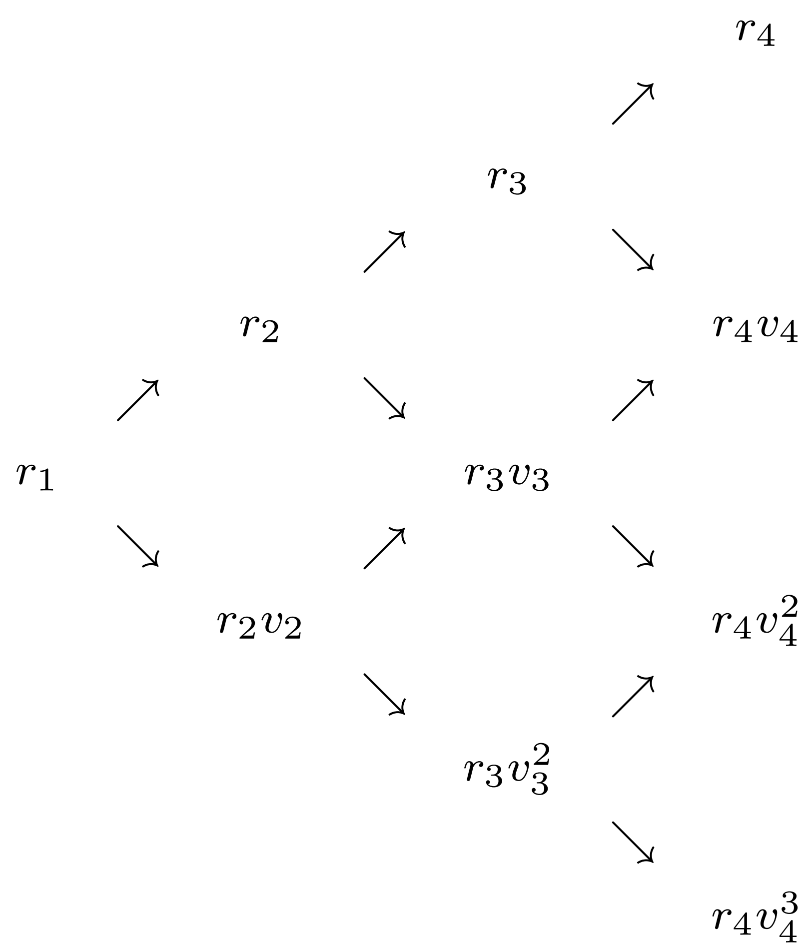
The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 905ff.[b]
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

---

[a] Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

[b] Repeated on next page.


r_4
r_3
r_2
$r_4 v_4$
r_1
$r_3 v_3$
$r_2 v_2$
$r_4 v_4^2$
$r_3 v_3^2$
$r_4 v_4^3$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- $P_{\rm u}$ is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_{\rm d}$ is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$y_{\mathrm{u}} \equiv P_{\mathrm{u}}^{-1/(i-1)} - 1,$$
$$y_{\mathrm{d}} \equiv P_{\mathrm{d}}^{-1/(i-1)} - 1.$$

- The yield volatility is defined as

$$\kappa_i \equiv \frac{\ln(y_{\mathrm{u}}/y_{\mathrm{d}})}{2}$$

(recall Eq. (114) on p. 955).

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \dots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to P_{u} or P_{d} one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}} + P_{\mathrm{d}}}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \tag{136}$$

- Obviously, P_{u} and P_{d} are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that $y_{\rm u}$ and $y_{\rm d}$ represent the spot rates at the up node and the down node, respectively (p. 1062).
- With κ_i^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\rm u}}^{-1/(i-1)} - 1}{{P_{\rm d}}^{-1/(i-1)} - 1} \right). \tag{137}$$

- Solving Eqs. (136)–(137) for r and v with backward induction takes $O(i^2)$ time.
  - That leads to a cubic-time algorithm.

The BDT Model: Calibration (continued)

- We next employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving *forward* in time, how much \$1 at a node contributes to the price (review p. 932(a)).
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

---

[a] Chen (`R84526007`) and Lyuu (1997); Lyuu (1999).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be
$$P_1, P_2, \ldots, P_i.$$
- They correspond to rates
$$r, rv, \ldots, rv^{i-1}$$
for period i, respectively.
- One dollar at time i has a present value of
$$f(r, v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- The yield volatility is

$$g(r, v) \equiv \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right).$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 1059).
- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 1059).

The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices.
- Now solve

$$
\begin{aligned}
f(r, v) &= \frac{1}{(1+y)^i}, \\
g(r, v) &= \kappa_i,
\end{aligned}
$$

for $r = r_i$ and $v = v_i$.

- This $O(n^2)$-time algorithm appears on p. 382 of the textbook.

Calibrating the BDT Model with the Differential Tree (in seconds)[a]

Number of years	Running time	Number of years	Running time	Number of years	Running time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

[a] Lyuu (1999).

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) \, dW.$$

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant volatility will not attain mean reversion.

The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

[a]Black and Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$
\begin{aligned}
t_2 &\equiv t_1 + \Delta t_1, \\
t_3 &\equiv t_2 + \Delta t_2.
\end{aligned}
$$

$$
\begin{array}{ccccc}
 & & & \nearrow & \\
 & & \ln r_{\rm d}(t_2) & & \\
 & \nearrow & & \searrow & \\
\ln r(t_1) & & & & \ln r_{\rm du}(t_3) = \ln r_{\rm ud}(t_3) \\
 & \searrow & & \nearrow & \\
 & & \ln r_{\rm u}(t_2) & & \\
 & & & \searrow &
\end{array}
$$

The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$\begin{aligned}\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\ \ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.\end{aligned}$$

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose

$$\begin{aligned}&\ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\ =\ &\ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.\end{aligned}$$

The Black-Karasinski Model: Discrete Time (continued)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{138}$$

- So from Δt_1, we can calculate the Δt_2 that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T$ (roughly).[a]

---

[a] As $\kappa(t), \theta(t), \sigma(t)$ are independent of r, the Δt_is will not depend on r.

The Black-Karasinski Model: Discrete Time (concluded)

- Unequal durations Δt_i are often necessary to ensure a combining tree.[a]

---

[a] Amin (1991); Chen (R98922127) (2011); Lok (D99922028) and Lyuu (2015).

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\,M(t)\,] = \infty$ for any finite t if they model the continuously compounded rate.[a]
- Hence periodic compounding should be used.
- Another issue is computational.
- Lognormal models usually do not give analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

[a] Hogan and Weintraub (1993).

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.[a]
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a]Hull and White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

- Like the Ho-Lee model, this is a normal model, and the inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a]Hull and White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

[a] Hull and White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value

$$r_0 + j\Delta r$$

for some integer j.

---

[a]Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with

$$\begin{aligned} t_i &\equiv i\Delta t, \\ r_j &\equiv r_0 + j\Delta r, \end{aligned}$$

  as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\,t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

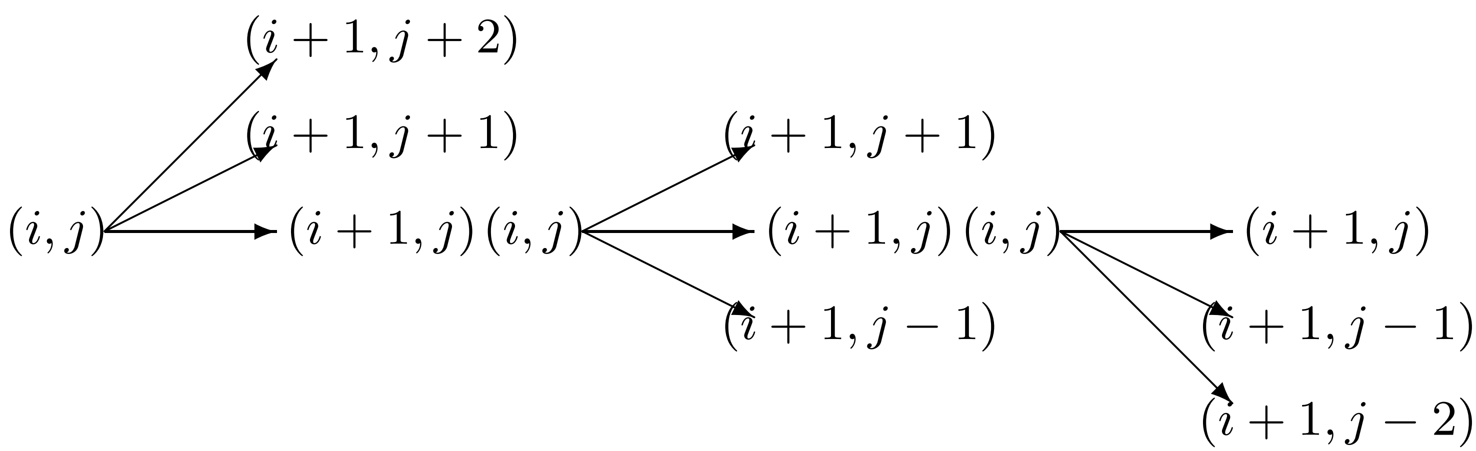
- Use

$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{139}$$

to denote the drift rate, or the expected change, of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1087.
- The interest rate movement described by the middle branch may be an increase of Δr, no change, or a decrease of Δr.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j)\\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j)\\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (139) on p. 1086.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.[a]

[a] Not $r(0, t_{i+1})$!

The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their accompanying probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.[b]

---

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

[b] A predecessor to Lyuu and Wu's (R90723065) (2003, 2005) mean-tracking idea, which is the precursor of the binomial-trinomial tree of Dai (B82506025, R86526008, D8852600) and Lyuu (2006, 2008, 2010).

The Hull-White Model: Calibration (continued)

- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to

$$r_j + \mu_{i,j} \Delta t.$$

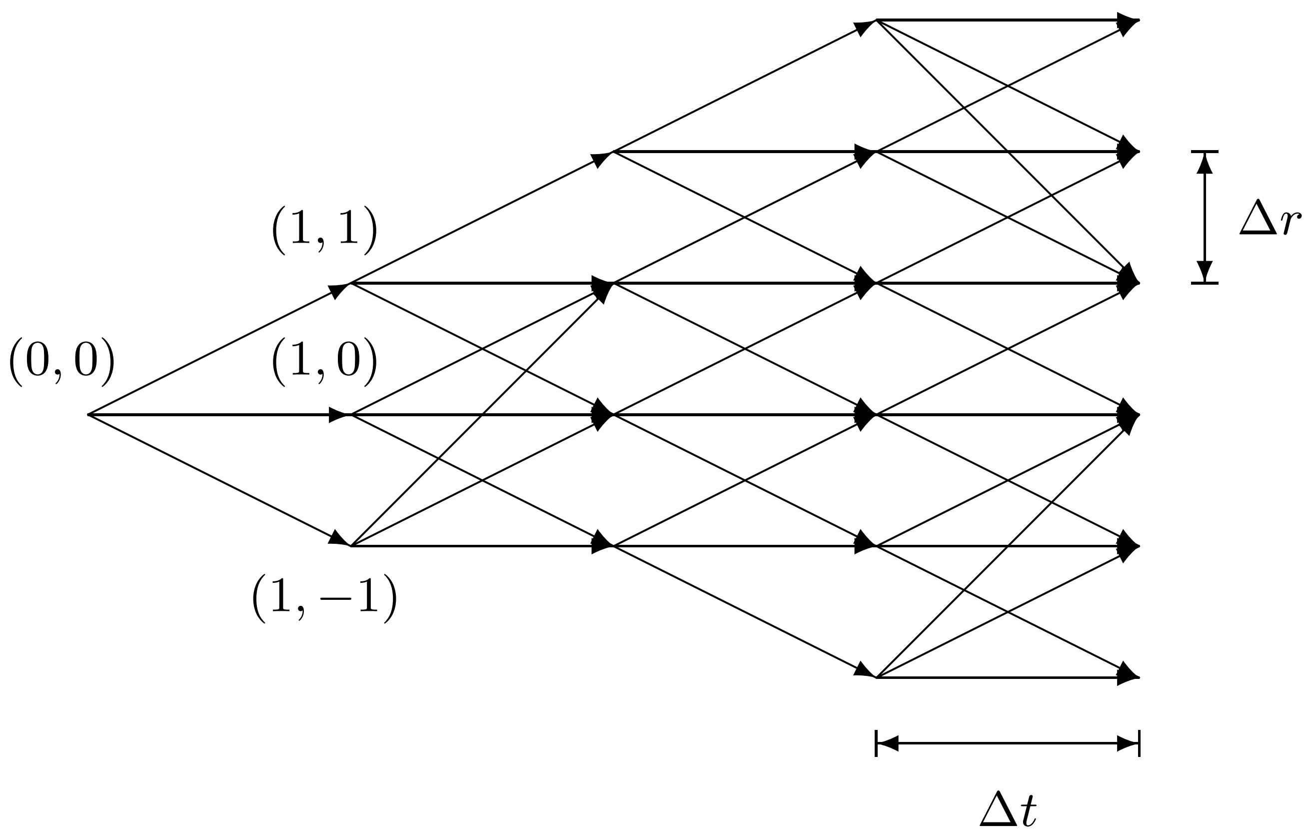
  – But note that $\mu_{i,j}$ is still not computed yet.

- Then the three nodes following node (i, j) are nodes

$$(i+1, k+1), (i+1, k), (i+1, k-1).$$

- See p. 1092 for a possible geometry.

- The resulting tree combines because of the constant jump sizes to reach k.


(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{140}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{140$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{140$''$}$$

where

$$\eta \equiv \mu_{i,j} \Delta t + (j - k)\, \Delta r.$$

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

  for the probabilities to lie between zero and one.
  - For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[a]
- Now it only remains to determine $\theta(t_i)$.

[a] Hull and White (1988).

The Hull-White Model: Calibration (continued)

- At this point at time t_i,

$$r(0, t_1), r(0, t_2), \ldots, r(0, t_{i+1})$$

  have already been matched.

- Let $Q(i, j)$ denote the value of the state contingent claim that pays one dollar at node (i, j) and zero otherwise.

- By construction, the state prices $Q(i, j)$ for all j are known by now.

- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= & \sum_j Q(i,j)\, e^{-r_j \Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \quad (141)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation in Eq. (141) can be approximated by

$$E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \approx e^{-r_j \Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (142)$$

  – This solves the chicken-egg problem!

- Substitute Eq. (142) into Eq. (141) and replace $\mu_{i,j}$ with $\theta(t_i) - ar_j$ to obtain

$$\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j \Delta t}\left(1 + ar_j(\Delta t)^2 + \sigma^2(\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j \Delta t}}.$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (142) is actually known analytically by Eq. (21) on p. 160:

$$\begin{aligned} & E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\ = \quad & e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}. \end{aligned}$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$,[a] the probabilities,[b] and finally the state prices at time t_{i+1}:

$$Q(i+1,j) = \sum_{(i,j^*) \text{ is connected to } (i+1,j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*}\Delta t} Q(i,j^*).$$

- There are at most 5 choices for j^* (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

[a]See Eq. (139) on p. 1086.
[b]See Eqs. (140) on p. 1093.

Comments on the Hull-White Model

- One can try different values of a and σ for each option.
- Or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- But suppose the demand is for a single set of parameters that replicate all option prices.
- Then the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull and White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1092).
  - So it is harder to program.
- The second shortcoming is again a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this chicken-egg dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (141) on p. 1096 that helps derive $\theta(t_i)$ later.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

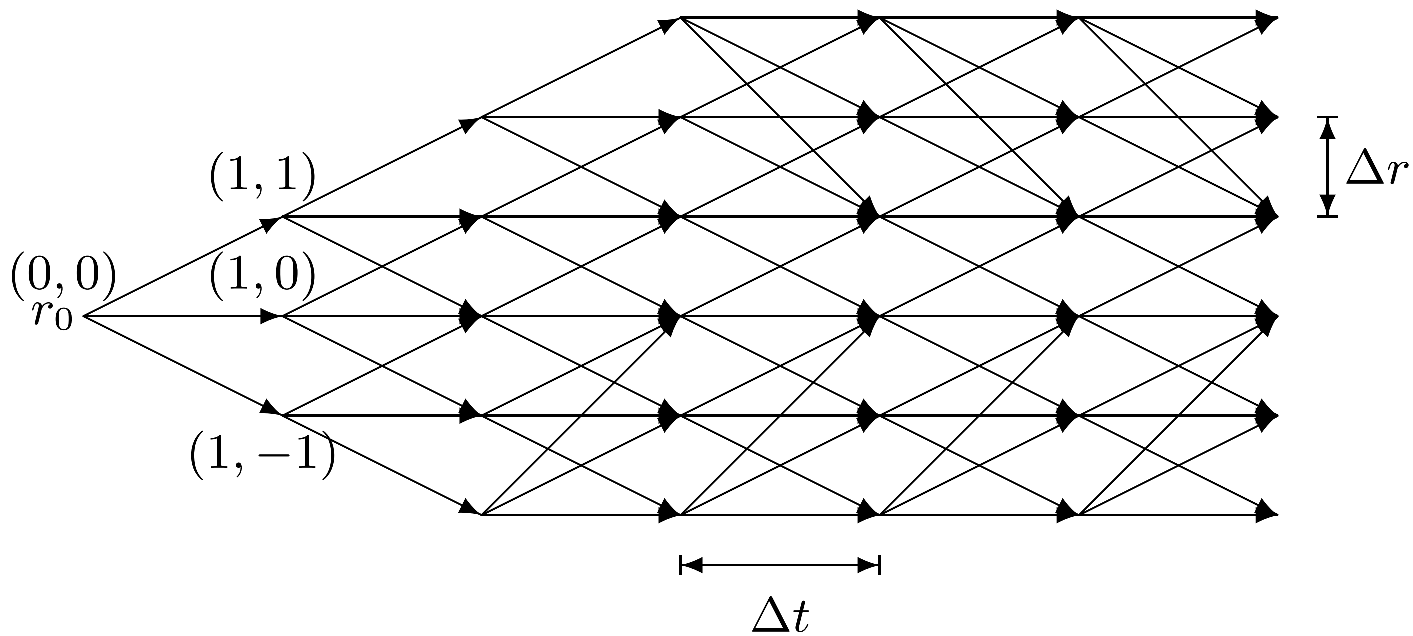
[a]Hull and White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (preview p. 1105).
  - The number of nodes above the r_0-line, $j_{\max}$, and that below the line, $j_{\min}$, will be picked so that the probabilities (140) on p. 1093 are positive for all nodes.
  - The tree's branches and probabilities are in place.



The short rate at node $(0,0)$ equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to *add* to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.[a]

[a] Contrast this with the previous algorithm, where it was the spot rate $r(0, t_{i+2})$ that is matched!

The Hull-White Model: Calibration

- Set $\Delta r = \sigma\sqrt{3\Delta t}$ and assume that $a > 0$.
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j\Delta r$.
- Hence the probabilities in Eqs. (140) on p. 1093 use

$$\eta \equiv -aj\Delta r\Delta t + (j - k)\,\Delta r.$$

- Recall that k denotes the middle branch.

The Hull-White Model: Calibration (continued)

- The probabilities become

$$p_1(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, \tag{143}$$

$$p_2(i,j) = \frac{2}{3} - \left[\, a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \,\right], \tag{144}$$

$$p_3(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. \tag{145}$$

- p_1: up move; p_2: flat move; p_3: down move.

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the same j.
- Furthermore, $p_1(i, j) = p_3(i, -j)$.

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{146}$$

  ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

  ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As $\frac{3-\sqrt{6}}{3} \approx 0.184$, a good choice is

$$j_{\max} = \lceil 0.184/(a\Delta t) \rceil.$$

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0,0) = 1$.
- Inductively, suppose that spot rates $r(0,t_1)$, $r(0,t_2)$, ..., $r(0,t_i)$ have already been matched at time t_i.

The Hull-White Model: Calibration (continued)

- By construction, the state prices $Q(i, j)$ for all j are known by now.
- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i,j)\, e^{-(\beta_i + r_j)\Delta t}$$

by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0, t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i,j)\, e^{-r_j \Delta t}}{\Delta t},$$

and the short rate at node (i, j) equals $\beta_i + r_j$.

The Hull-White Model: Calibration (concluded)

- The state prices at time t_{i+1},

$$Q(i+1, j),$$

where

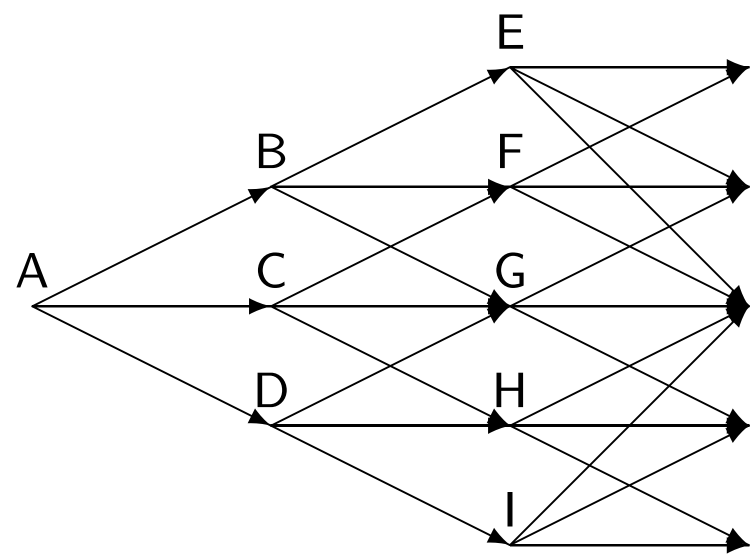
$$-\min(i+1, j_{\max}) \leq j \leq \min(i+1, j_{\max}),$$

can now be calculated as before.

- The total running time is $O(nj_{\max})$.
- The space requirement is $O(n)$.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\max} = 2$.
- The plot on p. 1115 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (143)–(145) on p. 1108 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	-1.73205	-3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve

$$0.08 - 0.05 \times e^{-0.18 \times t}.$$

- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2) \times 2 + \ln\left[\, Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \,\right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

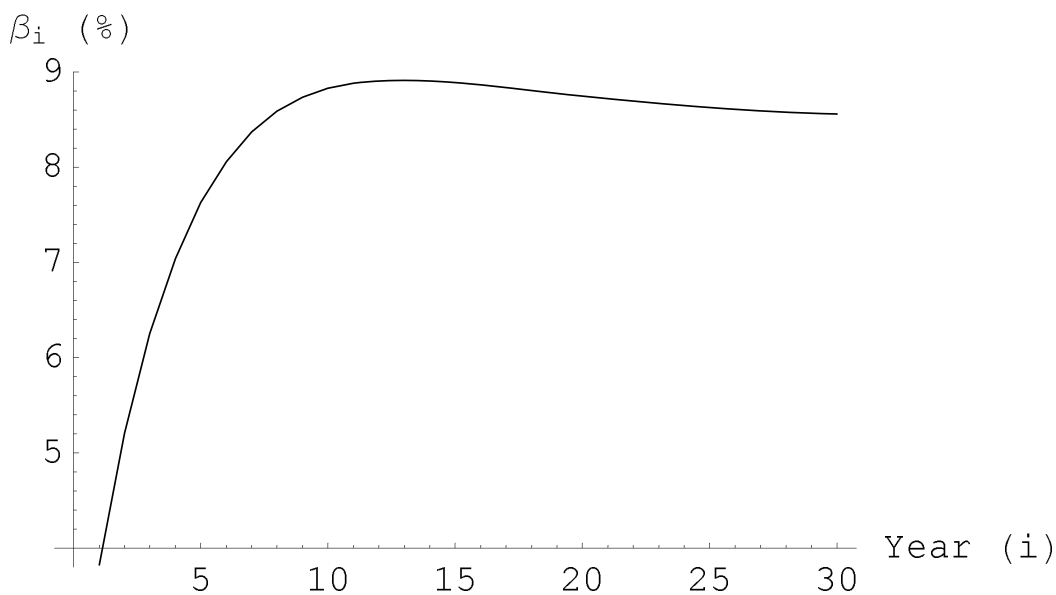
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1+r_1)}Q(1,1) = 0.018209,\\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1+r_1)}Q(1,1) + p_1(1,0)\, e^{-(\beta_1+r_0)}Q(1,0)\\
&= 0.199799,\\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1+r_1)}Q(1,1) + p_2(1,0)\, e^{-(\beta_1+r_0)}Q(1,0)\\
&\quad +p_1(1,-1)\, e^{-(\beta_1+r_{-1})}Q(1,-1) = 0.473597,\\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1+r_0)}Q(1,0) + p_2(1,-1)\, e^{-(\beta_1+r_{-1})}Q(1,-1)\\
&= 0.203263,\\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1+r_{-1})}Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0, 3)$ is matched by picking

$$\beta_2 = r(0,3) \times 3 + \ln \Big[Q(2,2)\, e^{-2\times\Delta r} + Q(2,1)\, e^{-\Delta r} + Q(2,0) + Q(2,-1)\, e^{\Delta r} + Q(2,-2)\, e^{2\times\Delta r} \Big] = 6.25359\%.$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1121 plots β_i for $i = 0, 1, \ldots, 29$.



β_i (%)
9
8
7
6
5
5
10
15
20
25
30
Year (i)

The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.

The Heath-Jarrow-Morton Model[a]

- This influential model is a forward rate model.
- It is also a popular model.
- The HJM model specifies the initial forward rate curve and the forward rate volatility structure, which describes the volatility of each forward rate for a given maturity date.
- Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

[a] Heath, Jarrow, and Morton (HJM) (1992).

Introduction to Mortgage-Backed Securities

Anyone stupid enough to promise to be responsible for a stranger's debts deserves to have his own property held to guarantee payment.

— Proverbs 27:13

Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- The lender — the mortgagee — can foreclose the loan by seizing the property if the borrower — the mortgagor — defaults, that is, fails to make the contractual payments.

Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.[a]
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.

[a]They can be traced to 1880s (Levy (2012)).

Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.[a]
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

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[a]Merton (1994).

Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
  3. Stripped mortgage-backed security (SMBS).

Problems Investing in Mortgages

- The mortgage sector is one of the largest in the debt market (see text).[a]
- Individual mortgages are unattractive for many investors.
- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.
- Most investors lack the resources and knowledge to assess the credit risk involved.

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[a]The outstanding balance was US$8.1 trillion as of 2012 vs. the US Treasury's US$10.9 trillion according to SIFMA.

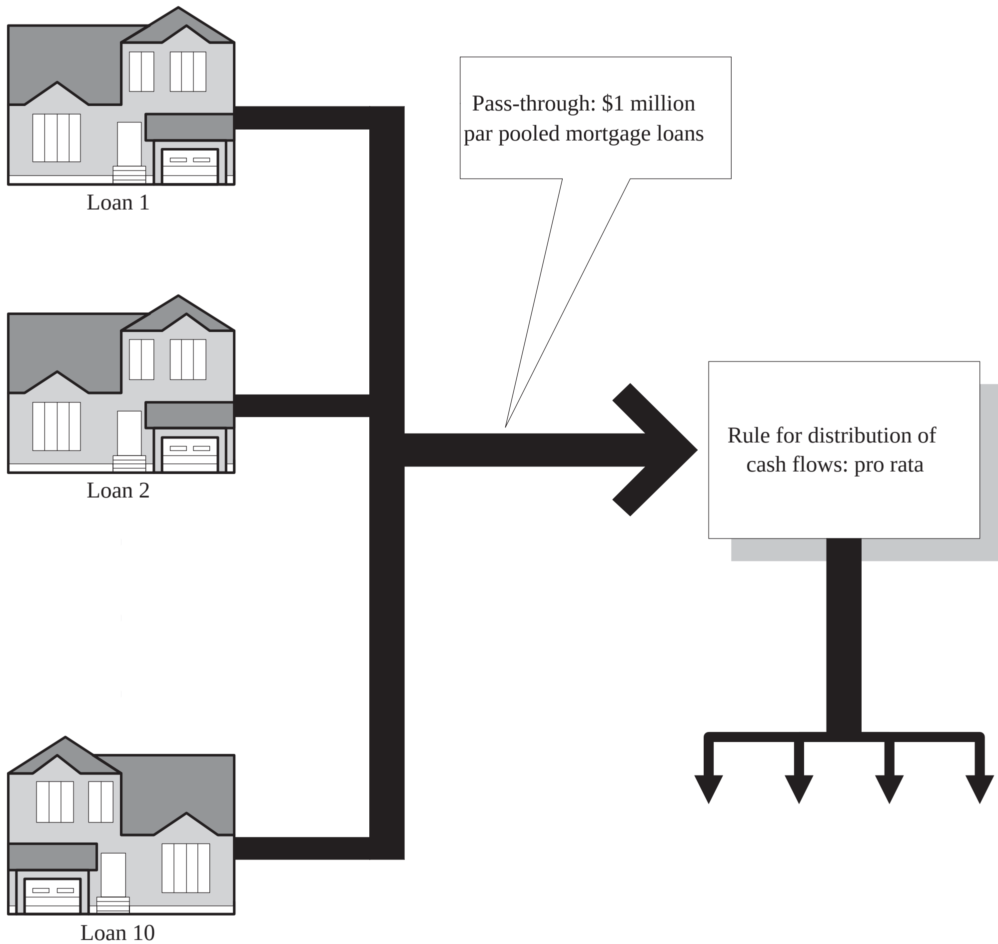
Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors, making the size and the timing of the cash flows unpredictable.

Mortgage Pass-Throughs[a]

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a \$25,000 certificate from a \$1 million pool is entitled to 2 1/2% (or 1/40th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.

[a]First issued by Ginnie Mae in 1970.


Pass-through: $1 million
par pooled mortgage loans
Loan 1
Loan 2
Loan 10
Rule for distribution of
cash flows: pro rata

Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.[a]
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

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[a] In June 1983 by Freddie Mac with the help of First Boston, which was acquired by Credit Suisse in 1990.

Collateralized Mortgage Obligations (CMOs) (continued)

- CMOs are *multiple*-maturity, *multi*class debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.
- The total prepayment risk is now divided among classes of bonds called classes or tranches.[a]
- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

[a] *Tranche* is a French word for "slice."

Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors
  - The outstanding balance of agency CMOs was US$1.1 trillion as of the first quarter of 2015.[a]

[a]SIFMA (2015).