Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,

$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$

- The computation time roughly doubles that for evaluating the derivative security itself.

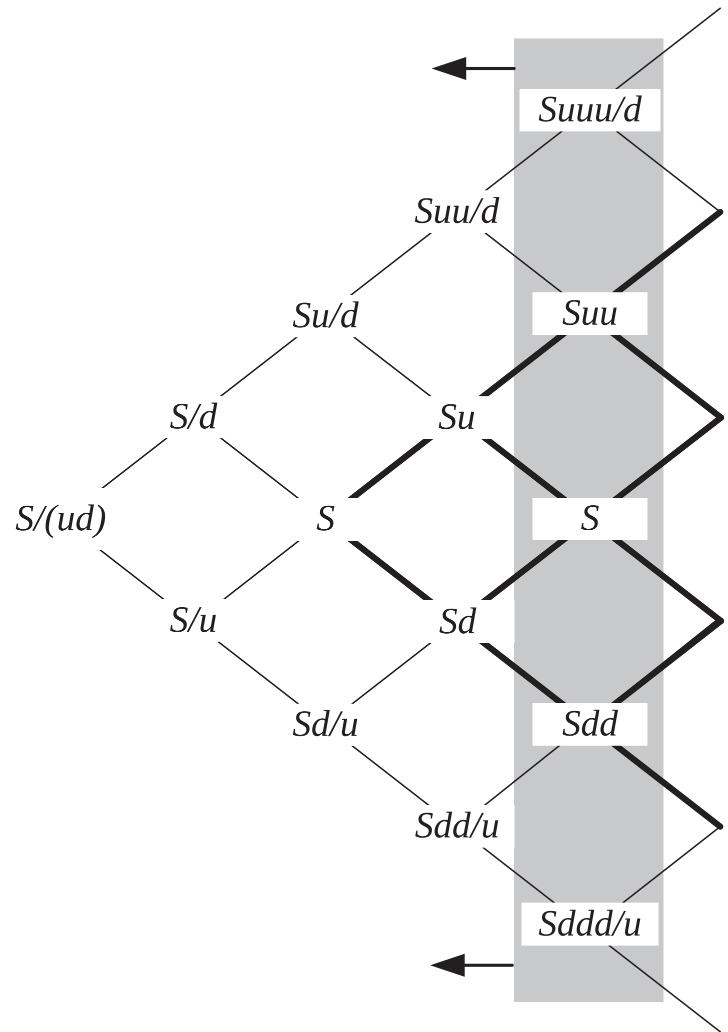
An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}.$$

- Almost zero extra computational effort.

[a] Pelsser and Vorst (1994).


Suuu/d
Suu/d
Su/d
Suu
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}.$$

- Alternative formulas exist (p. 601).

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.$$

- It does not work (see text for the reason).
- In general, calculating gamma is a hard problem numerically.
- But why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option can be derived from delta and gamma (p. 600).

- For vega and rho, there seems no alternative but to run the binomial tree algorithm twice.[a]

[a]But see pp. 959ff.

Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
  - The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a] Black and Scholes (1973); Merton (1974).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call[a] on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for a call on V.

[a]See p. 183.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus

$$\begin{aligned} nS &= C, \\ B &= V - C. \end{aligned}$$

- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 11 (p. 284) and the put-call parity,[a]

$$nS = VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{43}$$

$$B = VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \tag{44}$$

  – Above,

$$x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

[a] Merton (1974).

Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$
\begin{aligned}
&\frac{\ln(X/B)}{\tau} - r \\
= \quad & -\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}\,N(z - \sigma\sqrt{\tau})\right).
\end{aligned}
$$

  - $\omega \equiv Xe^{-r\tau}/V$.
  - $z \equiv (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate q and the bankruptcy costs are a constant proportion α of then remaining firm value.
- Then Eqs. (43)–(44) on p. 351 become, respectively,

$$
\begin{aligned}
nS &= Ve^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\
B &= (1-\alpha) Ve^{-q\tau} N(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}).
\end{aligned}
$$

  - Above,

$$
x \equiv \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
$$

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

$$B = 44,500 - 15,250 = 29,250$$

  dollars.

  – Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.[a]
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

---

[a]See p. 208.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for $1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility, dividend, and strike price are under partial control of the stockholders.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 358 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay

$$42,562.5 \times (15/45) = 14,187.5$$

dollars.

- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$

  - This is called claim dilution.[a]

[a] Fama and Miller (1972).

A Numerical Example (continued)

- Suppose the stockholders sell $(1/3) \times n$ Merck shares to fund a \$14,833.3 cash dividend.
- They now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30,000$.
- This is equivalent to owning 2/3 of a call on n Merck shares with a total strike price of \$45,000.
- n such calls are worth \$1,937.5 (p. 358).
- So the total market value of the XYZ.com stock is $(2/3) \times 1,937.5 = 1,291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

$$(2/3) \times n \times 44.5 - 1,291.67 = 28,375$$

  dollars.

- Hence the stockholders gain

$$14,833.3 + 1,291.67 - 15,250 \approx 875$$

  dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Further Topics

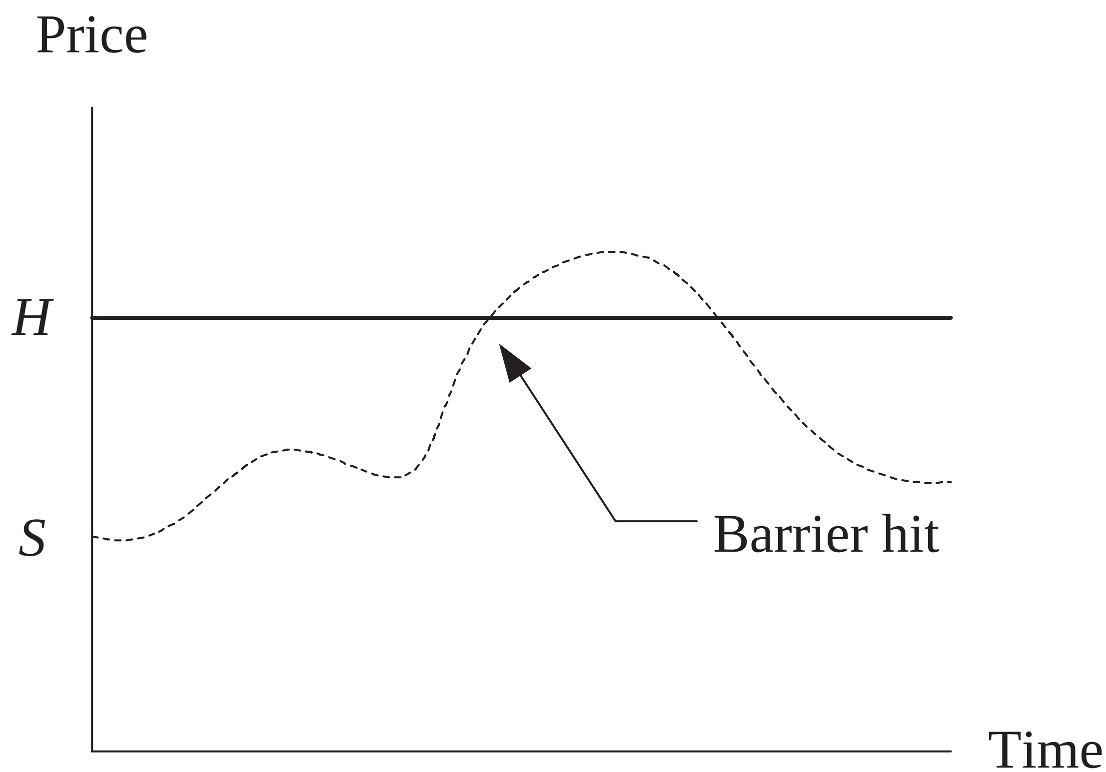
- Other Examples:
  - Subordinated debts as bull call spreads.
  - Warrants as calls.
  - Callable bonds as American calls with 2 strike prices.
  - Convertible bonds.
- Securities with a complex liability structure must be solved by trees.[a]

[a]Dai (B82506025, R86526008, D8852600), Lyuu, and Wang (F95922018) (2010).

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

[a]A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in HK as of April, 2006.


Price
H
S
Barrier hit
Time

Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all the possible barrier options mentioned above.[a]

[a] Haug (2006).

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{45}$$

  - $x \equiv \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \equiv (r - q + \sigma^2/2)/\sigma^2$.
- A European down-and-out call can be priced via the in-out parity (see text).

[a]Merton (1973).

A Formula for Up-and-In Puts[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- Again, a European up-and-out put can be priced via the in-out parity.

---

[a] Merton (1973).

Are American Options Barrier Options?[a]

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

---

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.

Interesting Observations

- Assume $H < X$.
- Replace S in the pricing formula Eq. (37) on p. 311 for the call with H^2/S.
- Equation (45) on p. 369 for the down-and-in call becomes Eq. (37) when $r - q = \sigma^2/2$.
- Equation (45) becomes S/H times Eq. (37) when $r - q = 0$.

Interesting Observations (concluded)

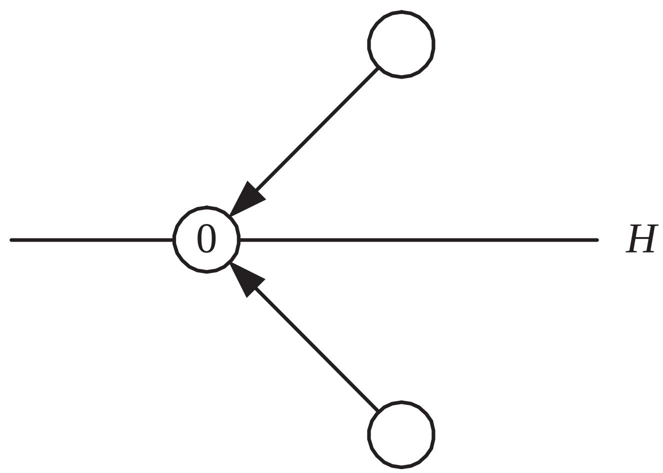
- Replace S in the pricing formula for the down-and-in call, Eq. (45), with H^2/S.
- Equation (45) becomes Eq. (37) when $r - q = \sigma^2/2$.
- Equation (45) becomes H/S times Eq. (37) when $r - q = 0$.[a]
- Why?

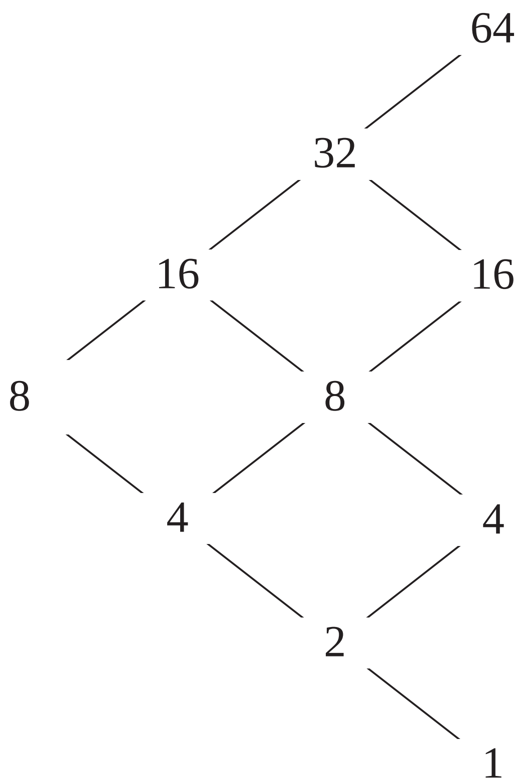
---

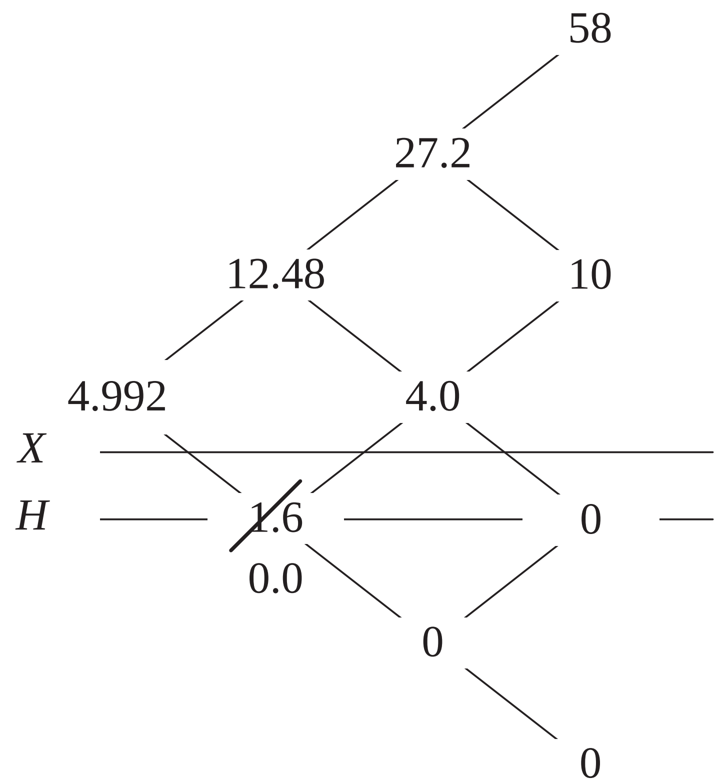
[a]Contributed by Mr. Chou, Ming-Hsin (`R02723073`) on April 24, 2014.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.







$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (continued)

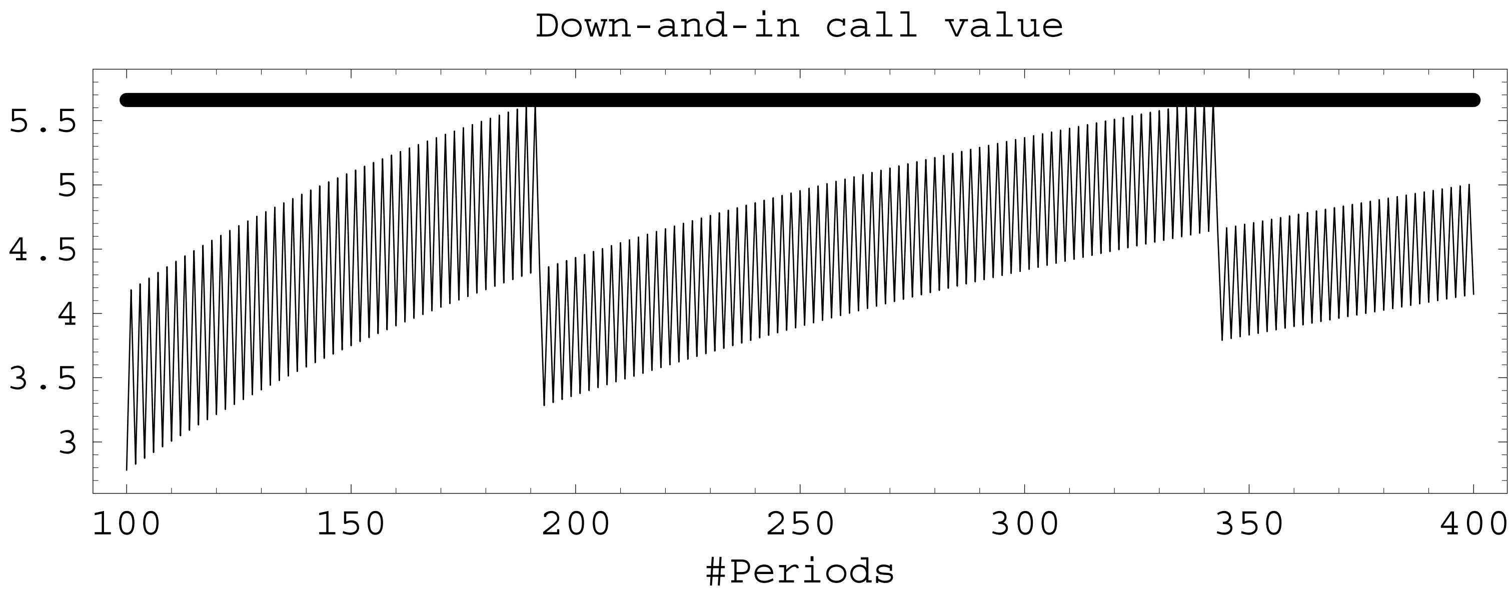
- But convergence is erratic because H is not at a price level on the tree (see plot on next page).[a]
  - The barrier H is moved to a node price.
  - This "effective barrier" changes as n increases.
- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.[b]
- Solutions will be presented later.

---

[a]Boyle and Lau (1994).

[b]Lin (`R95221010`) (2008).

Binomial Tree Algorithms (concluded)[a]



[a]Lyuu (1998).

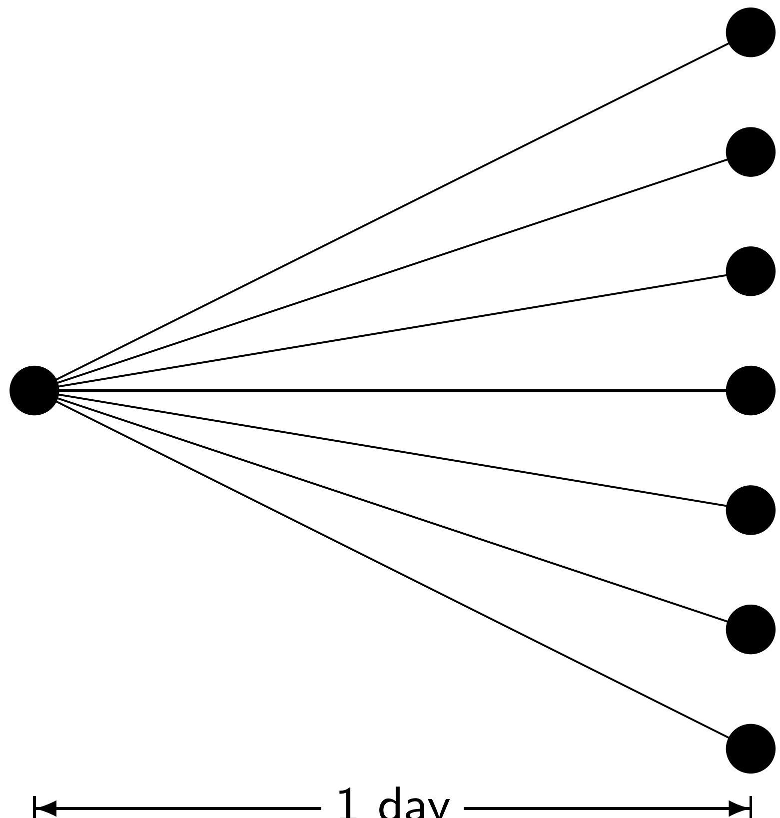
Daily Monitoring

- Almost all barrier options monitor the barrier only for daily closing prices.
- If so, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into d periods.
- Does this save time or space?[a]

---

[a] Contributed by Ms. Chen, Tzu-Chun (`R94922003`) and others on April 12, 2006.

A Heptanomial Tree (6 Periods Per Day)



Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
  - By that we mean the number of domestic currencies per unit of foreign currency.[a]
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.

---

[a]The market convention is the opposite: A/B = x means one unit of currency A (the reference currency) is equal to x units of currency B (the counter-value currency).

Foreign Currencies (concluded)

- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

Foreign Exchange Options

- In 2000 the total notional volume of foreign exchange options was US$13 trillion.[a]
  - 38.5% were vanilla calls and puts with a maturity less than one month.
  - 52.5% were vanilla calls and puts with a maturity between one and 18 months.
  - 4% were barrier options.
  - 1.5% were vanilla calls and puts with a maturity more than 18 months.
  - 1% were digital options (see p. 778).
  - 0.7% were Asian options (see p. 389).

[a]Lipton (2002).

Foreign Exchange Options (continued)

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases

$$\frac{100{,}000{,}000}{6{,}250{,}000} = 16$$

  puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for

$$100{,}000{,}000 \times .0088 = 880{,}000$$

  U.S. dollars.

Foreign Exchange Options (concluded)

- Assume the exchange rate S is lognormally distributed.
- The formulas derived for stock index options in Eqs. (37) on p. 311 apply with the dividend yield equal to $\hat{r}$:

$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{46}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x). \tag{46$'$}$$

  – Above,

$$x \equiv \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

Bar the roads!
Bar the paths!
Wert thou to flee from here, wert thou
to find all the roads of the world,
the way thou seekst
the path to that thou'dst find not[.]
— Richard Wagner (1813–1883), *Parsifal*

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends *critically* on the path.
- The (arithmetic) average-rate call has this terminal value:

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

- The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.[a]
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- The averaging clause is also common in convertible bonds and structured notes.

---

[a] As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen and Sandmann, 2003).

Path-Dependent Derivatives (continued)

- A lookback call option on the minimum has a terminal payoff of

$$S_n - \min_{0 \leq i \leq n} S_i.$$

- A lookback put on the maximum has a terminal payoff of

$$\max_{0 \leq i \leq n} S_i - S_n.$$

Path-Dependent Derivatives (concluded)

- The fixed-strike lookback option provides a payoff of
  - $\max(\max_{0 \leq i \leq n} S_i - X, 0)$ for the call.
  - $\max(X - \min_{0 \leq i \leq n} S_i, 0)$ for the put.
- Lookback calls and puts on the average (instead of a constant X) are called average-strike options.

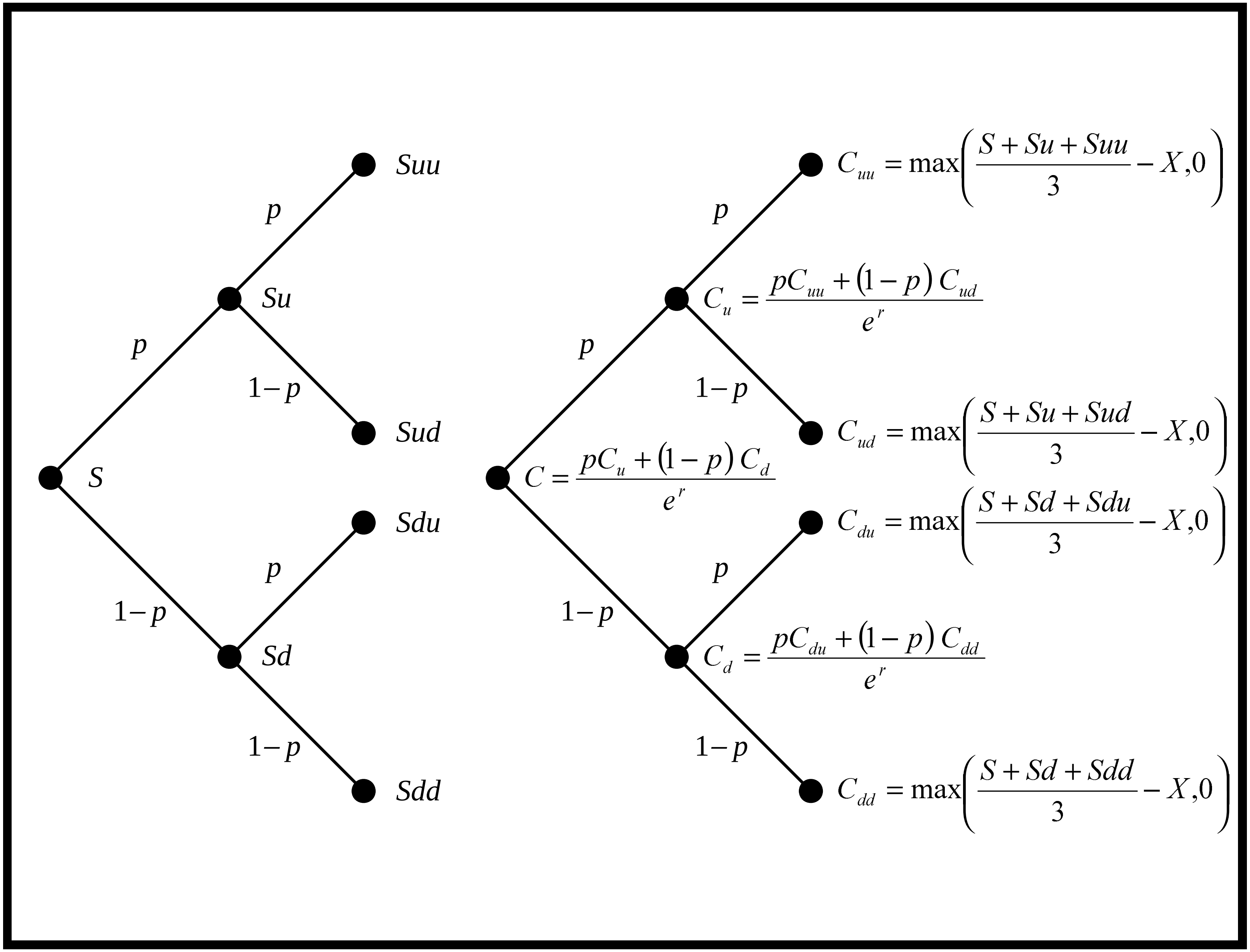
Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine (see next page).
- A naive algorithm enumerates the 2^n paths for an n-period binomial tree and then averages the payoffs.[a]
- But the complexity is exponential.
- The Monte Carlo method[b] and approximation algorithms are some of the alternatives left.

---

[a] Dai (B82506025, R86526008, D8852600) and Lyuu (2007) reduce it to $2^{O(\sqrt{n})}$.

[b] See pp. 765ff.


Suu
p
Su
p
$1-p$
Sud
S
Sdu
p
$1-p$
Sd
$1-p$
Sdd
$C_{uu} = \max\left(\frac{S + Su + Suu}{3} - X, 0\right)$
p
$C_u = \frac{pC_{uu} + (1-p)\,C_{ud}}{e^r}$
p
$1-p$
$C_{ud} = \max\left(\frac{S + Su + Sud}{3} - X, 0\right)$
$C = \frac{pC_u + (1-p)\,C_d}{e^r}$
$C_{du} = \max\left(\frac{S + Sd + Sdu}{3} - X, 0\right)$
p
$1-p$
$C_d = \frac{pC_{du} + (1-p)\,C_{dd}}{e^r}$
$1-p$
$C_{dd} = \max\left(\frac{S + Sd + Sdd}{3} - X, 0\right)$

States and Their Transitions

- The tuple

$$(i, S, P)$$

captures the state[a] for the Asian option.

  - i: the time.
  - S: the prevailing stock price.
  - P: the running sum.[b]

[a] A "sufficient statistic," if you will.

[b] When the average is a moving average, a different technique is needed (Kao (R89723057) and Lyuu, 2003).

States and Their Transitions (concluded)

- For the binomial model, the state transition is:

$$
\begin{array}{ccl}
 & & (i+1, Su, P+Su), \quad \text{for the up move} \\
 & \nearrow & \\
(i, S, P) & & \\
 & \searrow & \\
 & & (i+1, Sd, P+Sd), \quad \text{for the down move}
\end{array}
$$

- This leads to an exponential-time algorithm.

Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
  - Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$
$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas:

$$C = Se^{-q_{\rm a}\tau} N(x) - Xe^{-r\tau} N(x - \sigma_{\rm a}\sqrt{\tau}), \tag{47}$$

$$P = Xe^{-r\tau} N(-x + \sigma_{\rm a}\sqrt{\tau}) - Se^{-q_{\rm a}\tau} N(-x), \tag{47'}$$

  - With the volatility set to $\sigma_{\rm a} \equiv \sigma/\sqrt{3}$.
  - With the dividend yield set to $q_{\rm a} \equiv (r + q + \sigma^2/6)/2$.
  - $x \equiv \frac{\ln(S/X) + \left(r - q_{\rm a} + \sigma_{\rm a}^2/2\right)\tau}{\sigma_{\rm a}\sqrt{\tau}}$.

An Approximate Formula for Asian Calls[a]

$$C = e^{-r\tau}\left[\frac{S}{\tau}\int_0^{\tau} e^{\mu t+\sigma^2 t/2} N\left(\frac{-\gamma+(\sigma t/\tau)(\tau-t/2)}{\sqrt{\tau/3}}\right) dt \right.$$
$$\left. -XN\left(\frac{-\gamma}{\sqrt{\tau/3}}\right)\right],$$

where

- $\mu \equiv r-\sigma^2/2$.
- γ is the unique value that satisfies

$$\frac{S}{\tau}\int_0^{\tau} e^{3\gamma\sigma t(\tau-t/2)/\tau^2+\mu t+\sigma^2[\,t-(3t^2/\tau^3)(\tau-t/2)^2\,]/2}\, dt = X.$$

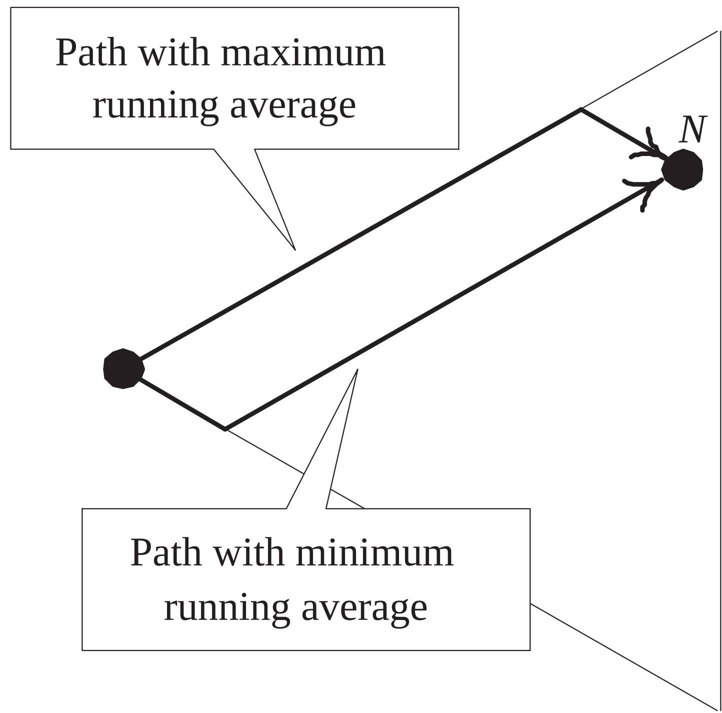
[a]Rogers and Shi (1995); Thompson (1999); Chen (`R92723061`) (2005); Chen (`R92723061`) and Lyuu (2006).

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j, i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i}d + \cdots + u^{j-i}d^i}^{j})$$

$$= S_0 \, \frac{1 - u^{j-i+1}}{1-u} + S_0 u^{j-i} d \, \frac{1-d^i}{1-d}.$$


Path with maximum running average
N
Path with minimum running average

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j+1$ and call it $A_{\max}(j,i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1+\overbrace{d+d^2+\cdots+d^i+d^i u+\cdots+d^i u^{j-i}}^{j})$$
$$= S_0\,\frac{1-d^{i+1}}{1-d}+S_0 d^i u\,\frac{1-u^{j-i}}{1-u}.$$

- Divide this value by $j+1$ and call it $A_{\min}(j,i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.

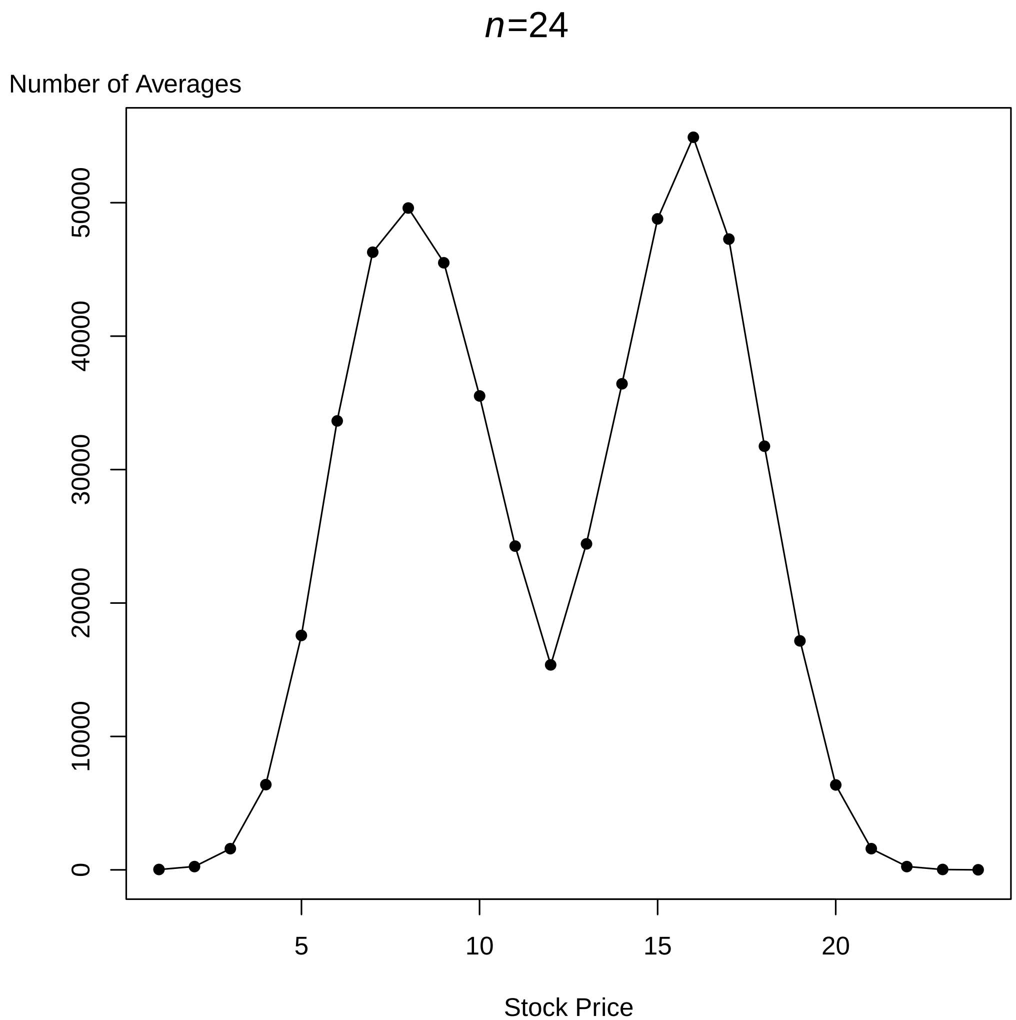
Approximation Algorithm for Asian Options (continued)

- The number of paths to $N(j,i)$ are far too many: $\binom{j}{i}$.
  - For example,

$$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)}\,.$$

- The number of distinct running averages for the nodes at any given time step n seems to be bimodal for n big enough.[a]
  - In the plot on the next page, $u = 5/4$ and $d = 4/5$.

[a]Contributed by Mr. Liu, Jun (`R99944027`) on April 15, 2014.



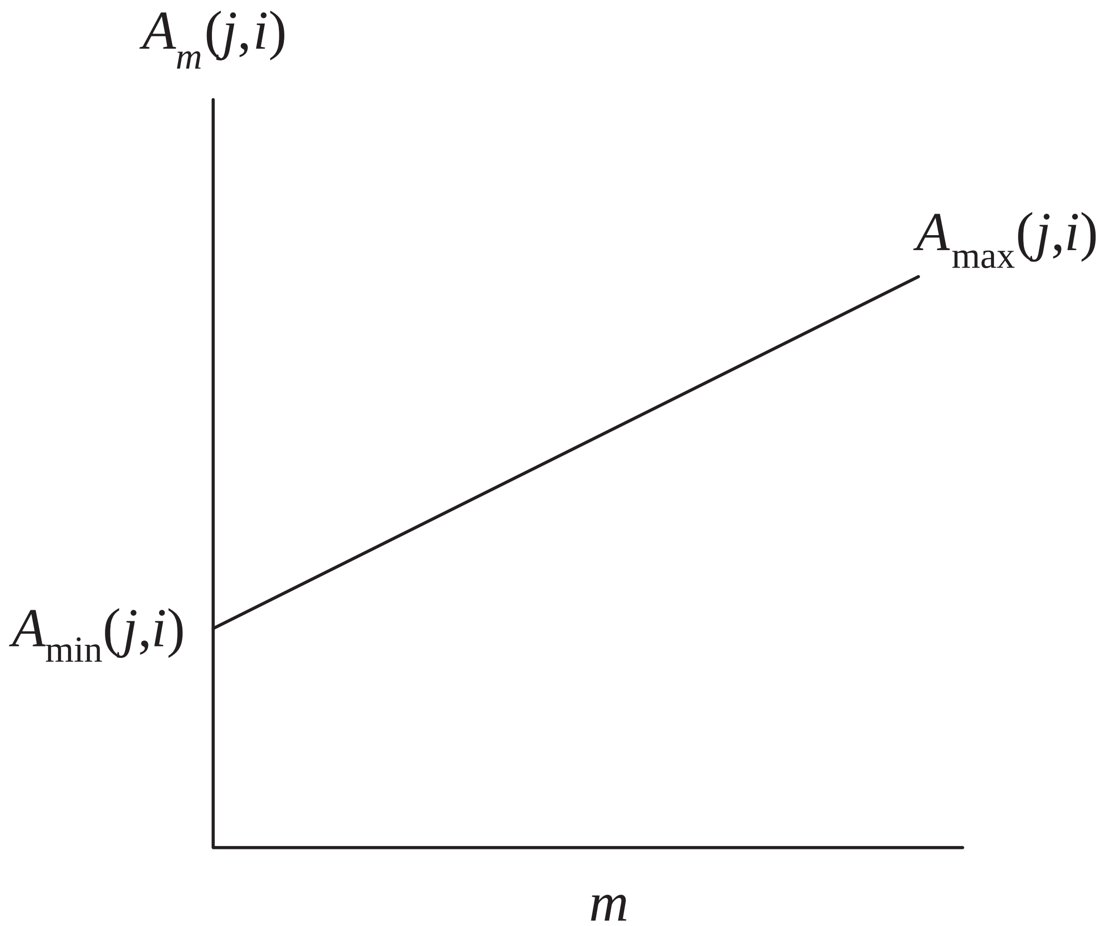
n=24
Number of Averages
0
10000
20000
30000
40000
50000
5
10
15
20
Stock Price

Approximation Algorithm for Asian Options (continued)

- But all averages must lie between $A_{\min}(j, i)$ and $A_{\max}(j, i)$.
- Pick $k + 1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j, i) \equiv \left(\frac{k-m}{k}\right) A_{\min}(j, i) + \left(\frac{m}{k}\right) A_{\max}(j, i)$$

for $m = 0, 1, \ldots, k$.



$A_m(j,i)$
$A_{\max}(j,i)$
$A_{\min}(j,i)$
m

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" introduces errors, but it works reasonably well in practice.[a]
- A better alternative picks values whose logarithms are equally spaced.[b]
- Still other alternatives are possible (considering the distribution of averages on p. 403).
- Generally, k must scale with at least n to show convergence behavior.[c]

---

[a] Hull and White (1993).

[b] Called log-linear interpolation.

[c] Dai (B82506025, R86526008, D8852600), Huang (F83506075), and Lyuu (2002).