Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma[a]

A smooth function of an Ito process is itself an Ito process.

Theorem 19 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

---

[a] Ito (1944).

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{72}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.

- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.

- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$.
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 20 (Higher-Dimensional Ito's Lemma) *Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$. Then $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

where $f_i \equiv \partial f / \partial X_i$ and $f_{ik} \equiv \partial^2 f / \partial X_i \partial X_k$.

Ito's Lemma (continued)

- The multiplication table for Theorem 20 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t\, dt + b_t\, dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$\begin{aligned} df &= \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(dX_t)^2 \\ &= \frac{\partial f}{\partial X_t}\,(a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(a_t\, dt + b_t\, dW_t)^2 \\ &= \left(\frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, b_t^2\right) dt \\ &\quad + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \end{aligned} \tag{73}$$

Ito's Lemma (continued)

Theorem 21 (Alternative Ito's Lemma) *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + b_i\, dW_i$*. Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider geometric Brownian motion $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
  - Hence $dX = \mu\, dt + \sigma\, dW$ by Eq. (67) on p. 525.
- Note that

$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

Geometric Brownian Motion (concluded)

- Ito's formula (72) on p. 556 implies

$$\begin{aligned} dY &= Y\,dX + (1/2)\,Y\,(dX)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\,(\mu\,dt + \sigma\,dW)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\sigma^2\,dt. \end{aligned}$$

- Hence

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right)dt + \sigma\,dW. \tag{74}$$

- The annualized *instantaneous* rate of return is $\mu + \sigma^2/2$ (not μ).[a]

[a] Consistent with Lemma 10 (p. 282).

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned} dY/Y &= a\,dt + b\,dW_Y, \\ dZ/Z &= f\,dt + g\,dW_Z. \end{aligned}$$

- Assume dW_Y and dW_Z have correlation ρ.
- Consider the Ito process $U \equiv YZ$.

Product of Geometric Brownian Motion Processes (continued)

- Apply Ito's lemma (Theorem 21 on p. 562):

$$
\begin{aligned}
dU &= Z\,dY + Y\,dZ + dY\,dZ \\
&= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\
&\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\
&= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z.
\end{aligned}
$$

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.

Product of Geometric Brownian Motion Processes (continued)

- Note that

$$
\begin{aligned}
Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\
Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\
U &= \exp\left[\,\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\,\right].
\end{aligned}
$$

  - There is no $bg\rho$ term in U!

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 566.
- Let $U \equiv Y/Z$.
- We now show that[a]

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{75}$$

- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a]Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 21 on p. 562) can be employed to show that

$$
\begin{aligned}
& dU \\
= {} & (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= {} & (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= {} & U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= {} & U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z .
\end{aligned}
$$

Forward Price

- Suppose S follows

$$\frac{dS}{S} = \mu \, dt + \sigma \, dW.$$

- Consider $F(S,t) \equiv S e^{y(T-t)}$.
- Observe that

$$\begin{aligned} \frac{\partial F}{\partial S} &= e^{y(T-t)}, \\ \frac{\partial^2 F}{\partial S^2} &= 0, \\ \frac{\partial F}{\partial t} &= -yS e^{y(T-t)}. \end{aligned}$$

Forward Prices (concluded)

- Then

$$\begin{aligned} dF &= e^{y(T-t)}\, dS - ySe^{y(T-t)}\, dt \\ &= Se^{y(T-t)}\, (\mu\, dt + \sigma\, dW) - ySe^{y(T-t)}\, dt \\ &= F(\mu - y)\, dt + F\sigma\, dW. \end{aligned}$$

  - One can also prove it by Eq. (73) on p. 561.

- Thus F follows

$$\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.$$

- This result has applications in forward and futures contracts.[a]

[a] It is also consistent with p. 515.

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- It is known that

$$\begin{aligned} E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\ \mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\ \mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)} \left[\, 1 - e^{-2\kappa(s-t_0)}\,\right] \\ &\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,], \end{aligned}$$

for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- A generalized version:

$$dX = \kappa(\mu - X)\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\, X(t)\,] &= \mu + (x_0 - \mu)\, e^{-\kappa(t-t_0)}, \qquad (76) \\ \mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa} \left[1 - e^{-2\kappa(t-t_0)} \right], \end{aligned}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Square-Root Process

- Suppose X is an Ornstein-Uhlenbeck process.
- Ito's lemma says $V \equiv X^2$ has the differential,

$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right) dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

$$dX = \kappa(\mu - X)\, dt + \sigma\sqrt{X}\, dW,$$

where $\kappa, \sigma \geq 0$ and $X(0)$ is a nonnegative constant.

- Like the Ornstein-Uhlenbeck process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rates.[a]
- The Ornstein-Uhlenbeck process, in contrast, allows negative interest rates.[b]
- The two processes are related (see p. 578).

---

[a] Cox, Ingersoll, and Ross (1985).

[b] But some rates have gone negative in Europe in 2015!

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \equiv (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a] William Feller (1906–1970) in 1951.

Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- The continuously compounded rate of return $X \equiv \ln S$ follows

$$dX = (\mu - \sigma^2/2)\, dt + \sigma\, dW$$

by Ito's lemma.[a]

---

[a]See also Eq. (74) on p. 565. Consistent with Lemma 10 (p. 282).

Local-Volatility Models

- The more general deterministic volatility model posits

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

where instantaneous volatility $\sigma(S, t)$ is called the local volatility function.[a]

- A (weak) solution exists if $S\sigma(S, t)$ is continuous and grows at most linearly in S and t.[b]

[a] Derman and Kani (1994); Dupire (1994).
[b] Skorokhod (1961).

Local-Volatility Models (continued)

- Theoretically,[a]

$$\sigma(X,T)^2 = 2\,\frac{\frac{\partial C}{\partial T} + (r_T - q_T)\,X\,\frac{\partial C}{\partial X} + q_T C}{X^2\,\frac{\partial^2 C}{\partial X^2}}. \tag{77}$$

- C is the call price at time $t = 0$ (today) with strike price X and time to maturity T.
- $\sigma(X,T)$ is the local volatility that will prevail at *future time* T and *stock price* $S_T = X$.

[a]Dupire (1994); Andersen and Brotherton-Ratcliffe (1998).

Local-Volatility Models (continued)

- For more general models, this equation gives the expectation as seen from today, under the risk-neural probability, of the instantaneous variance at time T given that $S_T = X$.[a]
- In practice, $\sigma(S,t)^2$ may have spikes, vary wildly, or even be negative.
- The term $\partial^2 C/\partial X^2$ in the denominator often results in numerical instability.
- Now, denote the implied volatility surface by $\Sigma(X,T)$ and the local volatility surface by $\sigma(S,t)$.

[a]Derman and Kani (1997).

Local-Volatility Models (continued)

- The relation between $\Sigma(X,T)$ and $\sigma(X,T)$ is[a]

$$
\begin{aligned}
\sigma(X,T)^2 &= \frac{\Sigma^2 + 2\Sigma\tau\left[\frac{\partial\Sigma}{\partial T} + (r_T - q_T)X\frac{\partial\Sigma}{\partial X}\right]}{\left(1 - \frac{Xy}{\Sigma}\frac{\partial\Sigma}{\partial X}\right)^2 + X\Sigma\tau\left[\frac{\partial\Sigma}{\partial X} - \frac{X\Sigma\tau}{4}\left(\frac{\partial\Sigma}{\partial X}\right)^2 + X\frac{\partial^2\Sigma}{\partial X^2}\right]}, \\
\tau &\equiv T - t, \\
y &\equiv \ln(X/S_t) + \int_t^T (q_s - r_s)\, ds.
\end{aligned}
$$

- Although this version may be more stable than Eq. (77) on p. 584, it is expected to suffer from similar problems.

[a]Andreasen (1996); Andersen and Brotherton-Ratcliffe (1998); Gatheral (2003); Wilmott (2006); Kamp (2009).

Local-Volatility Models (continued)

- Small changes to the implied volatility surface may produce big changes to the local volatility surface.
- In reality, option prices only exist for a finite set of maturities and strike prices.
- Hence interpolation and extrapolation may be needed to construct the volatility surface.[a]
- But some implied volatility surfaces generate option prices that allow arbitrage profits.

[a] Doing it to the option prices produces worse results (Li, 2000/2001).

Local-Volatility Models (continued)

- For example, consider the following implied volatility surface:[a]

$$\Sigma(X, T)^2 = a_{\mathrm{ATM}}(T) + b(X - S_0)^2, \quad b > 0.$$

- It generates higher prices for out-of-the-money options than in-the-money options for T large enough.[b]

---

[a] ATM: at-the-money.

[b] Rebonato (2004).

Local-Volatility Models (continued)

- Let $x \equiv \ln(X/S_0) - rT$.
- For X large enough,[a]

$$\Sigma(X,T)^2 < 2\,\frac{|\,x\,|}{T}.$$

- For X small enough,[b]

$$\Sigma(X,T)^2 < \beta\,\frac{|\,x\,|}{T} \quad \text{for any } \beta > 2.$$

---

[a]Lee (2004).
[b]Lee (2004).

Local-Volatility Models (concluded)

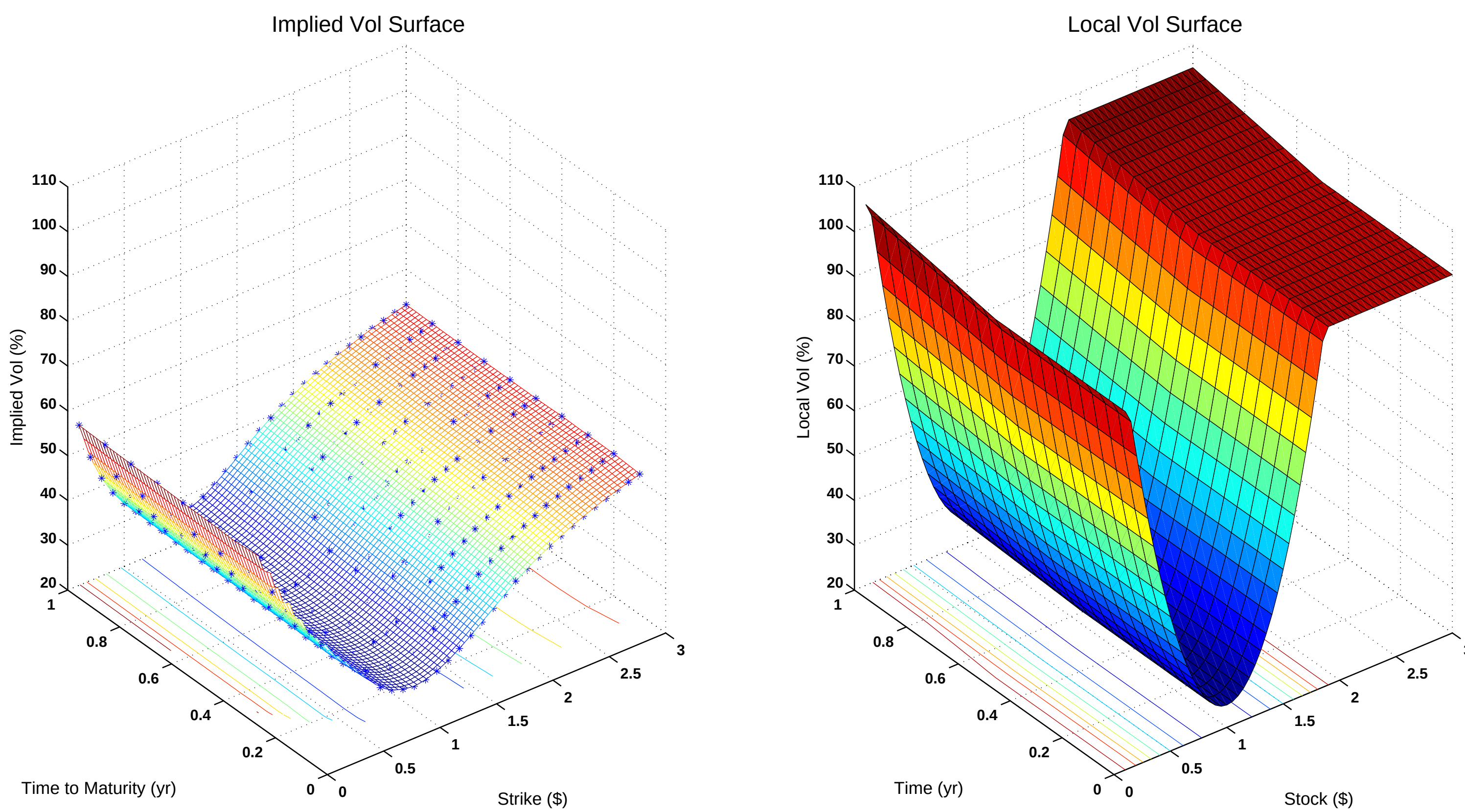
- There exist conditions for a set of option prices to be arbitrage-free.[a]
- For some vanilla equity options, the Black-Scholes model "seems" better than the local-volatility model.[b]

---

[a] Davis and Hobson (2007).

[b] Dumas, Fleming, and Whaley (1998).

Implied and Local Volatility Surfaces[a]



[a]Contributed by Mr. Lok, U Hou (D99922028) on April 5, 2014.

Implied Trees

- The trees for the local volatility model are called implied trees.[a]
- Their construction requires option prices at all strike prices and maturities.
  - That is, an implied volatility surface.
- The local volatility model does *not* require that the implied tree combine.

[a] Derman and Kani (1994); Dupire (1994); Rubinstein (1994).

Implied Trees (continued)

- How to construct a valid implied tree with efficiency has been open for a long time.[a]
  - Reasons may include: noise and nonsynchrony in data, arbitrage opportunities in the smoothed and interpolated/extrapolated implied volatility surface, wrong model, wrong algorithms, etc.
- Numerically, inversion is an ill-posed problem.[b]

---

[a]Rubinstein (1994); Derman and Kani (1994); Derman, Kani, and Chriss (1996); Jackwerth and Rubinstein (1996); Jackwerth (1997); Coleman, Kim, Li, and Verma (2000); Li (2000/2001); Moriggia, Muzzioli, and Torricelli (2009).

[b]Ayache, Henrotte, Nassar, and Wang (2004).

Implied Trees (concluded)

- It is solved for separable local volatilities σ.[a]
  - The local-volatility function $\sigma(S, V)$ is separable[b] if
$$\sigma(S,t) = \sigma_1(S)\,\sigma_2(t).$$
- A complete solution is close.[c]

[a]Lok (D99922028) and Lyuu (2015, 2016).
[b]Rebonato (2004); Brace, Gątarek, and Musiela (1997).
[c]Lok (D99922028) and Lyuu (2016).

The Hull-White Model

- Hull and White (1987) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sqrt{V}\,dW_1, \\ dV &= \mu_{\mathrm{v}} V\,dt + bV\,dW_2. \end{aligned}$$

- Above, V is the instantaneous variance.
- They assume μ_{v} depends on V and t (but not S).

The SABR Model

- Hagan, Kumar, Lesniewski, and Woodward (2002) postulate the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + S^{\theta} V\, dW_1, \\ dV &= bV\, dW_2, \end{aligned}$$

  for $0 \leq \theta \leq 1$.

- A nice feature of this model is that the implied volatility surface has a compact approximate closed form.

The Hilliard-Schwartz Model

- Hilliard and Schwartz (1996) postulate the following general model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + f(S)V^a\,dW_1, \\ dV &= \mu(V)\,dt + bV\,dW_2, \end{aligned}$$

for some well-behaved function $f(S)$ and constant a.

The Blacher Model

- Blacher (2002) postulates the following model,

$$\begin{aligned} \frac{dS}{S} &= r\,dt + \sigma \left[\, 1 + \alpha(S - S_0) + \beta(S - S_0)^2 \,\right] dW_1, \\ d\sigma &= \kappa(\theta - \sigma)\,dt + \epsilon\sigma\,dW_2. \end{aligned}$$

- So the volatility σ follows a mean-reverting process to level θ.

Heston's Stochastic-Volatility Model

- Heston (1993) assumes the stock price follows

$$\frac{dS}{S} = (\mu - q)\,dt + \sqrt{V}\,dW_1, \tag{78}$$

$$dV = \kappa(\theta - V)\,dt + \sigma\sqrt{V}\,dW_2. \tag{79}$$

  – V is the instantaneous variance, which follows a square-root process.

  – dW_1 and dW_2 have correlation ρ.

  – The riskless rate r is constant.

- It may be the most popular continuous-time stochastic-volatility model.[a]

[a] Christoffersen, Heston, and Jacobs (2009).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2\sqrt{V}$.
- So $\mu = r + b_2 V$.
- Define

$$\begin{aligned} dW_1^* &= dW_1 + b_2\sqrt{V}\,dt, \\ dW_2^* &= dW_2 + \rho b_2\sqrt{V}\,dt, \\ \kappa^* &= \kappa + \rho b_2 \sigma, \\ \theta^* &= \frac{\theta\kappa}{\kappa + \rho b_2 \sigma}. \end{aligned}$$

- dW_1^* and dW_2^* have correlation ρ.

Heston's Stochastic-Volatility Model (continued)

- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.
- Heston's model becomes, under probability measure Q,

$$\begin{aligned} \frac{dS}{S} &= (r-q)\,dt + \sqrt{V}\,dW_1^*, \\ dV &= \kappa^*(\theta^* - V)\,dt + \sigma\sqrt{V}\,dW_2^*. \end{aligned}$$

Heston's Stochastic-Volatility Model (continued)

- Define

$$
\begin{aligned}
\phi(u,\tau) &= \exp\left\{ \imath u(\ln S + (r-q)\,\tau) \right. \\
&\quad + \theta^*\kappa^*\sigma^{-2}\left[(\kappa^* - \rho\sigma u\imath - d)\,\tau - 2\ln\frac{1-ge^{-d\tau}}{1-g} \right] \\
&\quad \left. + \frac{v\sigma^{-2}(\kappa^* - \rho\sigma u\imath - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}} \right\}, \\
d &= \sqrt{(\rho\sigma u\imath - \kappa^*)^2 - \sigma^2(-\imath u - u^2)}, \\
g &= (\kappa^* - \rho\sigma u\imath - d)/(\kappa^* - \rho\sigma u\imath + d).
\end{aligned}
$$

Heston's Stochastic-Volatility Model (continued)

The formulas are[a]

$$
\begin{aligned}
C &= S\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right] \\
&\quad -Xe^{-r\tau}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\
P &= Xe^{-r\tau}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)du\right], \\
&\quad -S\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)du\right],
\end{aligned}
$$

where $\imath = \sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

[a]Contributed by Mr. Chen, Chun-Ying (D95723006) on August 17, 2008 and Mr. Liou, Yan-Fu (R92723060) on August 26, 2008.

Heston's Stochastic-Volatility Model (concluded)

- For American options, we will need a tree for Heston's model.[a]
- They are all $O(n^3)$-sized.

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[a]Leisen (2010); Beliaeva and Nawalka (2010); Chou (`R02723073`) (2015).

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?
- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.
- Merton (1976) proposed jump models.
- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.

[a]Eraker (2004).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.
- This assumption also has no impacts on option prices.
- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (78) on p. 599.
  - Closed-form formulas exist for GARCH-jump option pricing models.[a]

[a] Liou (R92723060) (2005).

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]
- Jumps *in volatility* are alternatives.[b]
  - E.g., add correlated jump processes to Eqs. (78) and Eq. (79) on p. 599.
- Such models allow high level of volatility caused by a jump to volatility.[c]

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[a] Bates (2000) and Pan (2002).

[b] Duffie, Pan, and Singleton (2000).

[c] Eraker, Johnnes, and Polson (2000); Lin (2007); Zhu and Lian (2012).

Complexities of Stochastic-Volatility Models

- A few stochastic-volatility models suffer from subexponential ($c^{\sqrt{n}}$) tree size.
- Examples include the Hull-White (1987), Hilliard-Schwartz (1996), and SABR (2002) models.[a]
- Future research may extend this negative result to more stochastic-volatility models.
  - We suspect many GARCH option pricing models entertain similar problems.[b]

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[a]Chiu (R98723059) (2012).

[b]Chen (R95723051) (2008); Chen (R95723051), Lyuu, and Wen (D94922003) (2011).

Complexities of Stochastic-Volatility Models (concluded)

- Calibration can be computationally hard.
  - Few have tried it on exotic options.[a]
- There are usually several local minima for the calibration error.[b]
  - They will give different prices to options not used in the calibration.
  - But which one captures the smile dynamics?

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[a] Ayache, Henrotte, Nassar, and Wang (2004).
[b] Ayache (2004).

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation (PDE).
- The key step is recognizing that the same random process drives both securities.
  - Their prices are perfectly correlated.
- We then figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.
- PDEs allow many numerical methods to be applicable.

Assumptions[a]

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

[a] Derman and Taleb (2005) summarizes criticisms on these assumptions and the replication argument.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 558),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

[a]Mathematically speaking, it is not quite right (Bergman, 1982).

Black-Scholes Differential Equation (continued)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S\, \frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS\, \frac{\partial C}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC.$$

- This is a backward equation, which describes the dynamics of a derivative's price *forward* in physical time.

Black-Scholes Differential Equation (concluded)

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC. \tag{80}$$

- The local-volatility model (77) on p. 584 is simply the dual of this equation:[a]

$$\frac{\partial C}{\partial T} + (r_T - q_T) X \frac{\partial C}{\partial X} - \frac{1}{2}\, \sigma(X, T)^2 X^2 \frac{\partial^2 C}{\partial X^2} = -q_T C.$$

- This is a forward equation, which describes the dynamics of a derivative's price *backward* in maturity time.

[a] Derman and Kani (1997).

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{81}$$

- Identity (81) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  – A definite relation thus exists between Γ and Θ.

Black-Scholes Differential Equation: An Alternative

- Perform the change of variable $V \equiv \ln S$.
- The option value becomes $U(V,t) \equiv C(e^V, t)$.
- Furthermore,

$$\begin{aligned} \frac{\partial C}{\partial t} &= \frac{\partial U}{\partial t}, \\ \frac{\partial C}{\partial S} &= \frac{1}{S}\,\frac{\partial U}{\partial V}, \\ \frac{\partial^2 C}{\partial^2 S} &= \frac{1}{S^2}\,\frac{\partial^2 U}{\partial V^2} - \frac{1}{S^2}\,\frac{\partial U}{\partial V}. \end{aligned} \tag{82}$$

- Equation (82) is an alternative way to calculate gamma.[a]

[a]See also Eq. (43) on p. 341.

Black-Scholes Differential Equation: An Alternative (concluded)

- The Black-Scholes differential equation (80) on p. 617 becomes

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 U}{\partial V^2}+\left(r-q-\frac{\sigma^2}{2}\right)\frac{\partial U}{\partial V}-rU+\frac{\partial U}{\partial t}=0$$

subject to $U(V,T)$ being the payoff such as $\max(X-e^V,0)$.

[Black] got the equation [in 1969] but then
was unable to solve it. Had he been a better
physicist he would have recognized it as a form
of the familiar heat exchange equation,
and applied the known solution. Had he been
a better mathematician, he could have
solved the equation from first principles.
Certainly Merton would have known exactly
what to do with the equation
had he ever seen it.
— Perry Mehrling (2005)

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T,S,A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T,S,A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a] Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 399ff.[a]
- But one-dimensional PDEs are available for Asian options.[b]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

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[a] See also Barraquand and Pudet (1996).

[b] Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.