump component Y_t.

Tree for Merton's Jump-Diffusion Model (continued)

- Motivated by decomposition (96) on p. 728, the tree construction divides each period into a diffusion phase followed by a jump phase.
- In the diffusion phase, X_t is approximated by the BOPM.
- Hence X_t can make an up move to $X_t + \sigma\sqrt{\Delta t}$ with probability p_u or a down move to $X_t - \sigma\sqrt{\Delta t}$ with probability p_d.

Tree for Merton's Jump-Diffusion Model (continued)

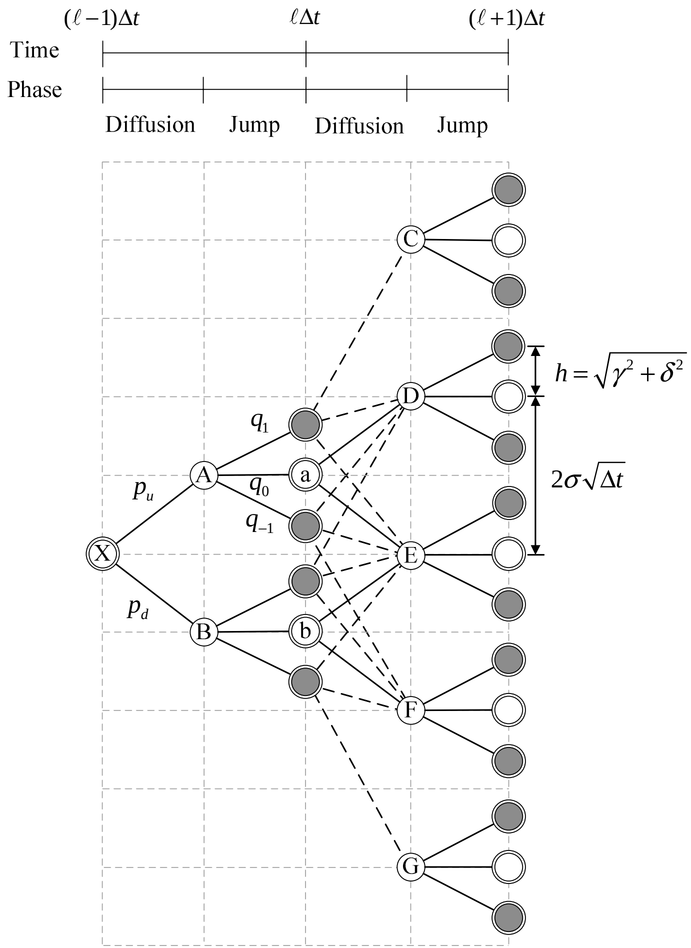
- According to BOPM,

$$
\begin{aligned}
p_u &= \frac{e^{\mu \Delta t} - d}{u - d}, \\
p_d &= 1 - p_u,
\end{aligned}
$$

  except that $\mu = r - \lambda \bar{k}$ here.

- The diffusion component gives rise to diffusion nodes.

- They are spaced at $2\sigma\sqrt{\Delta t}$ apart such as the white nodes A, B, C, D, E, F, and G on p. 731.



White nodes are diffusion nodes. Gray nodes are jump nodes. In the diffusion phase, the solid black lines denote the binomial structure of BOPM, whereas the dashed lines denote the trinomial structure. Here m is set to one here for simplicity. Only the double-circled nodes will remain after the construction. Note that a and b are diffusion nodes because no jump occurs in the jump phase.

Tree for Merton's Jump-Diffusion Model (concluded)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from *each* diffusion node to $2m$ jump nodes that match the first $2m$ moments of the lognormal jump.
- The m jump nodes above the diffusion node are spaced at h apart.
- The same holds for the m jump nodes below the diffusion node.
- The gray nodes at time $\ell\Delta t$ on p. 731 are jump nodes.
- After some work, the size of the tree is $O(n^{2.5})$.

Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff
$$\max\left(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0\right),$$
where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Multivariate Contingent Claims (concluded)[a]

Name	Payoff
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$
Better-off option	$\max(S_1(\tau), \ldots, S_k(\tau), 0)$
Worst-off option	$\min(S_1(\tau), \ldots, S_k(\tau), 0)$
Binary maximum option	$I\{\max(S_1(\tau), \ldots, S_k(\tau)) > X\}$
Maximum option	$\max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Minimum option	$\max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$
Basket average option	$\max((S_1(\tau) + \cdots + S_k(\tau))/k - X, 0)$
Multi-strike option	$\max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0)$
Pyramid rainbow option	$\max(\mid S_1(\tau) - X_1 \mid + \cdots + \mid S_k(\tau) - X_k \mid - X, 0)$
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0)$

[a] Lyuu and Teng (R91723054) (2011).

Correlated Trinomial Model[a]

- Two risky assets S_1 and S_2 follow

$$\frac{dS_i}{S_i} = r\,dt + \sigma_i\,dW_i$$

in a risk-neutral economy, $i = 1, 2$.

- Let

$$\begin{aligned} M_i &\equiv e^{r\Delta t}, \\ V_i &\equiv M_i^2(e^{\sigma_i^2 \Delta t} - 1). \end{aligned}$$

  - $S_i M_i$ is the mean of S_i at time Δt.
  - $S_i^2 V_i$ the variance of S_i at time Δt.

[a] Boyle, Evnine, and Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

- As usual, impose $u_i d_i = 1$.

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$
\begin{aligned}
1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\
S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\
S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2.
\end{aligned}
$$

Correlated Trinomial Model (concluded)

- Let $R \equiv M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$.
- Match the variances and covariance:

$$\begin{aligned}
S_1^2 V_1 &= (p_1+p_2)((S_1u_1)^2-(S_1M_1)^2)+p_5(S_1^2-(S_1M_1)^2)\\
&\quad +(p_3+p_4)((S_1d_1)^2-(S_1M_1)^2),\\
S_2^2 V_2 &= (p_1+p_4)((S_2u_2)^2-(S_2M_2)^2)+p_5(S_2^2-(S_2M_2)^2)\\
&\quad +(p_2+p_3)((S_2d_2)^2-(S_2M_2)^2),\\
S_1S_2R &= (p_1u_1u_2+p_2u_1d_2+p_3d_1d_2+p_4d_1u_2+p_5)\,S_1S_2.
\end{aligned}$$

- The solutions appear on p. 246 of the textbook.

Correlated Trinomial Model Simplified[a]

- Let $\mu'_i \equiv r - \sigma_i^2/2$ and $u_i \equiv e^{\lambda \sigma_i \sqrt{\Delta t}}$ for $i = 1, 2$.
- The following simpler scheme is good enough:

$$\begin{aligned}
p_1 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_2 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_3 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_4 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_5 &= 1 - \frac{1}{\lambda^2}.
\end{aligned}$$

[a]Madan, Milne, and Shefrin (1989).

Correlated Trinomial Model Simplified (continued)

- All of the probabilities lie between 0 and 1 if and only if

$$\begin{aligned} -1+\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}+\frac{\mu_2'}{\sigma_2}\right| \leq \quad & \rho \quad \leq 1-\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}-\frac{\mu_2'}{\sigma_2}\right|, & (99) \\ 1 \leq \quad & \lambda & (100) \end{aligned}$$

- We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as

$$-1+O(\sqrt{\Delta t})<\rho<1-O(\sqrt{\Delta t}),$$

such as the above one.[a]

[a] Kao (R98922093) (2011) and Kao (R98922093), Lyuu, and Wen (D94922003) (2014).

Correlated Trinomial Model Simplified (concluded)

- But this model cannot price 2-asset 2-barrier options accurately.[a]
- Few multivariate trees are both optimal and able to handle multiple barriers.[b]
- An alternative is to use orthogonalization.[c]

---

[a]See Chang (B89704039, R93922034), Hsu (R7526001, D89922012), and Lyuu (2006) and Kao (R98922093), Lyuu and Wen (D94922003) (2014) for solutions.

[b]See Kao (R98922093), Lyuu, and Wen (D94922003) (2014) for one.

[c]Hull and White (1990) and Dai (B82506025, R86526008, D8852600), Lyuu, and Wang (F95922018) (2012).

Extrapolation

- It is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{101}$$

- Assume c is an unknown constant independent of n.
  - Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and ignoring the smaller terms,

$$\begin{aligned} f(n_1) &= f + \frac{c}{n_1}, \\ f(n_2) &= f + \frac{c}{n_2}. \end{aligned}$$

- A better approximation to the desired f is

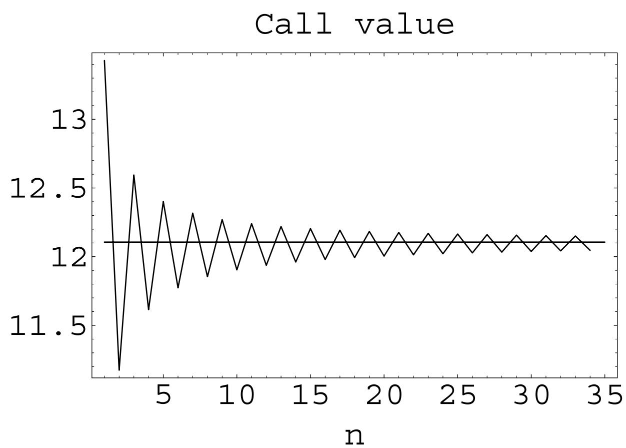
$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \tag{102}$$

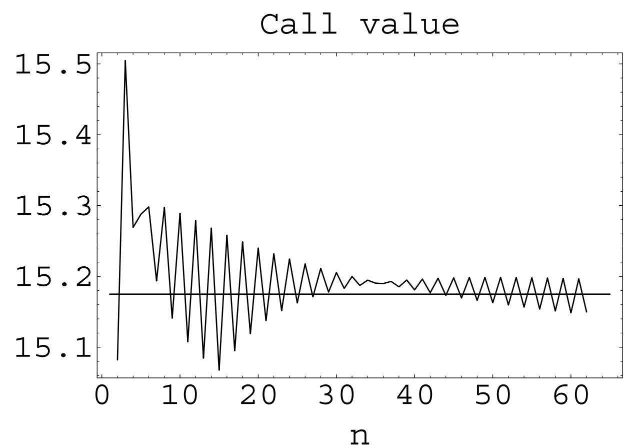
- This estimate should converge faster than $1/n$.[a]
- The Richardson extrapolation uses $n_2 = 2n_1$.

[a]It is identical to the forward rate formula (20) on p. 137!

Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (101) on p. 743.
- But the plots on p. 286 (redrawn on next page) demonstrate that convergence to the true option value oscillates with n.
- Extrapolation is inapplicable at this stage.

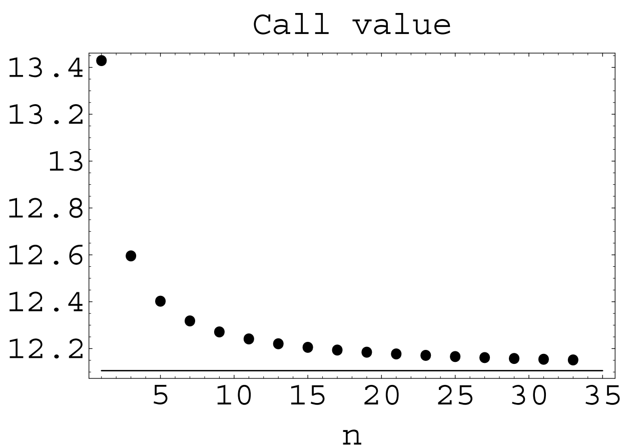

Call value
13
12.5
12
11.5
5 10 15 20 25 30 35
n


Call value
15.5
15.4
15.3
15.2
15.1
0 10 20 30 40 50 60
n

Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 746.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 748).[a]
- Apply extrapolation (102) on p. 744 with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 748.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

[a]This can be proved; see Chang and Palmer (2007).


Call value
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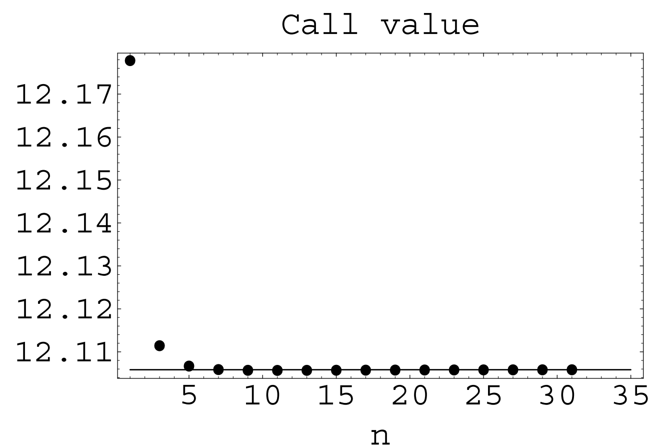
13.2
13
12.8
12.6
12.4
12.2
5 10 15 20 25 30 35
n

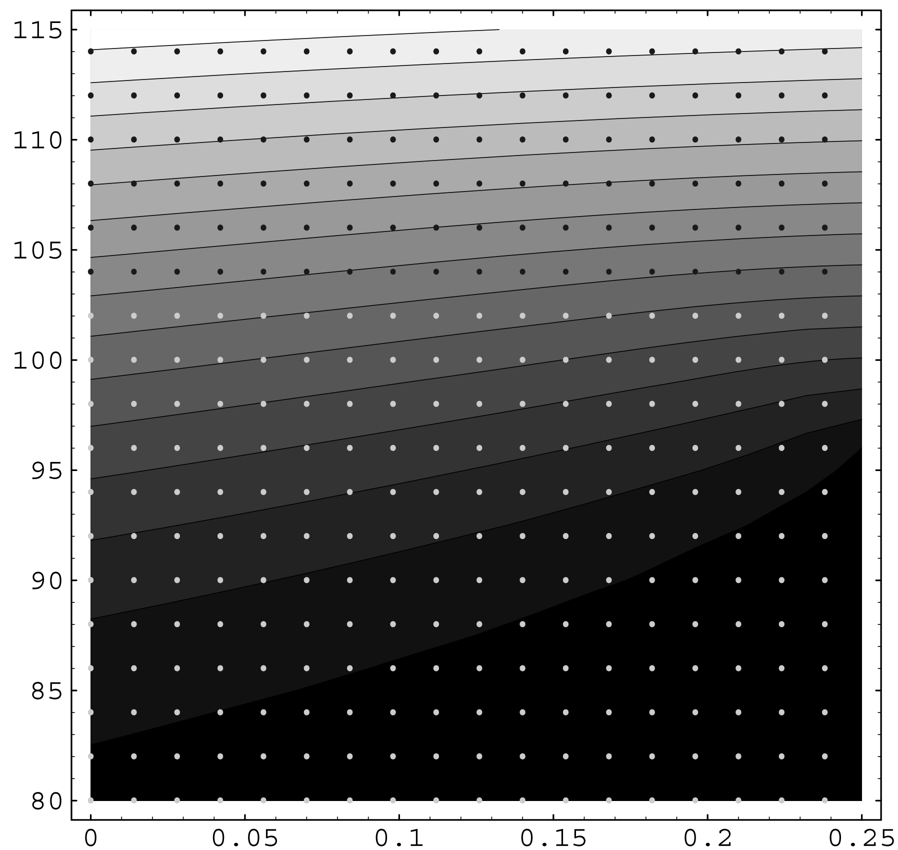

Call value
12.17
12.16
12.15
12.14
12.13
12.12
12.11
5 10 15 20 25 30 35
n

Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 752).
- Solve the equation numerically by introducing difference equations in place of derivatives.


115
110
105
100
95
90
85
80
0
0.05
0.1
0.15
0.2
0.25

Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite difference form is

$$\begin{aligned}-\rho(x_i, y_j) = {} & \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} \\ & + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.\end{aligned}$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$
\begin{aligned}
-h^2 \rho(x_i, y_j) &= \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) \\
&\quad + \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).
\end{aligned}
$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$, $D > 0$.
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x,t_{j+1}) - \theta(x,t_j)}{\Delta t} + \cdots, \tag{103}$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots. \tag{104}$$

Explicit Methods (continued)

- Next, assemble Eqs. (103) and (104) into a single equation at (x_i, t_j).
- But we need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (104), we might as well use x_i for x in Eq. (103).
- Two choices are possible for t in Eq. (104).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \tag{105}$$

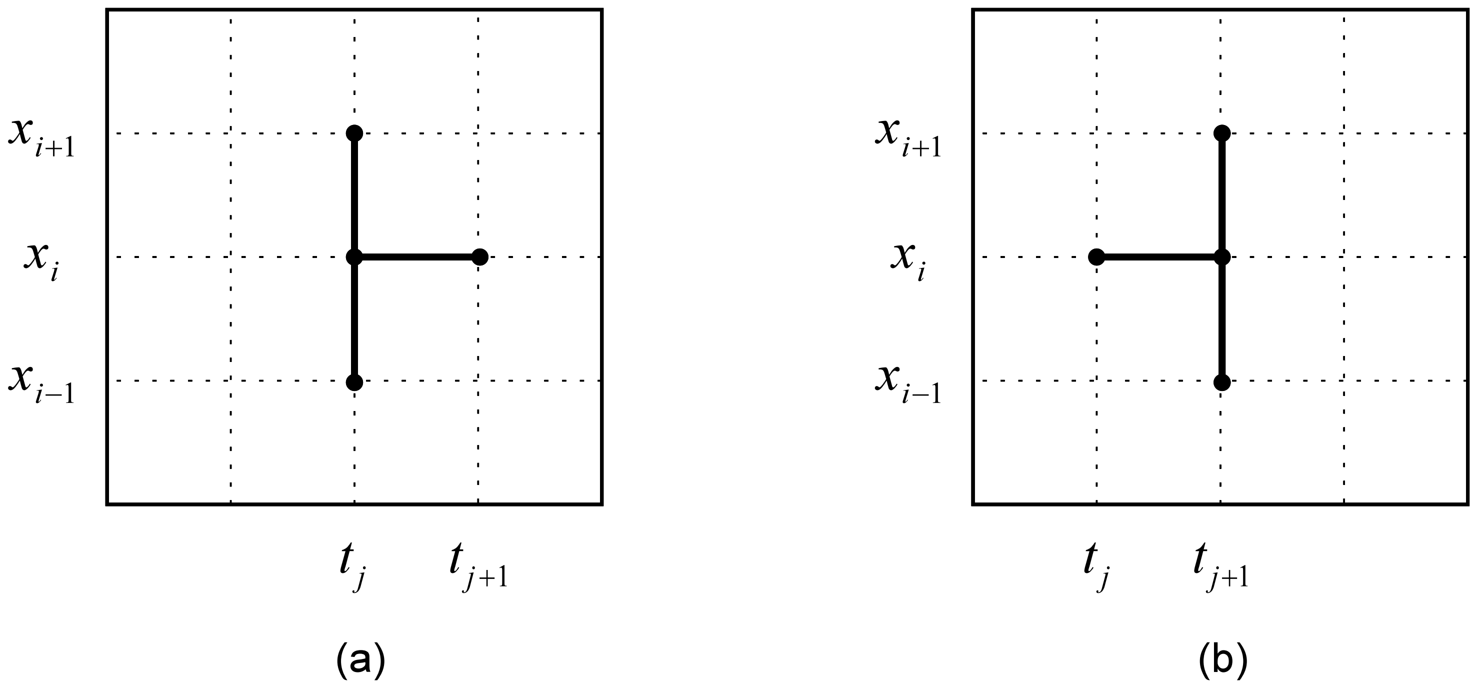
Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- Rearrange Eq. (105) on p. 756 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see exhibit (a) on next page).

Stencils



Explicit Methods (concluded)

- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

$$\theta_{i,1}, \quad i = 1, 2, \ldots .$$

- And then

$$\theta_{i,2}, \quad i = 1, 2, \ldots .$$

- And so on.

Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2/(2D).$$

  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

- The stability condition may lead to high running times and memory requirements.

- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time 8 times as much.

Explicit Method and Trinomial Tree

- Recall that

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!

Explicit Method and Trinomial Tree (concluded)

- The freedom in choosing Δx corresponds to similar freedom in the construction of trinomial trees.
- The explicit finite-difference equation is also identical to backward induction on a binomial tree.[a]
  - Let the binomial tree take 2 steps each of length $\Delta t/2$.
  - It is now a trinomial tree.

---

[a] Hilliard (2014).

Implicit Methods

- Suppose we use $t = t_{j+1}$ in Eq. (104) on p. 755 instead.
- The finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\,\frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}. \tag{106}$$

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is implicit:
  - The value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known.
  - See exhibit (b) on p. 758.

Implicit Methods (continued)

- Equation (106) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma)\,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma\theta_{i,j},$$

  where $\gamma \equiv (\Delta x)^2/(D\Delta t)$.

- This equation is unconditionally stable.

- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.

- After $\theta_{i,j}$ has been calculated for $i = 1, 2, \ldots, N$, the values of $\theta_{i,j+1}$ at time t_{j+1} can be computed as the solution to the following tridiagonal linear system,

Implicit Methods (continued)

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & a & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} -\gamma\theta_{1,j} - \theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j} - \theta_{N+1,j+1} \end{bmatrix},$$

where $a \equiv -2 - \gamma$.

Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
  - Never invert a matrix to solve a tridiagonal system.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.

Crank-Nicolson Method

- Take the average of explicit method (105) on p. 756 and implicit method (106) on p. 763:

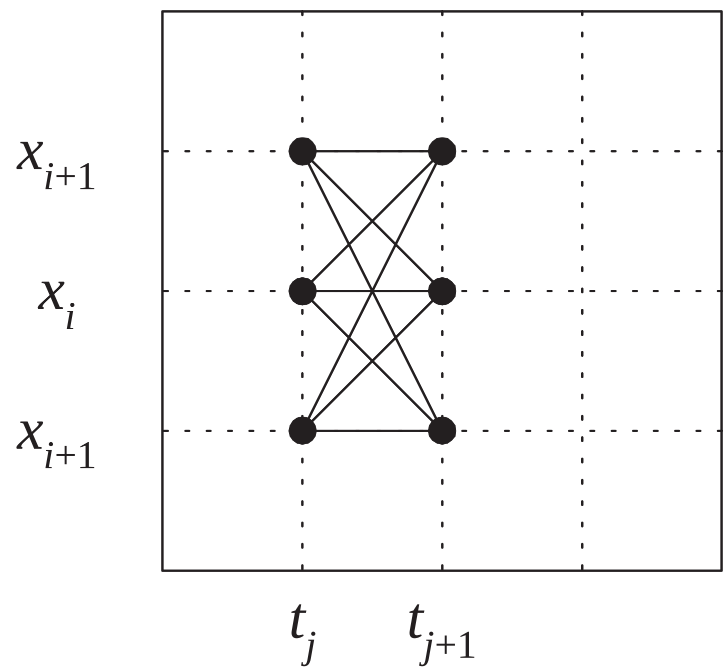
$$
\begin{aligned}
& \frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} \\
= \quad & \frac{1}{2} \left(D\, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2} + D\, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2} \right).
\end{aligned}
$$

- After rearrangement,

$$
\gamma\theta_{i,j+1} - \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{2} = \gamma\theta_{i,j} + \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{2}.
$$

- This is an unconditionally stable implicit method with excellent rates of convergence.

Stencil



Numerically Solving the Black-Scholes PDE (80) on p. 617

- See text.
- Brennan and Schwartz (1978) analyze the stability of the implicit method.