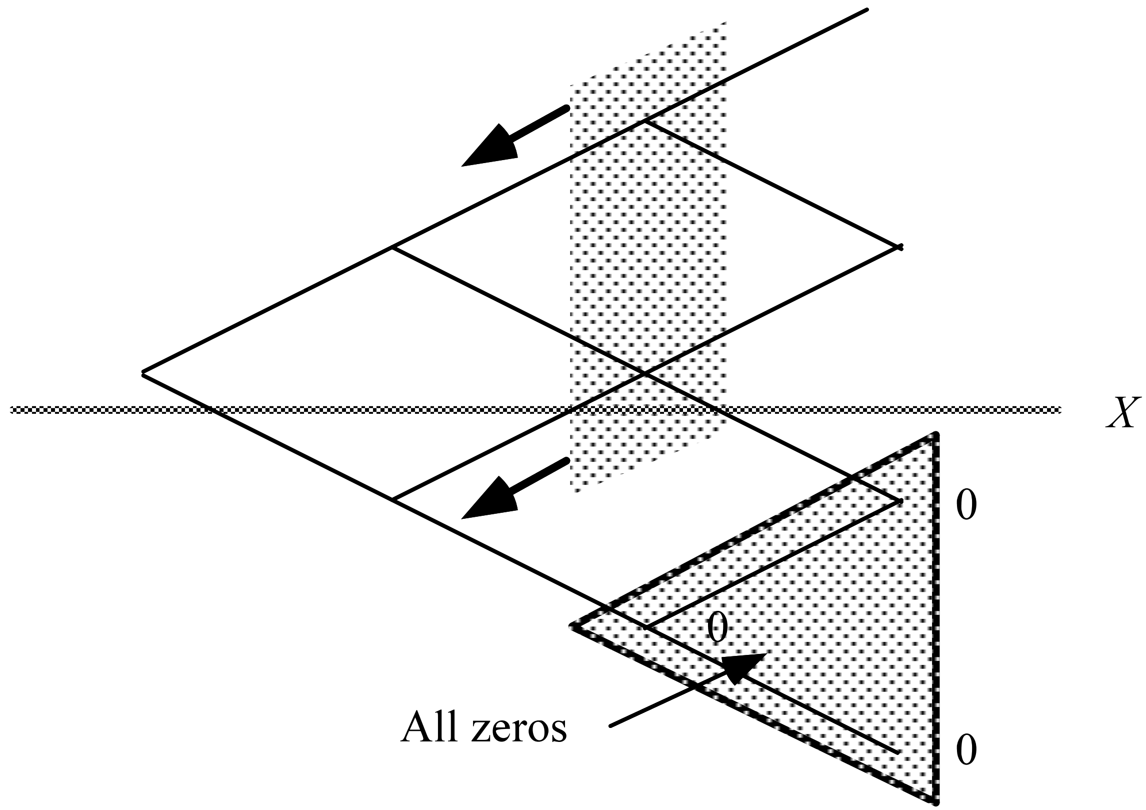
Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j;n,p) = \frac{p(n-j+1)}{(1-p)\,j}\,b(j-1;n,p).$$

Optimal Algorithm (continued)

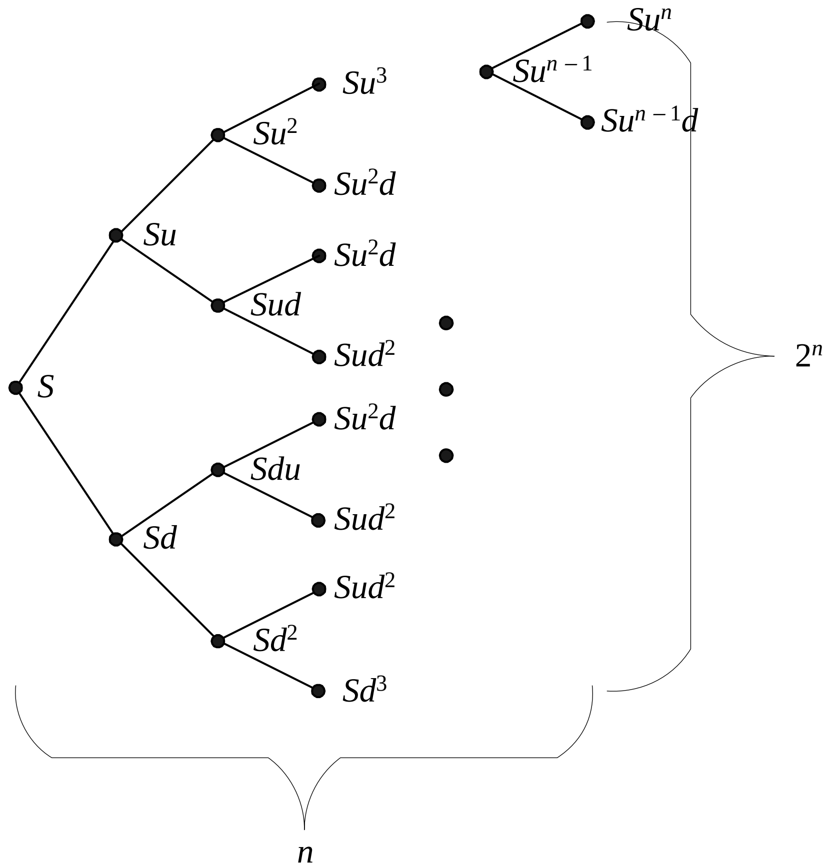
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
    $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (32) on p. 255 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.[a]
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

[a] Continuous-time trading may create arbitrage opportunities (Budish, Cramton, and Shim, 2015)!

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Let
$$\widehat{\mu} \equiv \frac{1}{n}\, E\left[\, \ln \frac{S_\tau}{S} \,\right]$$
denote the expected value of the continuously compounded rate of return per period.

- Let
$$\widehat{\sigma}^2 \equiv \frac{1}{n}\, \mathrm{Var}\left[\, \ln \frac{S_\tau}{S} \,\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$
\begin{aligned}
n\widehat{\mu} &= n[\,q\ln(u/d) + \ln d\,] \to \mu\tau, && (34)\\
n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau. && (35)
\end{aligned}
$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 267).
  - Other choices are possible (see text).

- Exact solutions for u, d, q are feasible if Eqs. (34)–(35) are replaced by equations: 3 equations for 3 variables.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \tag{36}$$

- With Eqs. (36), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The choices (36) result in the CRR binomial model.[a]
- A more common choice for the probability is actually

$$q = \frac{R-d}{u-d}.$$

by Eq. (29) on p. 236.

- Their numerical properties are essentially identical.

[a]Cox, Ross, and Rubinstein (1979).

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (36) on p. 278 or Eq. (29) on p. 236.
  - If this happens, the probabilities lie outside $[\,0, 1\,]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma \sqrt{\tau/n}} > e^{r\tau/n},$$

i.e., when $n > r^2 \tau / \sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions will be presented later.

[a]Many papers and programs forget to check this condition!

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

[a] The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.

Toward the Black-Scholes Formula (continued)

Lemma 10 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r - \sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \equiv (e^{r\tau/n} - d)/(u - d).$$

- Let $n \to \infty$.[a]

[a] See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return[b] is thus the riskless rate r.

---

[a]By Lemma 10 (p. 282) and Eq. (25) on p. 162.

[b]In the sense of $(1/\tau) \ln E[\, S_\tau/S \,]$ (arithmetic average rate of return) not $(1/\tau)E[\, \ln(S_\tau/S) \,]$ (geometric average rate of return). In the latter case, it would be $r - \sigma^2/2$ by Lemma 10.

Toward the Black-Scholes Formula (concluded)[a]

Theorem 11 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

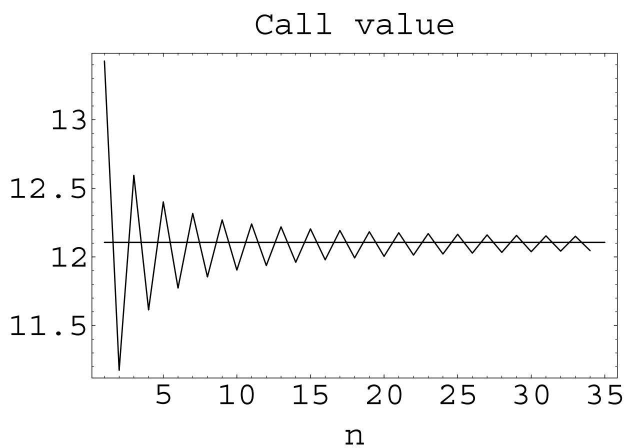
(See Eq. (33) on p. 255 for the meaning of x.)

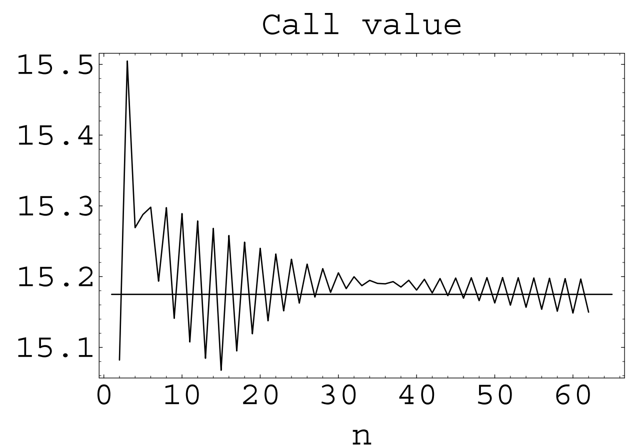
[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\tau/n}}, \\
d &= e^{-\sigma\sqrt{\tau/n}}, \\
\hat{r} &= r\tau/n.
\end{aligned}
$$





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of u and d.[b]

---

[a]Chang and Palmer (2007).

[b]See Exercise 9.3.8 of the textbook.

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

[a] Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

[a] Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.
- So?

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the continuation value.
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Time-Dependent Instantaneous Volatility[a]

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of σ.
- In the limit, the variance of $\ln(S_\tau / S)$ is

$$\int_0^\tau \sigma^2(t)\, dt$$

rather than $\sigma^2 \tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\frac{\int_0^\tau \sigma^2(t)\, dt}{\tau}}\,.$$

[a]Merton (1973).

Time-Dependent Instantaneous Volatility (concluded)

- There is no guarantee that the implied volatility is constant.
- For the binomial model, u and d depend on time:

$$\begin{aligned} u &= e^{\sigma(t)\sqrt{\tau/n}}, \\ d &= e^{-\sigma(t)\sqrt{\tau/n}}. \end{aligned}$$

- How to make the binomial tree combine?[a]

[a] Amin (1991); Chen (R98922127) (2011).

Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.
- The riskless rate r in the Black-Scholes formula should be the spot rate with a time to maturity equal to τ.
- In other words,
$$r = \frac{\sum_{i=0}^{n-1} r_i}{\tau},$$
where r_i is the continuously compounded short rate measured in periods for period i.[a]
- Will the binomial tree fail to combine?

[a] That is, one-period forward rate.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a] Fama (1965); K. French (1980); K. French and Roll (1986).

[b] Recall p. 147 about dating issues.

Trading Days and Calendar Days (continued)

- Think of σ as measuring the *annualized* volatility of stock price *one year from now*.
- Suppose a year has m (say 253) trading days.
- We can replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

[a]D. French (1984).

Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?[a]

[a]Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\, u$, $(Su - D)\, d$, $(Sd - D)\, u$, $(Sd - D)\, d$.
  - The binomial tree no longer combines.

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & (Su - D)\,u \\
 & Su - D \arrow[ur] \arrow[dr] & \\
 & & (Su - D)\,d \\
S \arrow[uur] \arrow[ddr] & & \\
 & & (Sd - D)\,u \\
 & Sd - D \arrow[ur] \arrow[dr] & \\
 & & (Sd - D)\,d
\end{tikzcd}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

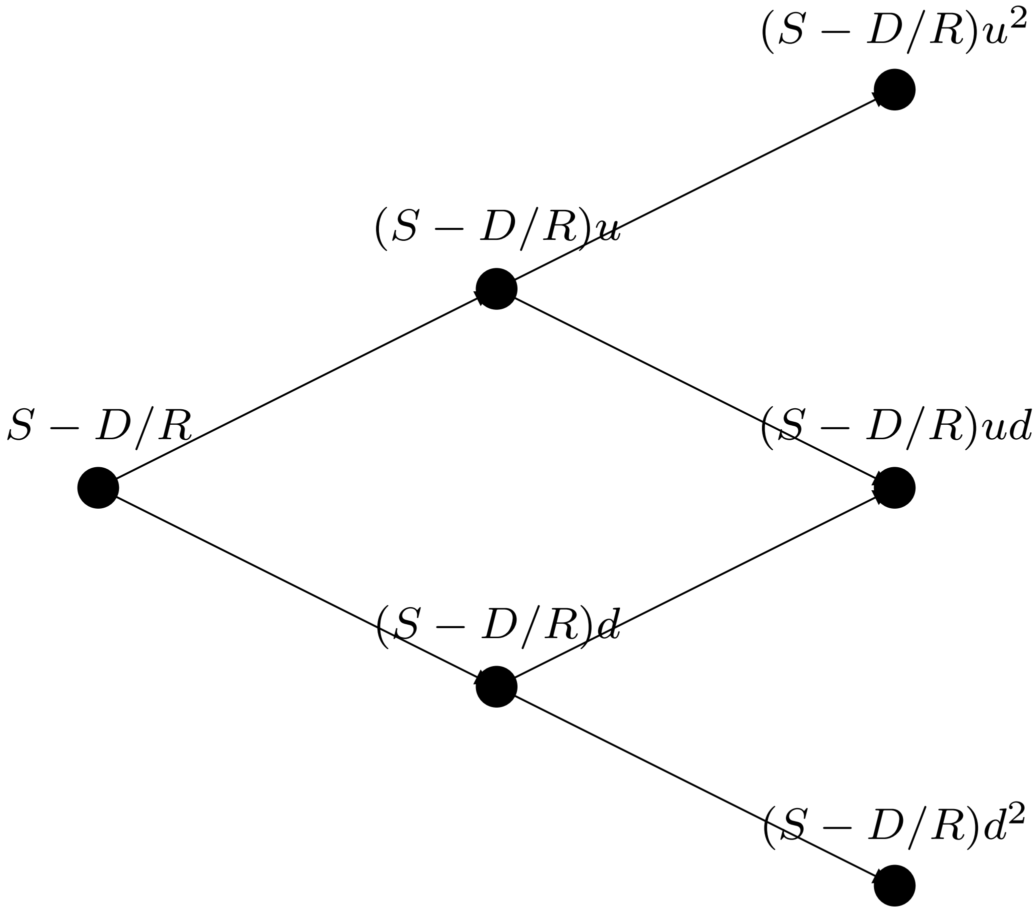
---

[a]Roll (1977).

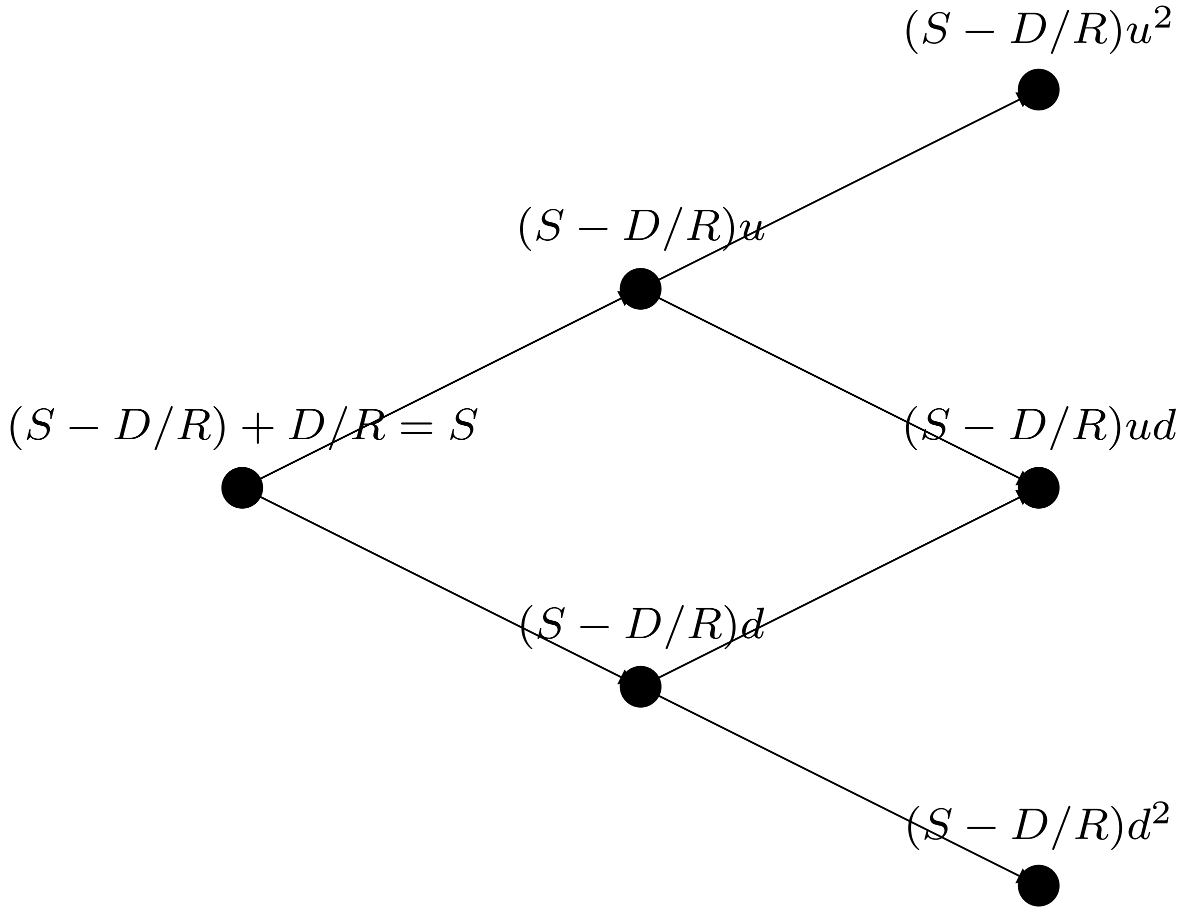
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 302 (Step 1)



The Ad-Hoc Approximation vs. P. 302 (Step 2)



The Ad-Hoc Approximation vs. P. 302[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\,u, (Su - D)\,d, (Sd - D)\,u, (Sd - D)\,d$ (p. 302).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned} (Su - D)\,u &> (S - D/R)u^2, \\ (Sd - D)\,d &< (S - D/R)d^2, \end{aligned}$$

[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 302 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually increased when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 707ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.[b]

---

[a] Dai (B82506025, R86526008, D8852600) and Lyuu (2004).

[b] Geske and Shastri (1985). It works well for American options but not European options (Dai (B82506025, R86526008, D8852600), 2009).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays *no* dividends.[a]

[a] In pricing European options, only the distribution of S_τ matters.

Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{37}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{37'}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (37) and (37′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \equiv \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

[a] Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{38}$$

where $\Delta t \equiv \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Other than the change in Eq. (38), binomial tree algorithms stay the same *as if there were no dividends*.

Risk Reversals[a]

- From formulas (37) and (37′) on p. 311, one can verify that $C = P$ when

$$X = Se^{(r-q)\tau}.$$

- A risk reversal consists of a short out-of-the-money put and a long out-of-the-money call with the same maturity.
- Furthermore, the portfolio has zero value.
- A short risk reversal position is also called a collar.[b]

---

[a] Neftci (2008).
[b] Bennett (2014).

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \equiv \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 284).
- Recall that

$$N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,$$

the density function of standard normal distribution.

Delta

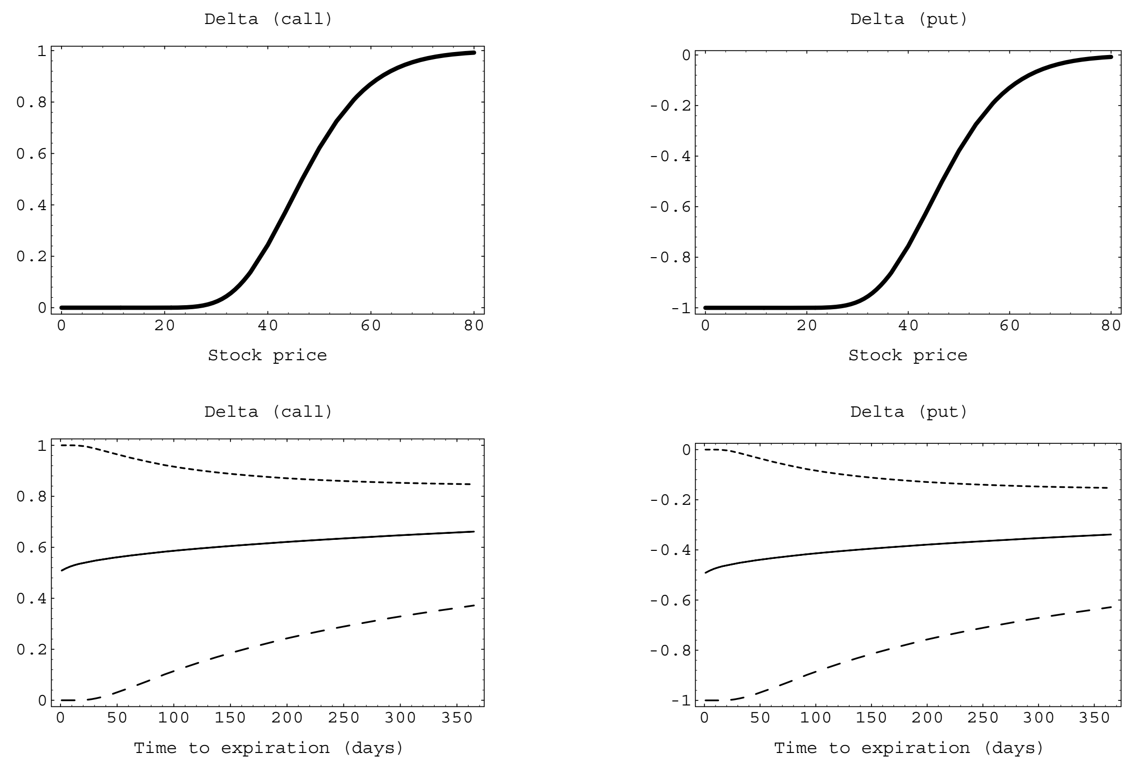
- Defined as
$$\Delta \equiv \frac{\partial f}{\partial S}.$$
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.[a]
- The delta used in the BOPM (p. 230) is the discrete analog.
- The delta of a long stock is apparently 1.

[a]Elementary calculus.

Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0.$$
- So the deltas of a call and an otherwise identical put cancel each other when $N(x) = 1/2$, i.e., when[a]
$$X = Se^{(r+\sigma^2/2)\,\tau}. \tag{39}$$

[a] The straddle (p. 192) $C + P$ then has zero delta!



Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money options.

Dashed curves: out-of-the-money calls or in-the-money puts.

Delta (continued)

- Suppose the stock pays a continuous dividend yield of q.
- Let

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \tag{40}$$

(recall p. 311).

- Then

$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-q\tau} N(x) > 0, \\ \frac{\partial P}{\partial S} &= -e^{-q\tau} N(-x) < 0. \end{aligned}$$

Delta (continued)

- Consider an X_1-strike call and an X_2-strike put, $X_1 \geq X_2$.
- They are otherwise identical.
- Let

$$x_i \equiv \frac{\ln(S/X_i) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{41}$$

- Then their deltas sum to zero when $x_1 = -x_2$.[a]
- That implies

$$\frac{S}{X_1} = \frac{X_2}{S}\, e^{-(2r-2q+\sigma^2)\,\tau}. \tag{42}$$

[a]The strangle (p. 194) $C + P$ then has zero delta!

Delta (continued)

- Suppose we demand $X_1 = X_2 = X$ and have a straddle.
- Then
$$X = Se^{(r-q+\sigma^2/2)\,\tau}$$
leads to a straddle with zero delta.
  - This generalizes Eq. (39) on p. 319.
- When $C(X_1)$'s delta and $P(X_2)$'s delta sum to zero, does the portfolio $C(X_1) - P(X_2)$ have zero value?

Delta (concluded)

- The risk reversal has value

$$
\begin{aligned}
& Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma\sqrt{\tau}) \\
& -X_2 e^{-r\tau} N(-x_2 + \sigma\sqrt{\tau}) + Se^{-q\tau} N(-x_2) \\
= \quad & 2Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma\sqrt{\tau}) - X_2 e^{-r\tau} N(x_1 + \sigma\sqrt{\tau}) \\
= \quad & 2Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma\sqrt{\tau}) \\
& -\frac{S^2}{X_1} e^{(r-2q+\sigma^2)\,\tau} N(x_1 + \sigma\sqrt{\tau}).
\end{aligned}
$$

- This is not identically zero so not a risk reversal (p. 314).

- E.g., with $r = q = 0$ and τ large, it is about $2S - (S^2/X_1)\, e^{\sigma^2 \tau} = 2S - X_2$.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
  - A delta-neutral portfolio is immune to *small* price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
  - Long Δ shares of stock to hedge a short call.
- In general, hedge a position in a security with delta Δ_1 by shorting Δ_1/Δ_2 units of a security with delta Δ_2.

Theta (Time Decay)

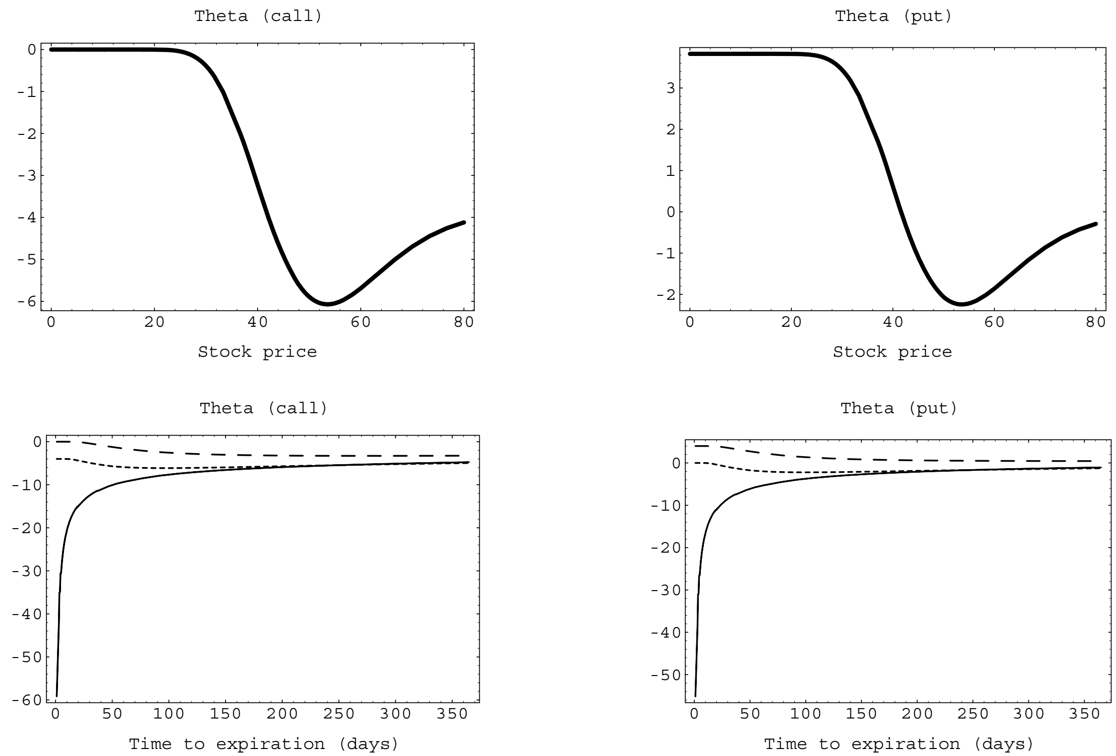
- Defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial f / \partial \tau = \partial f / \partial t$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau} N(x - \sigma\sqrt{\tau}) < 0.$$

  – The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau} N(-x + \sigma\sqrt{\tau}).$$

  – Can be negative or positive.



Dotted curve: in-the-money call or out-of-the-money put.

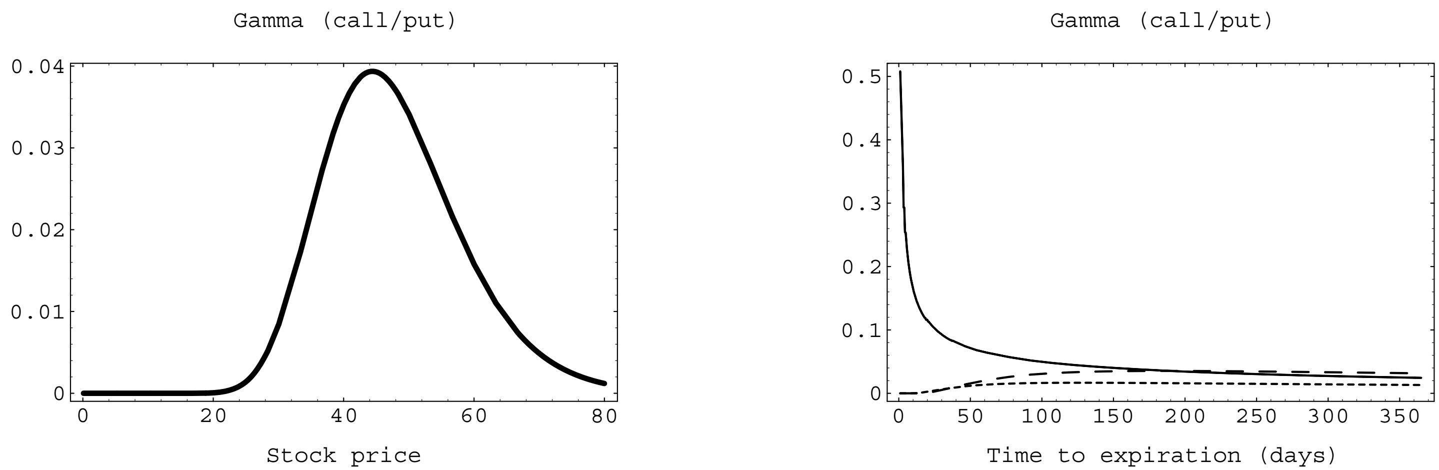
Solid curves: at-the-money options.

Dashed curve: out-of-the-money call or in-the-money put.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$



Dotted lines: in-the-money call or out-of-the-money put.

Solid lines: at-the-money option.

Dashed lines: out-of-the-money call or in-the-money put.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of a securitys value with respect to the volatility of the underlying asset

$$\Lambda \equiv \frac{\partial f}{\partial \sigma}.$$

- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes or estimation error in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - So higher volatility always increases the option value.

[a]Vega is not Greek.

Vega (continued)

- If the stock pays a continuous dividend yield of q, then

$$\Lambda = Se^{-q\tau}\sqrt{\tau}\, N'(x),$$

  where x is defined in Eq. (40) on p. 321.

- Vega is maximized when $x = 0$, i.e., when

$$S = Xe^{-(r-q+\sigma^2/2)\,\tau}.$$

- Vega declines very fast as S moves away from that peak.

Vega (continued)

- Now consider a risk reversal (p. 314) consisting of an X_1-strike call C and a short X_2-strike put P, $X_1 \geq X_2$.
- The options' vegas cancel out when
$$x_1 = -x_2,$$
where x_i are defined in Eq. (41) on p. 322.
- This leads to Eq. (42) on p. 322.
  - The same condition leads to zero delta for the strangle $C + P$ (p. 322).

Vega (concluded)

- Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 288 for the Black-Scholes model).

- The Black-Scholes formula (p. 284) implies

$$\begin{aligned} C &\to S, \\ P &\to Xe^{-r\tau}, \end{aligned}$$

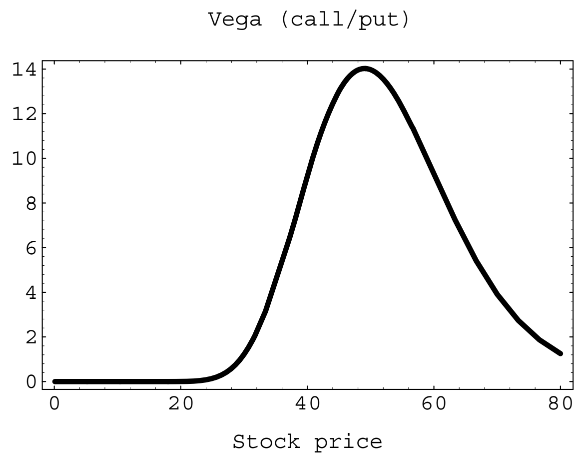
as $\sigma \to \infty$.

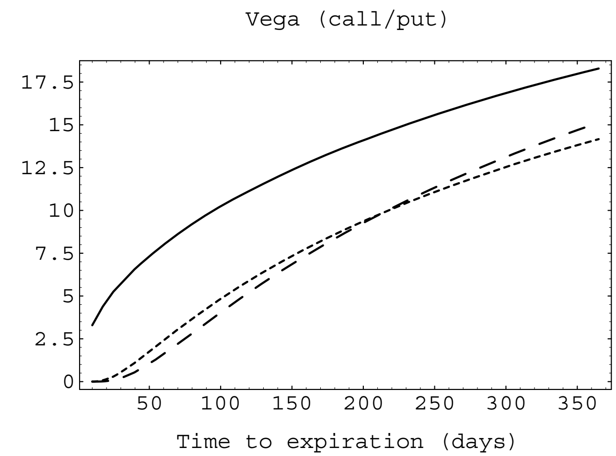
- These boundary conditions may be handy for certain numerical methods.

Variance Vega[a]

- Defined as the rate of change of a securitys value with respect to the variance (square of volatility) of the underlying asset
$$V \equiv \frac{\partial f}{\partial \sigma^2}.$$
- It is easy to verify that
$$V = \frac{\Lambda}{2\sigma}.$$

[a]Demeterfi, Derman, Kamal, and Zou (1999).





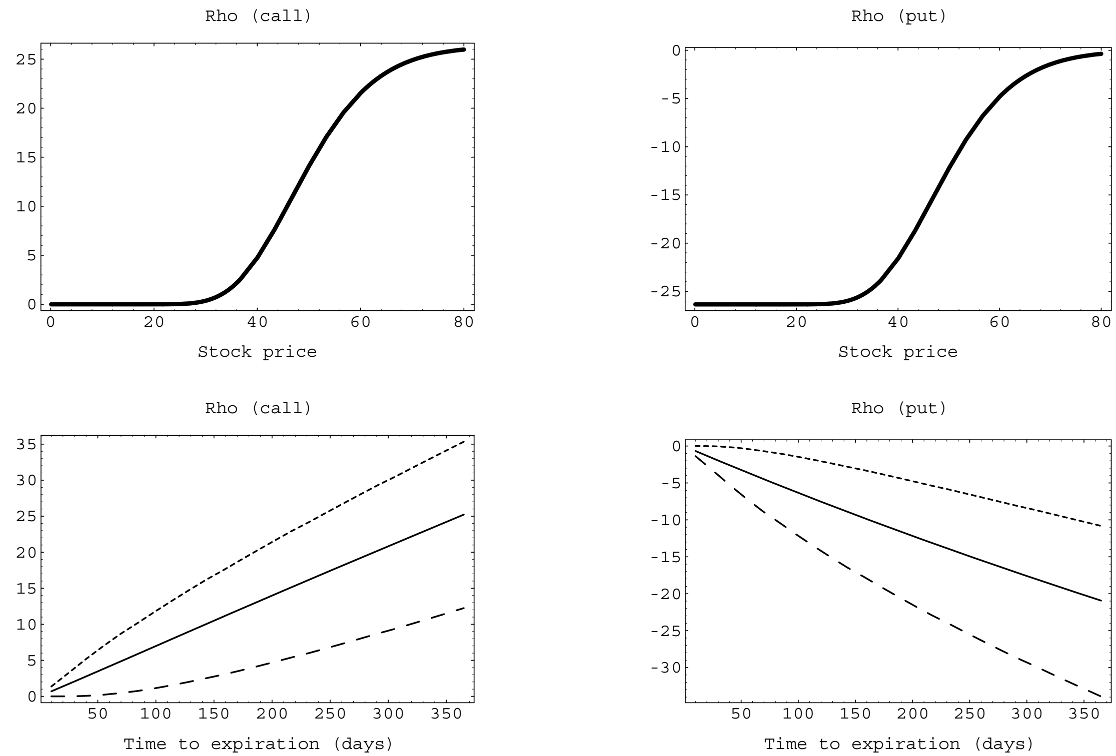
Dotted curve: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curve: out-of-the-money call or in-the-money put.

Rho

- Defined as the rate of change in its value with respect to interest rates
$$\rho \equiv \frac{\partial f}{\partial r}.$$
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$



Dotted curves: in-the-money call or out-of-the-money put.

Solid curves: at-the-money option.

Dashed curves: out-of-the-money call or in-the-money put.