

Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.
- Invest \$1 for j periods to end up with $[1 + S(j)]^j$ dollars at time j .
 - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[1 + S(i)]^i [1 + S(i, j)]^{j-i}$ dollars at time j .
 - $S(i, j)$: $(j - i)$ -period spot rate i periods from now.
 - The rollover strategy.

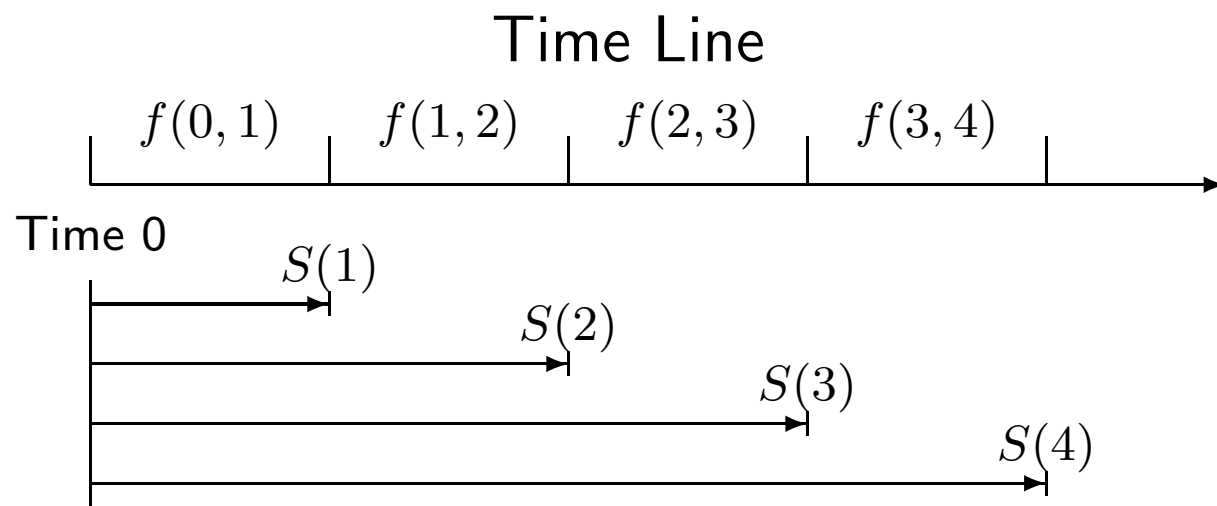
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \equiv \left[\frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \quad (17)$$

we will end up with $[1 + S(j)]^j$ dollars again.

- By definition, $f(0, j) = S(j)$.
- $f(i, j)$ is called the (implied) forward rates.
 - More precisely, the $(j - i)$ -period forward rate i periods from now.



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
 - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.

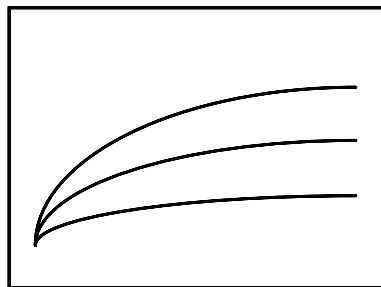
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

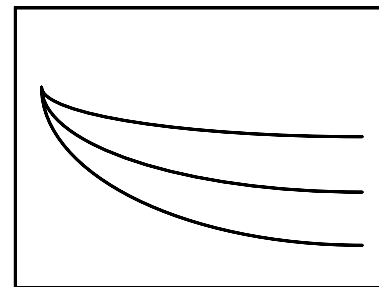
$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$



(a)



(b)

Forward Rates \equiv Spot Rates \equiv Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n -period zero-coupon bonds and receive

$$[1 + S(n)]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)].$$

Forward Rates \equiv Spot Rates \equiv Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)]\}^{1/n} - 1. \quad (18)$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

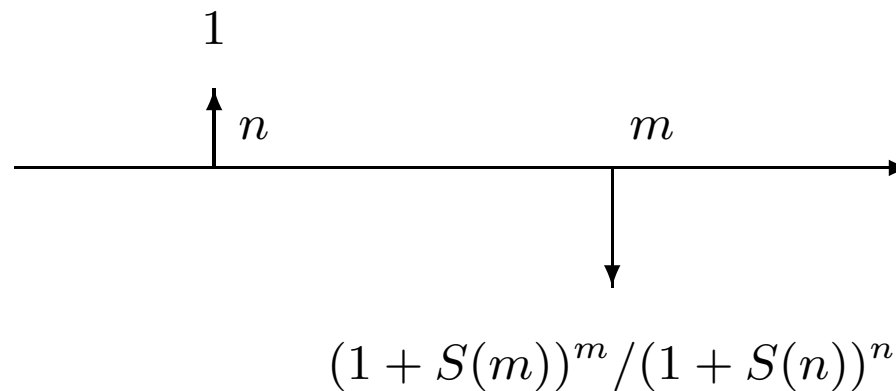
$$f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \quad (19)$$

Locking in the Forward Rate $f(n, m)$

- Buy one n -period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m / (1 + S(n))^n$ m -period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m / (1 + S(n))^n$ dollars.

Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the rate $f(n, m)$ between times n and m .



Forward Contracts

- We had generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to
 - Borrow money at time n in the future, and
 - Repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

- Can the spot rate curve be an arbitrary curve?^a

^aContributed by Mr. Dai, Tian-Shyr (B82506025, R86526008, D88526006) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^n C e^{-iS(i)} + F e^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,^a

$$S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}.$$

^aCompare it with Eq. (18) on p. 132.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \quad (20)$$

- Compare the above formula with Eq. (17) on p. 126.

- The one-period forward rate:

$$f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}.$$

- Compare the above formula with Eq. (19) on p. 132.

Spot and Forward Rates under Continuous Compounding (concluded)

- Now,

$$\begin{aligned} f(T) &\equiv \lim_{\Delta T \rightarrow 0} f(T, T + \Delta T) \\ &= S(T) + T \frac{\partial S}{\partial T}. \end{aligned}$$

- So $f(T) > S(T)$ if and only if $\partial S / \partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S / \partial T)$, then $f(T) < 0$.^a

^aContributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[S(a, b)]. \quad (21)$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
 - $f(j, j + 1) > S(j + 1)$ if and only if $S(j + 1) > S(j)$ from Eq. (17) on p. 126.
 - So $E[S(j, j + 1)] > S(j + 1) > \dots > S(1)$ if and only if $S(j + 1) > \dots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

A “Bad” Expectations Theory

- The expected returns^a on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1)) E[1 + S(1, 2)] \quad (22)$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

^aMore precisely, the one-plus returns.

A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
 - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
 - The expected return is

$$E[(1 + S(1, 2))^{-1}] / (1 + S(2))^{-2}.$$

- Strategy two buys a one-period bond with a return of $1 + S(1)$.

A “Bad” Expectations Theory (concluded)

- The theory says the returns are equal:

$$\frac{1 + S(1)}{(1 + S(2))^2} = E \left[\frac{1}{1 + S(1, 2)} \right].$$

- Combine this with Eq. (22) on p. 141 to obtain

$$E \left[\frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}.$$

- But this is impossible save for a certain economy.
 - Jensen’s inequality states that $E[g(X)] > g(E[X])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
 - Use $g(x) \equiv (1 + x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E \left[(1 + S(1, n))^{-(n-1)} \right]}{(1 + S(n))^{-n}} = 1 + S(1) \quad \text{for all } n > 1.$$

- This theory is the basis of many interest rate models.

Duration in Practice

- To handle more general types of spot rate curve changes, define a vector $[c_1, c_2, \dots, c_n]$ that characterizes the perceived type of change.
 - Parallel shift: $[1, 1, \dots, 1]$.
 - Twist: $[1, 1, \dots, 1, -1, \dots, -1]$,
 $[1.8\%, 1.6\%, 1.4\%, 1\%, 0\%, -1\%, -1.4\%, \dots]$, etc.
 -
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let

$$P(y) \equiv \sum_i C_i / (1 + S(i) + yc_i)^i$$

be the price associated with the cash flow C_1, C_2, \dots

- Define duration as

$$-\left. \frac{\partial P(y)/P(0)}{\partial y} \right|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y) - P(-\Delta y)}{2P(0)\Delta y}.$$

- Modified duration equals the above when

$$\begin{aligned} [c_1, c_2, \dots, c_n] &= [1, 1, \dots, 1], \\ S(1) &= S(2) = \dots = S(n). \end{aligned}$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\text{Var}[X] \equiv E[(X - E[X])^2].$$

- The covariance between random variables X and Y is

$$\text{Cov}[X, Y] \equiv E[(X - \mu_X)(Y - \mu_Y)],$$

where μ_X and μ_Y are the means of X and Y , respectively.

- Random variables X and Y are uncorrelated if

$$\text{Cov}[X, Y] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \equiv \sqrt{\text{Var}[X]}.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \equiv \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.^a

^aPaul Wilmott (2009), “the correlations between financial quantities are notoriously unstable.” It may even break down “at high-frequency time intervals” (Budish, Cramton, and Shim, 2015).

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\text{Var} \left[\sum_{i=1}^n a_i X_i \right] = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{Cov}[X_i, X_j].$$

- It becomes

$$\sum_{i=1}^n a_i^2 \text{Var}[X_i]$$

when X_i are uncorrelated.

Conditional Expectation

- “ $X | I$ ” denotes X conditional on the information set I .
- The information set can be another random variable’s value or the past values of X , say.
- The conditional expectation $E[X | I]$ is the expected value of X conditional on I ; it is a random variable.
- The law of iterated conditional expectations:

$$E[X] = E[E[X | I]].$$

- If I_2 contains at least as much information as I_1 , then

$$E[X | I_1] = E[E[X | I_2] | I_1]. \quad (23)$$

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function

$$\text{Prob}[X \leq z] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx.$$

Moment Generating Function

- The moment generating function of random variable X is defined as

$$\theta_X(t) \equiv E[e^{tX}].$$

- The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp \left[\mu t + \frac{\sigma^2 t^2}{2} \right]. \quad (24)$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N \left(\sum_i \mu_i, \sum_i \sigma_i^2 \right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N \left(\sum_{i=1}^n t_i \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \text{Cov}[X_i, X_j] \right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$.^a

- X_i are said to have a multivariate normal distribution.

^aCorrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1]$ so that

$$\text{Prob}[X \leq x] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where^a

$$c \equiv \sqrt{-2(\ln \omega)/\omega}.$$

^aAs they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate^a

$$\left(\sum_{i=1}^{12} \xi_i \right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

^aJäckel (2002), “this is not a highly accurate approximation and should only be used to establish ballpark estimates.”

A Dirty Trick and a Right Attitude (concluded)

- The general formula is

$$\frac{(\sum_{i=1}^n \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.^a
- Always blame your random number generator last.^b
- Instead, check your programs first.

^aContributed by Mr. Chen, Shih-Hang (R02723031) on March 5, 2014.

^b“The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings.” William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated as follows.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$\begin{aligned}U &\equiv aX_1, \\V &\equiv \rho U + \sqrt{1 - \rho^2} aX_2.\end{aligned}$$

Generation of Bivariate Normal Distributions (concluded)

- U and V are the desired random variables with

$$\begin{aligned}\text{Var}[U] &= \text{Var}[V] = a^2, \\ \text{Cov}[U, V] &= \rho a^2.\end{aligned}$$

- Note that the mapping between (X_1, X_2) and (U, V) is one-to-one.

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu + \sigma^2/2} \quad \text{and} \quad \sigma_Y^2 = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1), \quad (25)$$

respectively.

- They follow from $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$.

The Lognormal Distribution (continued)

- Conversely, suppose Y is lognormally distributed with mean μ and variance σ^2 .
- Then

$$\begin{aligned}E[\ln Y] &= \ln(\mu/\sqrt{1 + (\sigma/\mu)^2}), \\ \text{Var}[\ln Y] &= \ln(1 + (\sigma/\mu)^2).\end{aligned}$$

- If X and Y are joint-lognormally distributed, then

$$\begin{aligned}E[XY] &= E[X] E[Y] e^{\text{Cov}[\ln X, \ln Y]}, \\ \text{Cov}[X, Y] &= E[X] E[Y] \left(e^{\text{Cov}[\ln X, \ln Y]} - 1 \right).\end{aligned}$$

The Lognormal Distribution (concluded)

- Let Y be lognormally distributed such that $\ln Y \sim N(\mu, \sigma^2)$.
- Then

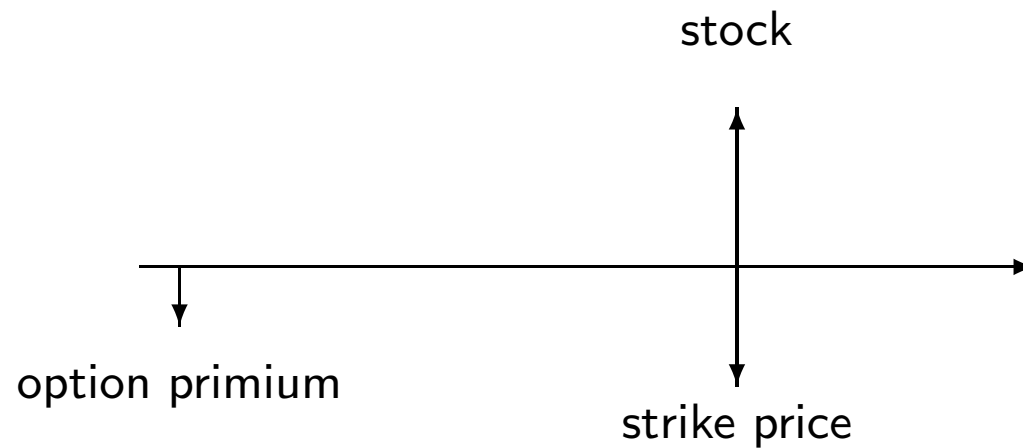
$$\int_a^\infty y f(y) dy = e^{\mu + \sigma^2/2} N\left(\frac{\mu - \ln a}{\sigma} + \sigma\right).$$

Option Basics

The shift toward options as
the center of gravity of finance [...]
— Merton H. Miller (1923–2000)

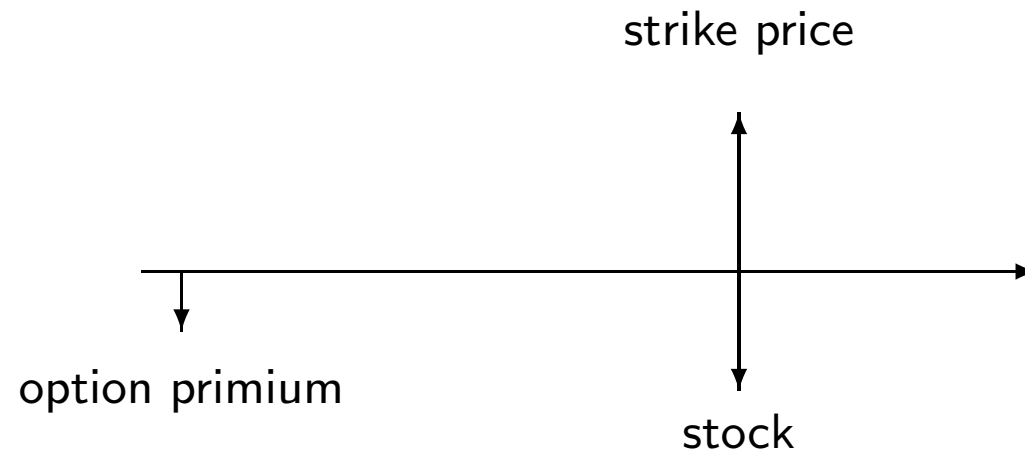
Calls and Puts

- A call gives its holder the right to *buy* a unit of the underlying asset by paying a strike price.



Calls and Puts (continued)

- A put gives its holder the right to *sell* a unit of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
 - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C : call value.
- P : put value.
- X : strike price.
- S : stock price.
- D : dividend.

Payoff, Mathematically Speaking

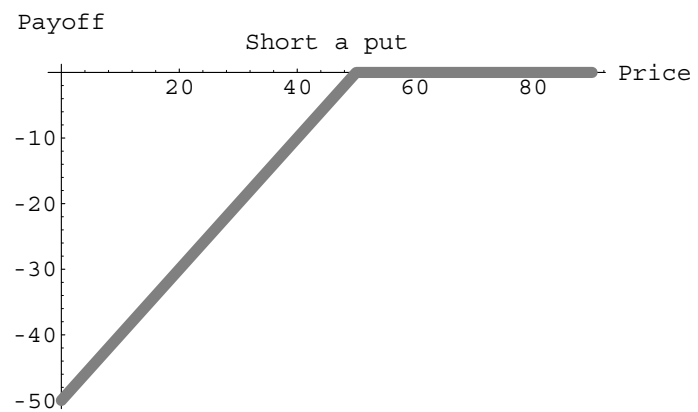
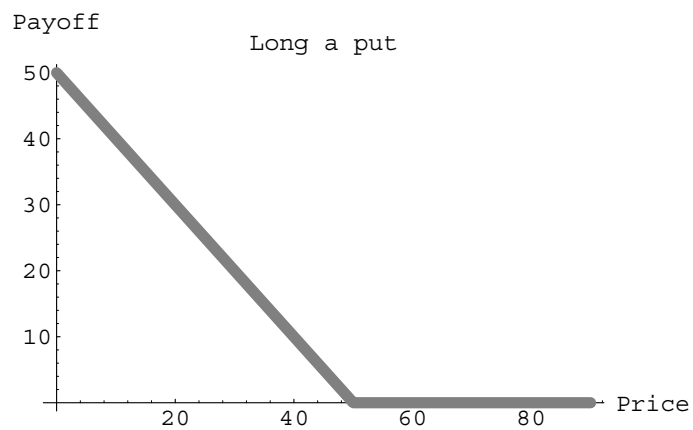
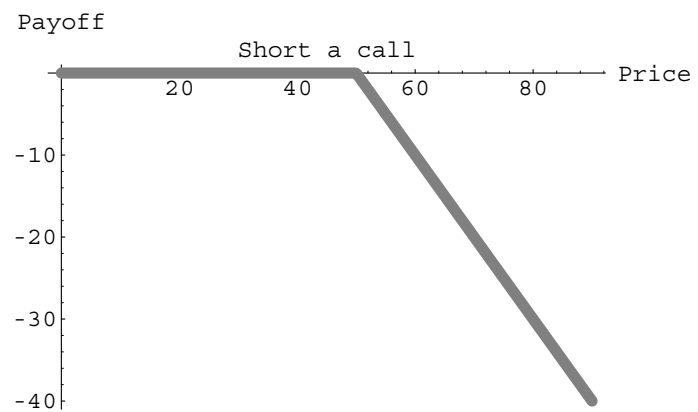
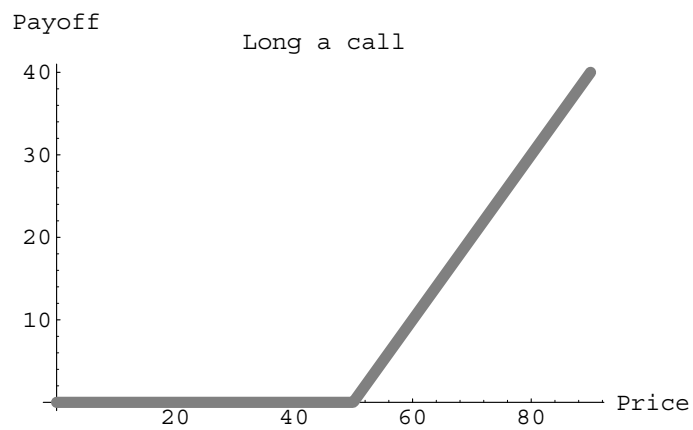
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.



Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

the intrinsic value of a call.

- At any time t before the expiration date, we call

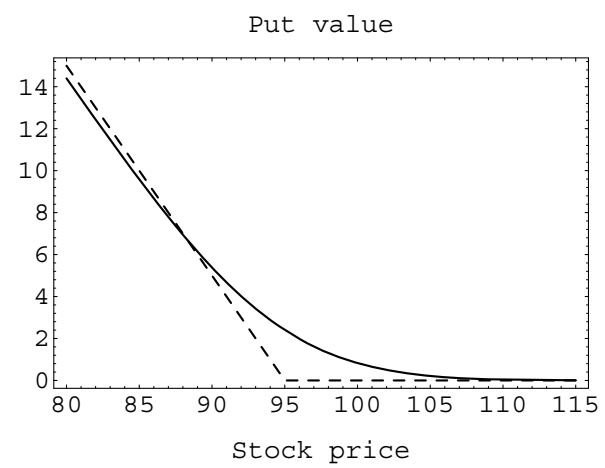
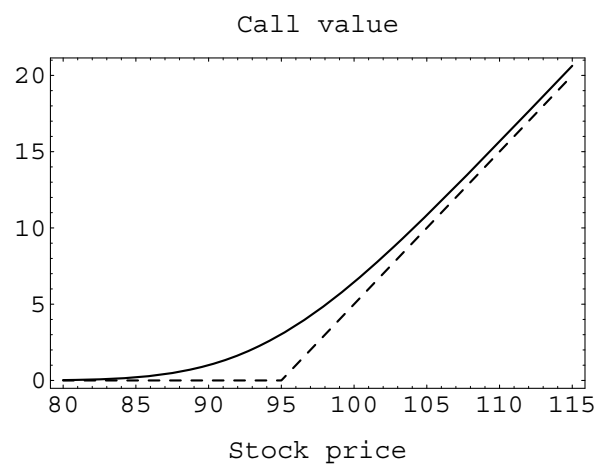
$$\max(0, X - S_t)$$

the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.^a
- Finding an option's value at any time before expiration is a major intellectual breakthrough.

^a11% of option holders let in-the-money options expire worthless.



Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
 - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n -for- m stock split, m shares become n shares.
- Accordingly, the strike price is only m/n times its previous value, and the number of shares covered by one contract becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- We assume options are unprotected.

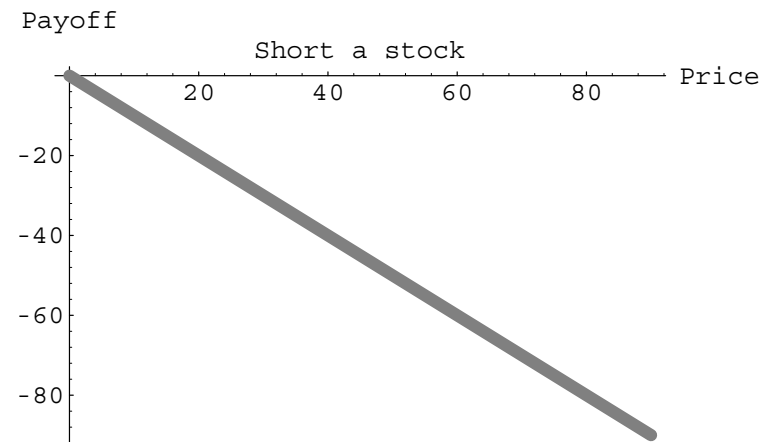
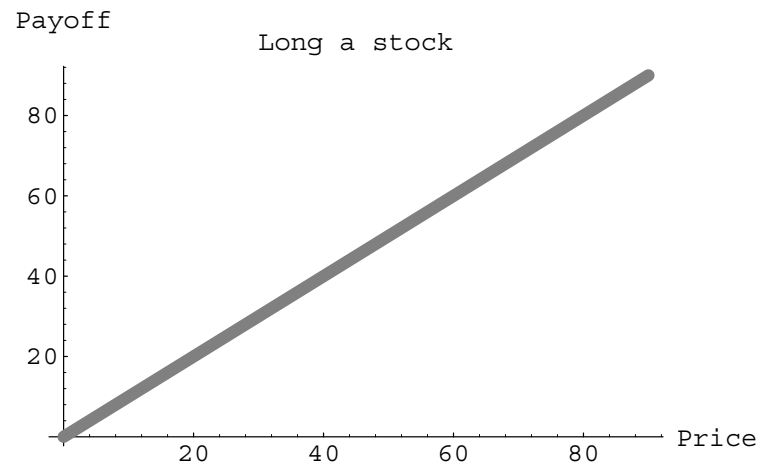
Example

- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
 - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
 - This action generates proceeds for the investor.
 - The investor can close out the short position by buying 1,000 XYZ shares.
 - Clearly, the investor profits if the stock price falls.
- Not all assets can be shorted.

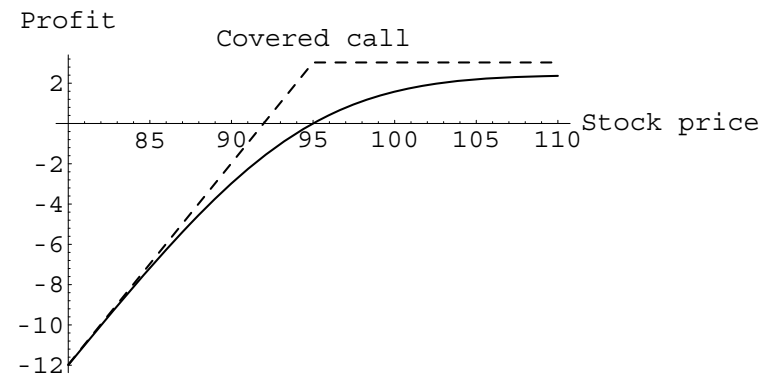
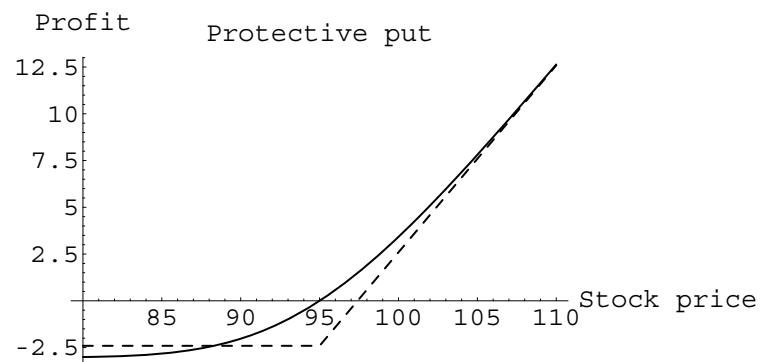
Payoff of Stock



Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Covered call: A long position in stock with a short call.^a
 - It is “covered” because the stock can be delivered to the buyer of the call if the call is exercised.
- Protective put: A long position in stock with a long put.
- Both strategies break even only if the stock price rises, so they are bullish.

^aA short position has a payoff opposite in sign to that of a long position.



Solid lines are profits of the portfolio one month before maturity, assuming the portfolio is set up when $S = 95$ then.

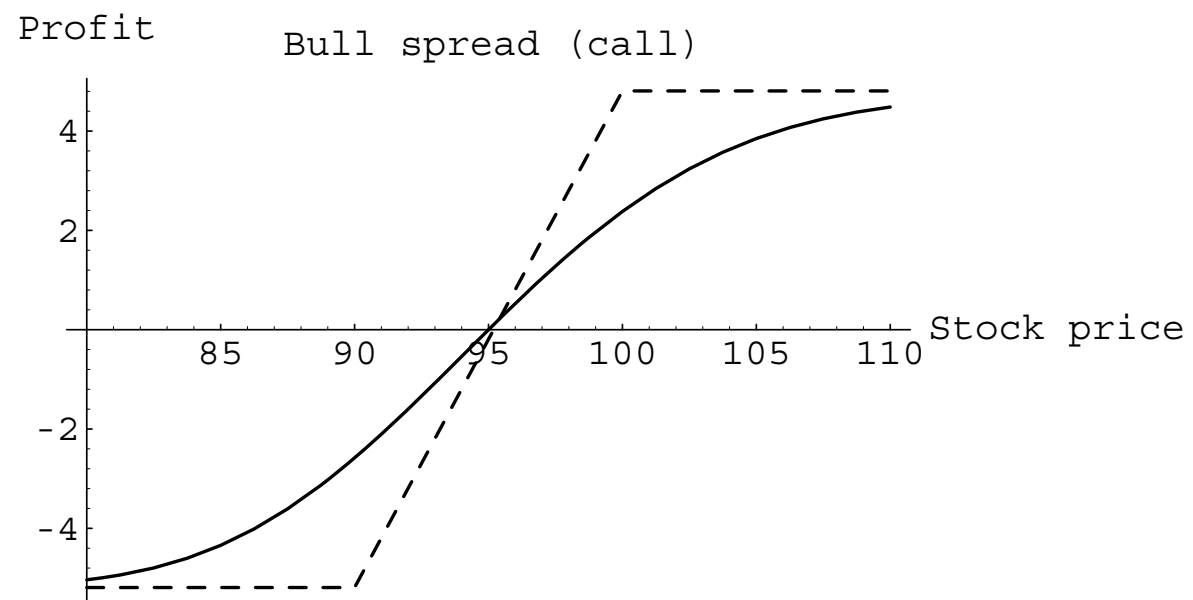
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L , X_M , and X_H to denote the strike prices with

$$X_L < X_M < X_H.$$

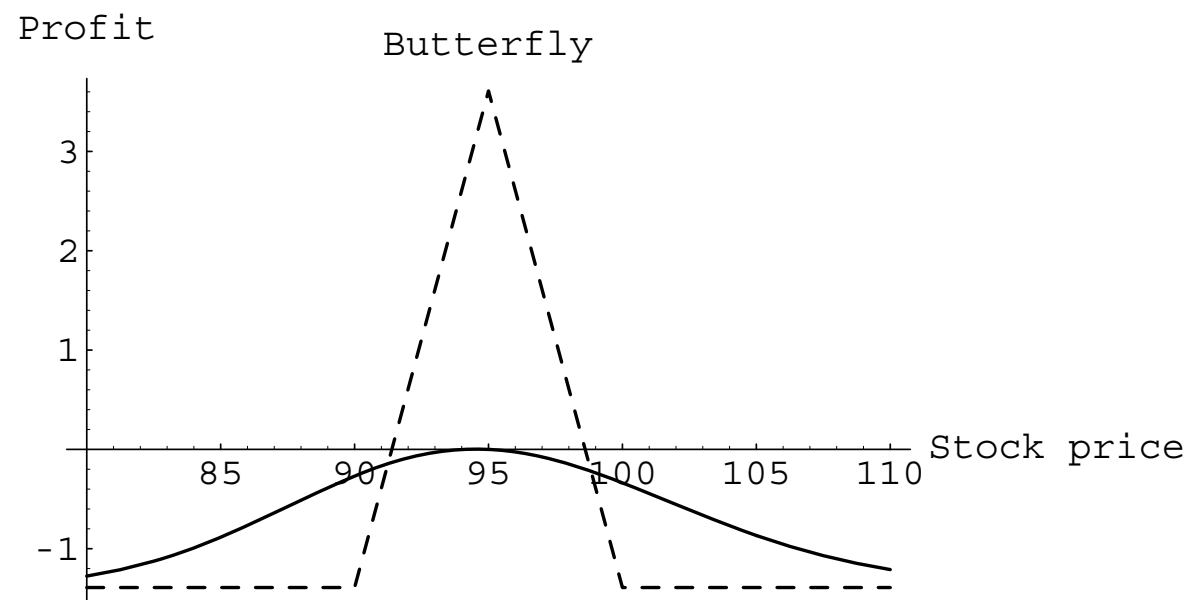
Covered Position: Spread (continued)

- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
 - The initial investment is $C_L - C_H$.
 - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
 - * When both are exercised at expiration.
 - The maximum loss is $C_L - C_H$.
 - * When neither is exercised at expiration.
 - If we buy $(X_H - X_L)^{-1}$ units of the bull call spread and $X_H - X_L \rightarrow 0$, a (Heaviside) step function emerges as the payoff.



Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
 - The spread is long one X_L call, long one X_H call, and short two X_M calls.



Covered Position: Spread (continued)

- A butterfly spread pays a positive amount at expiration only if the asset price falls between X_L and X_H .
- Take a position in $(X_M - X_L)^{-1}$ units of the butterfly spread.
- When $X_H - X_L \rightarrow 0$, it approximates a state contingent claim,^a which pays \$1 only when the state $S = X_M$ happens.^b

^aAlternatively, Arrow security.

^bSee Exercise 7.4.5 of the textbook.

Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The (undiscounted) state price equals

$$\frac{\partial^2 C}{\partial X^2}.$$

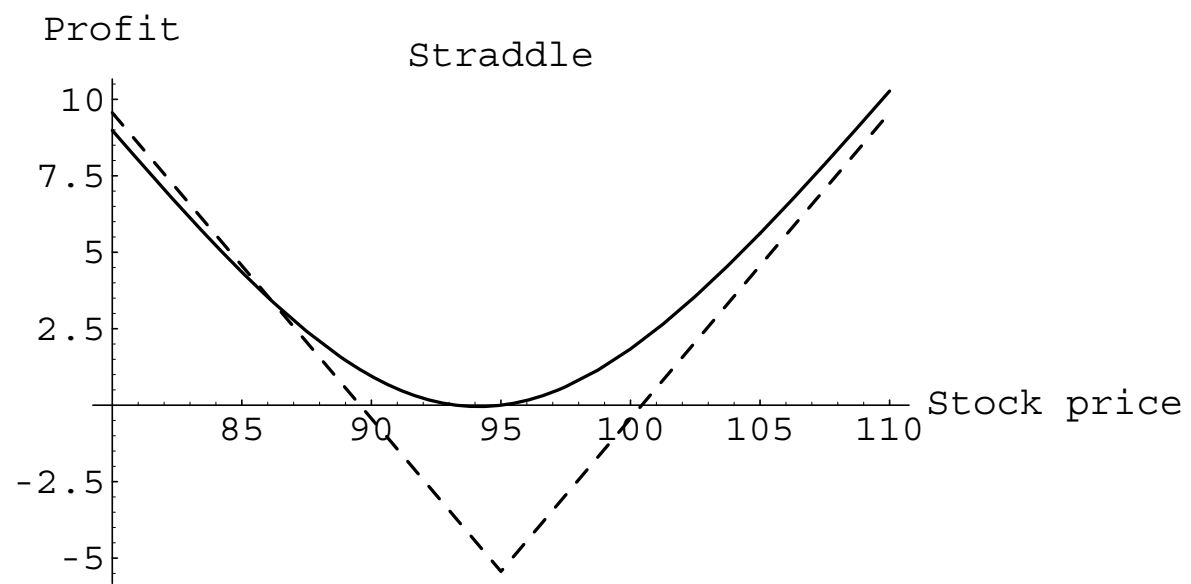
- Recall that C is the call's price.^a
- In fact, the PV of $\partial^2 C / \partial X^2$ is the “probability” density of the stock price $S_T = X$ at option's maturity.^b

^aOne can also use the put (see Exercise 9.3.6 of the textbook).

^bBreeden and Litzenberger (1978).

Covered Position: Combination

- A combination consists of options of different types on the same underlying asset.
 - These options must be either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
 - Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
 - Selling a straddle benefits from low volatility.



Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.

