Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.

— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader [$\cdots$]
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later.[a]

$P(t, T)$**:** the present value at time t of one dollar at time T.

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[a] Alternatively, the instantaneous spot rate, or short rate, at time t.

Standard Notations (continued)

$r(t,T)$**:** the $(T-t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period.[a]

$F(t,T,M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

[a]In other words, the $(T-t)$-period spot rate at time t.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t,T,M)=\frac{P(t,M)}{P(t,T)},\quad T\leq M. \tag{115}$$

  - The forward price equals the future value at time T of the underlying asset.[a]

- Equation (115) holds whether the model is discrete-time or continuous-time.

[a] See Exercise 24.2.1 of the textbook for proof.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1 \tag{116}$$

  in discrete time.

  - The analog to Eq. (116) under simple compounding is

$$f(t,T,L) = \frac{1}{L}\left(\frac{P(t,T)}{P(t,T+L)} - 1\right).$$

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{117}$$

by Eq. (115) on p. 969.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{118}$$

- Because Eq. (118) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{119}$$

the spot rate curve is

$$r(t,T) = \frac{\int_t^T f(t,s)\,ds}{T-t}.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (119) is

$$P(t,T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}.$$

- The short rate and the market discount function are related by

$$r(t) = - \left. \frac{\partial P(t,T)}{\partial T} \right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{120}$$

  - Relation (120) in fact follows from the risk-neutral valuation principle.[a]

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[a]Theorem 17 on p. 503.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (120) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned}
& P(t,T) \\
= \quad & E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \quad & E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \quad & E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right]. \qquad (121)
\end{aligned}$$

Risk-Neutral Pricing (concluded)

- Equation (120) on p. 974 can also be expressed as

$$E_t[\, P(t+1, T)\,] = F(t, t+1, T).$$

  - Verify that with, e.g., Eq. (115) on p. 969.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.[a]

[a] But the forward rate is not an unbiased estimator of the expected future short rate (p. 925).

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{122}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the *floating-rate payer*.
- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= \; & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= \; & \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= \; & P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{123}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Term Structure Equation

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow
$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$
- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\ = \;& (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\ & +(-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW. \end{aligned}$$

- Pick

$$\alpha \equiv \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \equiv \lambda(r,t) \tag{124}$$

  for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r, t)\, dt + \sigma(r, t)\, dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 of the textbook,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r, t)\, \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2}\, \frac{\partial^2 P}{\partial r^2} \right) / P, \tag{125}$$

$$\sigma_p = \left(\sigma(r, t)\, \frac{\partial P}{\partial r} \right) / P, \tag{125'}$$

subject to $P(\,\cdot\,, T, T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (124) on p. 987 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{126}$$

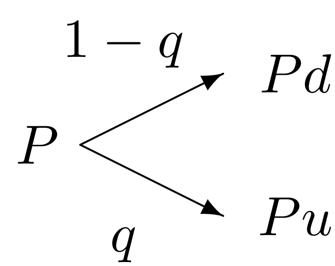
- This is called the term structure equation.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

- Equation (126) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:



The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{127}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{128}$$

The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate r.
- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- As in the continuous-time case, it can be shown that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \qquad (129)$$

which is independent of bond maturity and q.

  - Recall the BOPM.

- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

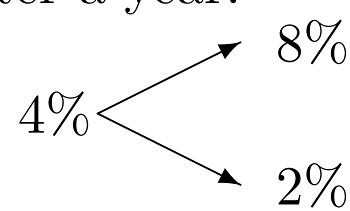
- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



Numerical Examples (continued)

- *No* real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 996.

Numerical Examples (continued)

$$90.703 \nearrow 92.593\ (= 100/1.08), \quad \searrow 98.039\ (= 100/1.02)$$

$$96.154 \nearrow 100, \quad \searrow 100$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

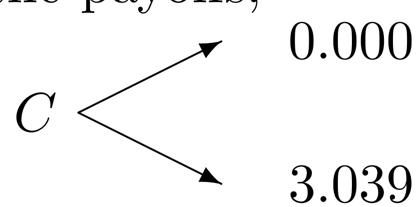
  where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,



- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

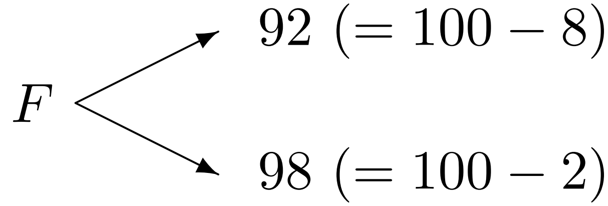
- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p)\times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:



- As the futures price F is the expected future payoff,[a]

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

[a]See Exercise 13.2.11 of the textbook or p. 504.

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

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[a] By Eq. (115) on p. 969.

[b] Recall p. 448.

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

the discount function $P(t,T)$ suffices to establish the spot rate curve.

- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\,r(T)\,|\,r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$

from Eq. (64) on p. 562.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{130}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

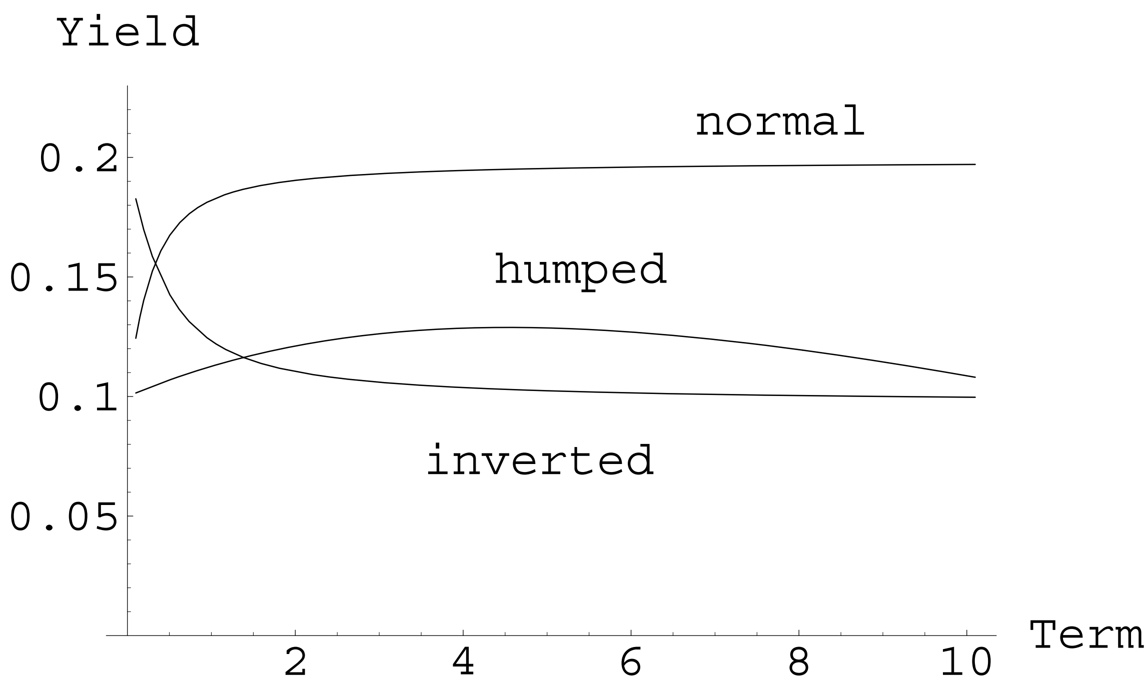
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve

$$(\partial r(t,T)/\partial r)\,\sigma = \sigma B(t,T)/(T-t).$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Indeed, higher β leads to greater attenuation of volatility with maturity.


Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
Term
2
4
6
8
10

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$\begin{aligned} x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\ \sigma_v &\equiv v(t,T)\,B(T,s), \\ v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}. \end{aligned}$$

- By the put-call parity, the price of a European put is

$$XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[\,0, T\,]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.$$

[a]Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \tag{131}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

  where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- This new process has a constant volatility, and its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation

$$r = f(x) \equiv \frac{x^2 \sigma^2}{4}$$

(see p. 1021).

- When $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.[a]

[a] Nawalkha and Beliaeva (2007).

$$
\begin{array}{ccccc}
 & & & & x + 2\sqrt{\Delta t} \\
 & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & \\
 & \nearrow & & \searrow & \\
x & & & & x \\
 & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & \\
 & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t}
\end{array}
\qquad
\begin{array}{ccccc}
 & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & \\
 & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & \\
f(x) & & & & f(x) \\
 & \searrow & & \nearrow & \\
 & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & \\
 & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{132}$$

  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

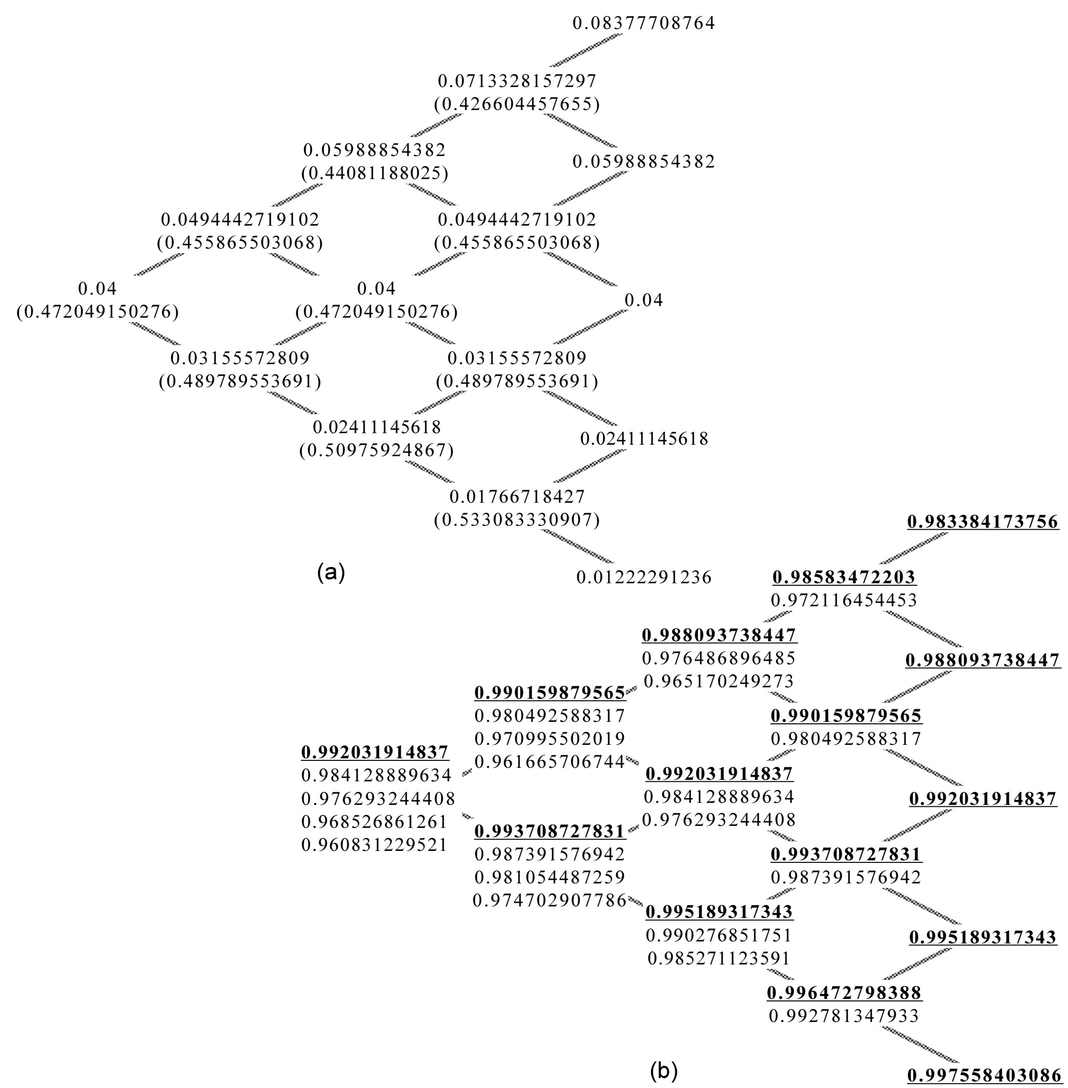
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 1024(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.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(a)
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0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate

$$\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.$$

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
  - I suspect that
  $$p(r) = A\sqrt{\frac{\Delta t}{r}} + B - C\sqrt{r \Delta t}$$
  for some $A, B, C > 0$.[a]
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

[a] Thanks to a lively class discussion on May 28, 2014.

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y, t)\, dt + \sigma(y, t)\, dW.$$

- Need to make sure the binomial model's drift and diffusion converge to the above process.
- Set the probability of an up move to

$$\frac{\alpha(y, t)\, \Delta t + y - y_{\rm d}}{y_{\rm u} - y_{\rm d}}.$$

- Here $y_{\rm u} \equiv y + \sigma(y, t)\sqrt{\Delta t}$ and $y_{\rm d} \equiv y - \sigma(y, t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.

[a] Nelson and Ramaswamy (1990).

A General Method (continued)

- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.
- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t+\Delta t)\sqrt{\Delta t} \\ \neq \; & -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t+\Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.
- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows

$$dx = m(y,t)\, dt + dW$$

for some $m(y,t)$.[a]

- The diffusion term is now a constant, and the binomial tree for x combines.

---

[a]See Exercise 25.2.13 of the textbook.

A General Method (concluded)

- The transformation is unique.[a]
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\rm d}(x,t)}{y_{\rm u}(x,t) - y_{\rm d}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that

$$\begin{aligned} y_{\rm u}(x,t) &\equiv y(x+\sqrt{\Delta t}, t+\Delta t),\\ y_{\rm d}(x,t) &\equiv y(x-\sqrt{\Delta t}, t+\Delta t). \end{aligned}$$

[a]Chiu (R98723059) (2012).

Examples

- The transformation is

$$\int^{r} (\sigma\sqrt{z})^{-1}\, dz = \frac{2\sqrt{r}}{\sigma}$$

  for the CIR model.

- The transformation is

$$\int^{S} (\sigma z)^{-1}\, dz = \frac{\ln S}{\sigma}$$

  for the Black-Scholes model.

- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.

Options on Coupon Bonds[a]

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.

---

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- The payoff for the option is

$$\max\left\{\left[\sum_{i=1}^{n} c_i P(r(T), T, t_i)\right] - X, 0\right\}.$$

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.

- This r^* can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for r.

Options on Coupon Bonds (continued)

- The solution is unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let
$$X_i \equiv P(r^*, T, t_i),$$
the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.
- Note that $P(r, T, t_i) \geq X_i$ if and only if $\leq r^*$.

Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$
\begin{aligned}
& \max\left\{ \left[\sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - \left[\sum_i c_i X_i \right], 0 \right\} \\
= & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).
\end{aligned}
$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

[a] Ho and Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

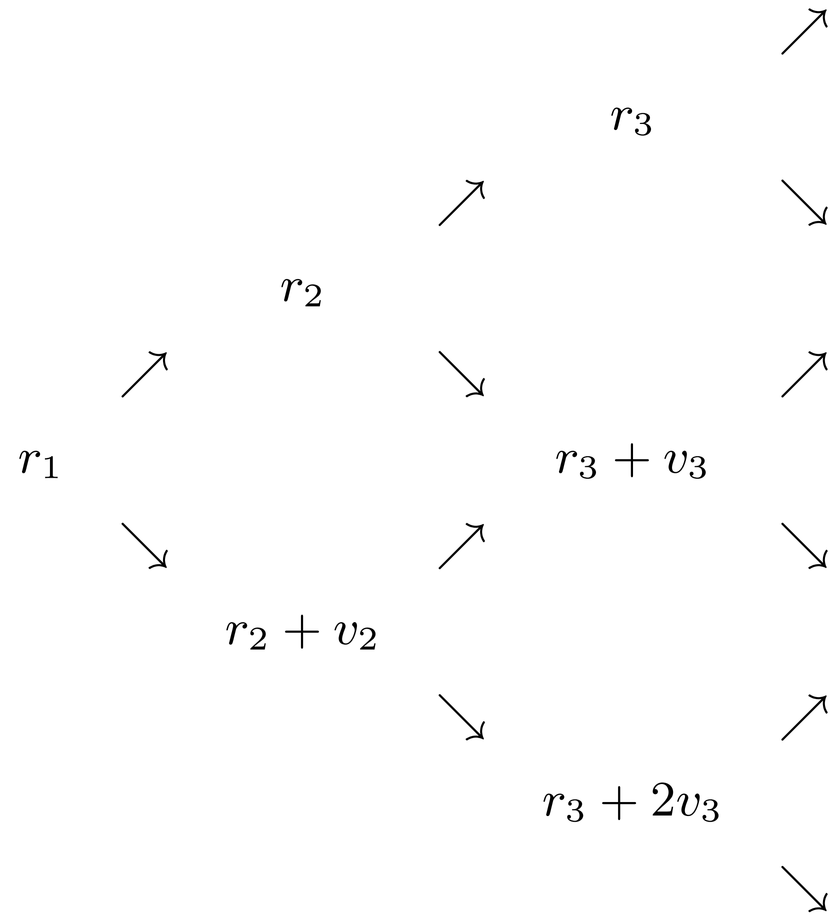
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] Ho and Lee (1986).


r_3
r_2
r_1
$r_3 + v_3$
$r_2 + v_2$
$r_3 + 2v_3$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\rm d}(t+1, t+2), P_{\rm d}(t+1, t+3), \ldots & \text{if short rate moves down} \\ P_{\rm u}(t+1, t+2), P_{\rm u}(t+1, t+3), \ldots & \text{if short rate moves up} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$P_{\rm u}(t+1, t+n) = P_{\rm d}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{133}$$

(see p. 376 of the textbook).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$
\begin{aligned}
y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\
y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1}
\end{aligned}
$$

- The volatility of the yield to maturity for this bond is therefore

$$
\begin{aligned}
\kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\
&= \sqrt{p(1-p)}\ (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\
&= \sqrt{p(1-p)}\ \frac{v_2 + \cdots + v_n}{n-1}.
\end{aligned}
$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{134}$$

- The variance of the short rate therefore equals

$$p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2,$$

where r_{u} and r_{d} are the two successor rates.[a]

---

[a] Contrast this with the lognormal model (108) on p. 909.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

$$\kappa_2, \kappa_3, \ldots .$$

  - It is independent of

$$r_2, r_3, \ldots .$$

- It is easy to compute the v_is from the volatility structure, and vice versa (review p. 1047).
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t{+}n) = (pP_{\rm u}(t{+}1, t{+}n) {+} (1{-}p)\, P_{\rm d}(t{+}1, t{+}n))\, P(t, t{+}1).$$

- Combine the above with Eq. (133) on p. 1046 and assume $p = 1/2$ to obtain[a]

$$P_{\rm d}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[\, v_2 + \cdots + v_n\,]}{1 + \exp[\, v_2 + \cdots + v_n\,]}, \tag{135}$$

$$P_{\rm u}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[\, v_2 + \cdots + v_n\,]}. \tag{135$'$}$$

[a]In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$\begin{aligned} P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\ P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}. \end{aligned}$$

- Short rate volatility $\sigma = v/2$ by Eq. (134) on p. 1048.
- Price derivatives by taking expectations under the risk-neutral probability.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \equiv \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its value is either $\ln \frac{P_{\rm d}(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_{\rm u}(t+1,t+n)}{P(t,t+n)}$.
- Thus the variance of return is

$$\text{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2 \sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is[a]

$$(n-1)(m-1)\,\sigma^2.$$

- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

[a] See Exercise 26.2.7 of the textbook.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

$$dr = \theta(t)\, dt + \sigma(t)\, dW.$$

- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.