Numerical Examples

- Assume
  - $S_0 = 100$, $y_0 = \ln S_0 = 4.60517$.
  - $r = 0$.
  - $n = 1$.
  - $h_0^2 = 0.0001096$, $\gamma = h_0 = 0.010469$.
  - $\gamma_n = \gamma/\sqrt{n} = 0.010469$.
  - $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, and $c = 0$.

Numerical Examples (continued)

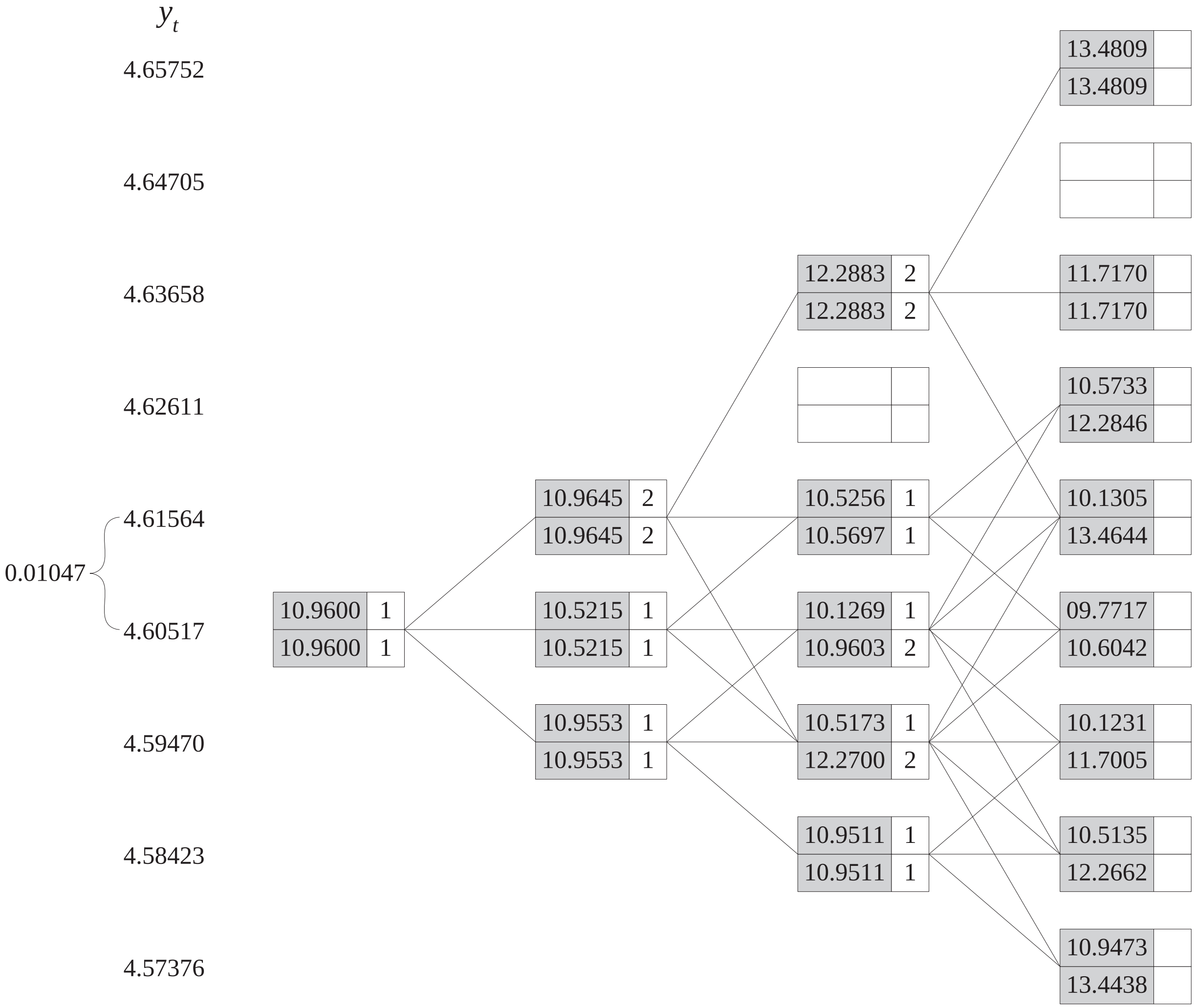
- A daily variance of 0.0001096 corresponds to an annual volatility of

$$\sqrt{365 \times 0.0001096} \approx 20\%.$$

- Let $h^2(i, j)$ denote the variance at node (i, j).
- Initially, $h^2(0, 0) = h_0^2 = 0.0001096$.

Numerical Examples (continued)

- Let $h^2_{\max}(i, j)$ denote the maximum variance at node (i, j).
- Let $h^2_{\min}(i, j)$ denote the minimum variance at node (i, j).
- Initially, $h^2_{\max}(0, 0) = h^2_{\min}(0, 0) = h^2_0$.
- The resulting three-day tree is depicted on p. 861.



y_t

4.65752

4.64705

4.63658

4.62611

4.61564

0.01047

4.60517

4.59470

4.58423

4.57376

10.9600 1
10.9600 1

10.9645 2
10.9645 2

10.5215 1
10.5215 1

10.9553 1
10.9553 1

12.2883 2
12.2883 2

10.5256 1
10.5697 1

10.1269 1
10.9603 2

10.5173 1
12.2700 2

10.9511 1
10.9511 1

13.4809
13.4809

11.7170
11.7170

10.5733
12.2846

10.1305
13.4644

09.7717
10.6042

10.1231
11.7005

10.5135
12.2662

10.9473
13.4438

- A top number inside a gray box refers to the minimum variance h^2_{min} for the node.
- A bottom number inside a gray box refers to the maximum variance h^2_{max} for the node.
- Variances are multiplied by 100,000 for readability.
- A top number inside a white box refers to the η corresponding to h^2_{min}.
- A bottom number inside a white box refers to the η corresponding to h^2_{max}.

Numerical Examples (continued)

- Let us see how the numbers are calculated.
- Start with the root node, node $(0, 0)$.
- Try $\eta = 1$ in Eqs. (102)–(104) on p. 845 first to obtain

$$\begin{aligned} p_u &= 0.4974, \\ p_m &= 0, \\ p_d &= 0.5026. \end{aligned}$$

- As they are valid probabilities, the three branches from the root node use single jumps.

Numerical Examples (continued)

- Move on to node $(1, 1)$.
- It has one predecessor node—node $(0, 0)$—and it takes an up move to reach the current node.
- So apply updating rule (106) on p. 851 with $\ell = 1$ and $h_t^2 = h^2(0, 0)$.
- The result is $h^2(1, 1) = 0.000109645$.

Numerical Examples (continued)

- Because $\lceil h(1,1)/\gamma \rceil = 2$, we try $\eta = 2$ in Eqs. (102)–(104) on p. 845 first to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7499, \\ p_d &= 0.1264. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(1,1)$ use double jumps.

Numerical Examples (continued)

- Carry out similar calculations for node $(1, 0)$ with $\ell = 0$ in updating rule (106) on p. 851.
- Carry out similar calculations for node $(1, -1)$ with $\ell = -1$ in updating rule (106).
- Single jump $\eta = 1$ works for both nodes.
- The resulting variances are

$$
\begin{aligned}
h^2(1, 0) &= 0.000105215, \\
h^2(1, -1) &= 0.000109553.
\end{aligned}
$$

Numerical Examples (continued)

- Node $(2, 0)$ has 2 predecessor nodes, $(1, 0)$ and $(1, -1)$.
- Both have to be considered in deriving the variances.
- Let us start with node $(1, 0)$.
- Because it takes a middle move to reach the current node, we apply updating rule (106) on p. 851 with $\ell = 0$ and $h_t^2 = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000101269$.

Numerical Examples (continued)

- Now move on to the other predecessor node $(1, -1)$.
- Because it takes an up move to reach the current node, apply updating rule (106) on p. 851 with $\ell = 1$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000109603$.
- We hence record

$$\begin{aligned} h_{\min}^2(2,0) &= 0.000101269, \\ h_{\max}^2(2,0) &= 0.000109603. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2,0)$ first.
- Because $\lceil h_{\max}(2,0)/\gamma \rceil = 2$, we first try $\eta = 2$ in Eqs. (102)–(104) on p. 845 to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7500, \\ p_d &= 0.1263. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Now consider state $h^2_{\min}(2,0)$.
- Because $\lceil h_{\min}(2,0)/\gamma \rceil = 1$, we first try $\eta = 1$ in Eqs. (102)–(104) on p. 845 to obtain

$$\begin{aligned} p_u &= 0.4596, \\ p_m &= 0.0760, \\ p_d &= 0.4644. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the minimum variance use single jumps.

Numerical Examples (continued)

- Node $(2, -1)$ has 3 predecessor nodes.
- Start with node $(1, 1)$.
- Because it takes a down move to reach the current node, we apply updating rule (106) on p. 851 with $\ell = -1$ and $h_t^2 = h^2(1, 1)$.[a]
- The result is $h_{t+1}^2 = 0.0001227$.

---

[a] Note that it is *not* $\ell = -2$.

Numerical Examples (continued)

- Now move on to predecessor node $(1, 0)$.
- Because it also takes a down move to reach the current node, we apply updating rule (106) on p. 851 with $\ell = -1$ and $h_t^2 = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000105609$.

Numerical Examples (continued)

- Finally, consider predecessor node $(1, -1)$.
- Because it takes a middle move to reach the current node, we apply updating rule (106) on p. 851 with $\ell = 0$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000105173$.
- We hence record

$$\begin{aligned} h_{\min}^2(2, -1) &= 0.000105173, \\ h_{\max}^2(2, -1) &= 0.0001227. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2,-1)$.
- Because $\lceil\, h_{\max}(2,-1)/\gamma\,\rceil = 2$, we first try $\eta = 2$ in Eqs. (102)–(104) on p. 845 to obtain

$$\begin{aligned} p_u &= 0.1385, \\ p_m &= 0.7201, \\ p_d &= 0.1414. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,-1)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Next, consider state $h^2_{\min}(2, -1)$.
- Because $\lceil h_{\min}(2, -1)/\gamma \rceil = 1$, we first try $\eta = 1$ in Eqs. (102)–(104) on p. 845 to obtain

$$\begin{aligned} p_u &= 0.4773, \\ p_m &= 0.0404, \\ p_d &= 0.4823. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2, -1)$ with the minimum variance use single jumps.

Numerical Examples (concluded)

- Other nodes at dates 2 and 3 can be handled similarly.
- In general, if a node has k predecessor nodes, then up to $2k$ variances will be calculated using the updating rule.
  - This is because each predecessor node keeps two variance numbers.
- But only the maximum and minimum variances will be kept.

Negative Aspects of the RT Algorithm Revisited[a]

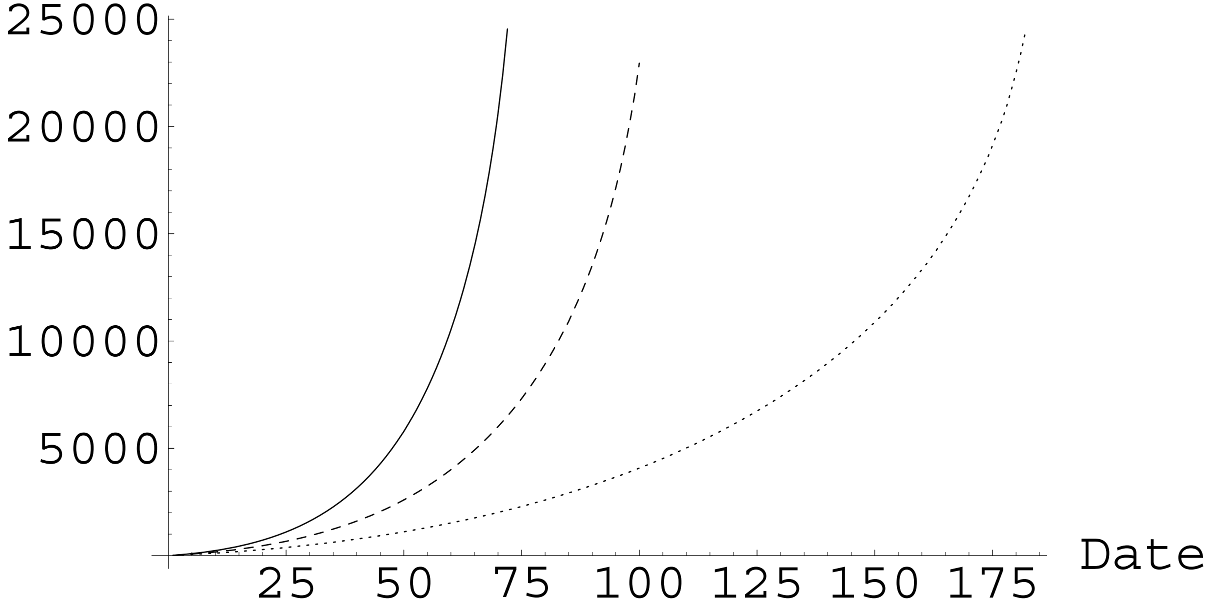
- Recall the problems mentioned on p. 857.
- In our case, combinatorial explosion occurs when

  $$n > \frac{1-\beta_1}{\beta_2} = \frac{1-0.9}{0.04} = 2.5$$

  (see the next plot).
- Suppose we are willing to accept the exponential running time and pick $n = 100$ to seek accuracy.
- But the problem of shortened maturity forces the tree to stop at date 9!

---

[a] Lyuu and Wu (`R90723065`) (2003, 2005).



Dotted line: $n = 3$; dashed line: $n = 4$; solid line: $n = 5$.

Backward Induction on the RT Tree

- After the RT tree is constructed, it can be used to price options by backward induction.
- Recall that each node keeps two variances $h^2_{\max}$ and $h^2_{\min}$.
- We now increase that number to K equally spaced variances between $h^2_{\max}$ and $h^2_{\min}$ at each node.
- Besides the minimum and maximum variances, the other $K-2$ variances in between are linearly interpolated.[a]

---

[a] In practice, log-linear interpolation works better (Lyuu and Wu (`R90723065`) (2005)). Log-cubic interpolation works even better (Liu (`R92922123`) (2005)).

Backward Induction on the RT Tree (continued)

- For example, if $K = 3$, then a variance of

$$10.5436 \times 10^{-6}$$

  will be added between the maximum and minimum variances at node $(2, 0)$ on p. 861.[a]

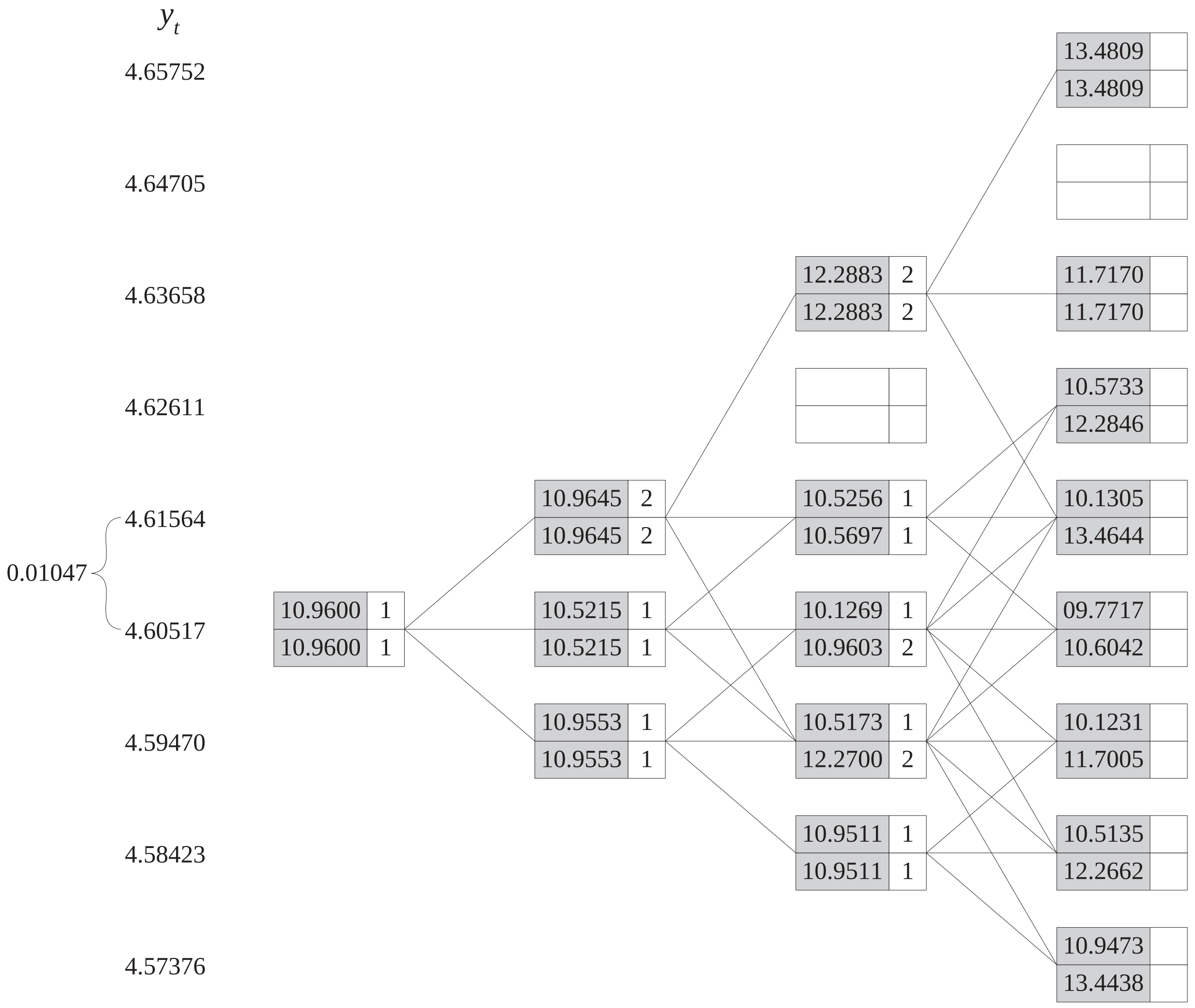
- In general, the kth variance at node (i, j) is

$$h^2_{\min}(i,j)+k\,\frac{h^2_{\max}(i,j)-h^2_{\min}(i,j)}{K-1}, \quad k=0,1,\ldots,K-1.$$

- Each interpolated variance's jump parameter and branching probabilities can be computed as before.

---

[a] Repeated on p. 881.

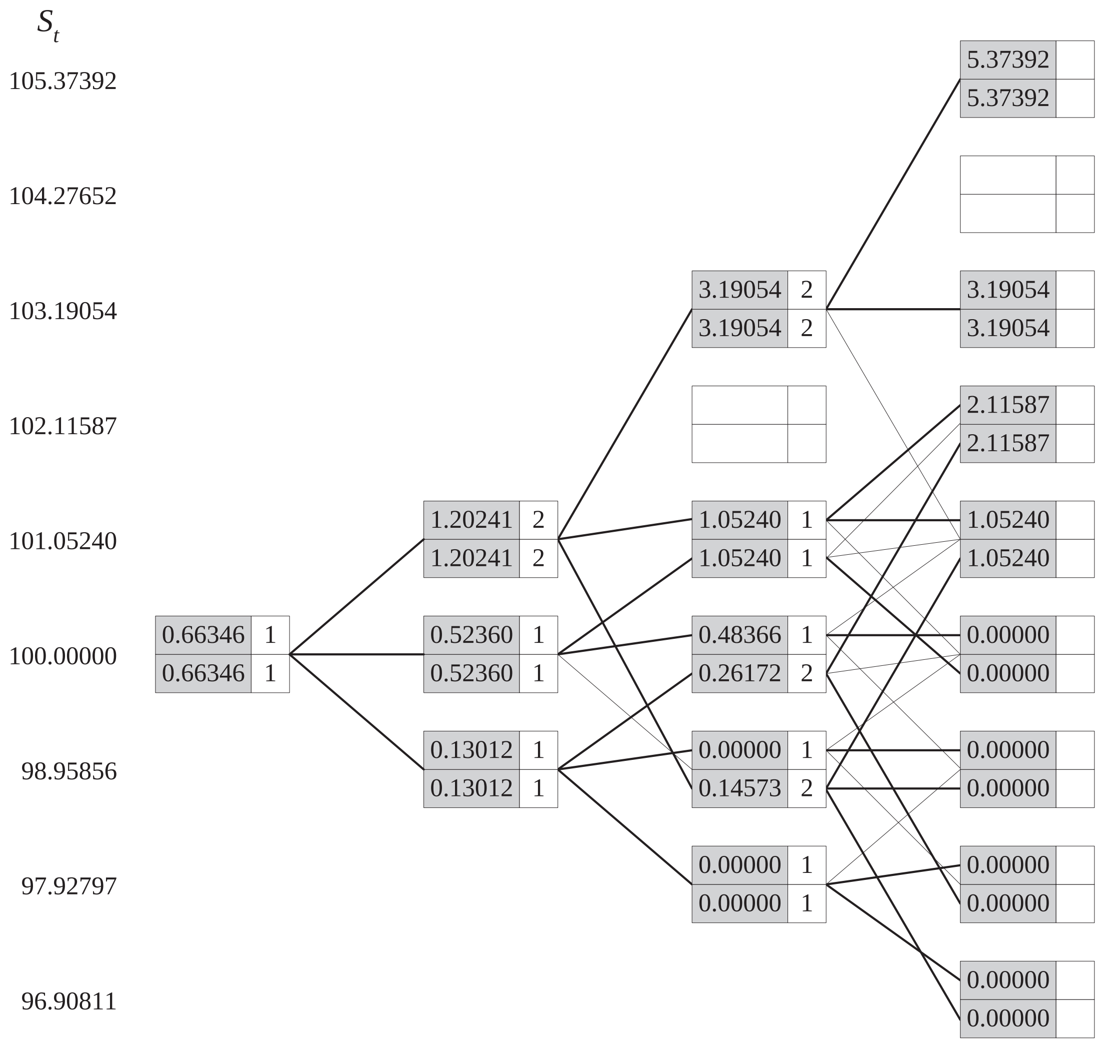

y_t
4.65752
4.64705
4.63658
4.62611
4.61564
0.01047
4.60517
4.59470
4.58423
4.57376
10.9600 1
10.9600 1
10.9645 2
10.9645 2
10.5215 1
10.5215 1
10.9553 1
10.9553 1
12.2883 2
12.2883 2
10.5256 1
10.5697 1
10.1269 1
10.9603 2
10.5173 1
12.2700 2
10.9511 1
10.9511 1
13.4809
13.4809
11.7170
11.7170
10.5733
12.2846
10.1305
13.4644
09.7717
10.6042
10.1231
11.7005
10.5135
12.2662
10.9473
13.4438

Backward Induction on the RT Tree (concluded)

- Suppose a variance falls between two of the K variances during backward induction.
- Linear interpolation of the option prices corresponding to the two bracketing variances will be used as the approximate option price.
- The above ideas are reminiscent of the ones on p. 391, where we dealt with Asian options.

Numerical Examples

- We next use the numerical example on p. 881 to price a European call option with a strike price of 100 and expiring at date 3.
- Recall that the riskless interest rate is zero.
- Assume $K = 2$; hence there are no interpolated variances.
- The pricing tree is shown on p. 884 with a call price of 0.66346.
  - The branching probabilities needed in backward induction can be found on p. 885.


S_t
105.37392
104.27652
103.19054
102.11587
101.05240
100.00000
98.95856
97.92797
96.90811
0.66346 1
0.66346 1
1.20241 2
1.20241 2
0.52360 1
0.52360 1
0.13012 1
0.13012 1
3.19054 2
3.19054 2
1.05240 1
1.05240 1
0.48366 1
0.26172 2
0.00000 1
0.14573 2
0.00000 1
0.00000 1
5.37392
5.37392
3.19054
3.19054
2.11587
2.11587
1.05240
1.05240
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000

rb[i][0]
rb[i][1]

rb[0][ ]	rb[1][ ]	rb[2][ ]	rb[3][ ]
0 0	–1 1	–2 3	–3 5

h^2[i][j][0]
h^2[i][j][1]

j	h^2[0][ ][ ]	h^2[1][ ][ ]	h^2[2][ ][ ]	h^2[3][ ][ ]
5				13.4809 13.4809
4				
3			12.2883 12.2883	11.7170 11.7170
2				10.5733 12.2846
1		10.9645 10.9645	10.5256 10.5697	10.1305 13.4644
0	10.9600 10.9600	10.5215 10.5215	10.1269 10.9603	09.7717 10.6042
–1		10.9553 10.9553	10.5173 12.2700	10.1231 11.7005
–2			10.9511 10.9511	10.5135 12.2662
–3				10.9473 13.4438

η[i][j][0]
η[i][j][1]

j	η[0][ ][ ]	η[1][ ][ ]	η[2][ ][ ]
3			2 2
2			
1		2 2	1 1
0	1 1	1 1	1 2
–1		1 1	1 2
–2			1 1

p[i][j][0][1]	p[i][j][1][1]
p[i][j][0][0]	p[i][j][1][0]
p[i][j][0][–1]	p[i][j][1][–1]

j	p[0][ ][ ][ ]	p[1][ ][ ][ ]	p[2][ ][ ][ ]
3			0.1387 0.1387 0.7197 0.7197 0.1416 0.1416
2			
1		0.1237 0.1237 0.7499 0.7499 0.1264 0.1264	0.4777 0.4797 0.0396 0.0356 0.4827 0.4847
0	0.4974 0.4974 0.0000 0.0000 0.5026 0.5026	0.4775 0.4775 0.0400 0.0400 0.4825 0.4825	0.4596 0.1237 0.0760 0.7500 0.4644 0.1263
–1		0.4972 0.4972 0.0004 0.0004 0.5024 0.5024	0.4773 0.1385 0.0404 0.7201 0.4823 0.1414
–2			0.4970 0.4970 0.0008 0.0008 0.5022 0.5022

Numerical Examples (continued)

- Let us derive some of the numbers on p. 884.
- A gray line means the updated variance falls strictly between $h^2_{\max}$ and $h^2_{\min}$.
- The option price for a terminal node at date 3 equals $\max(S_3 - 100, 0)$, independent of the variance level.
- Now move on to nodes at date 2.
- The option price at node $(2, 3)$ depends on those at nodes $(3, 5)$, $(3, 3)$, and $(3, 1)$.
- It therefore equals

$$0.1387 \times 5.37392 + 0.7197 \times 3.19054 + 0.1416 \times 1.05240 = 3.19054.$$

Numerical Examples (continued)

- Option prices for other nodes at date 2 can be computed similarly.
- For node $(1, 1)$, the option price for both variances is

$$0.1237 \times 3.19054 + 0.7499 \times 1.05240 + 0.1264 \times 0.14573 = 1.20241.$$

- Node $(1, 0)$ is most interesting.
- We knew that a down move from it gives a variance of 0.000105609.
- This number falls between the minimum variance 0.000105173 and the maximum variance 0.0001227 at node $(2, -1)$ on p. 881.

Numerical Examples (continued)

- The option price corresponding to the minimum variance is 0.
- The option price corresponding to the maximum variance is 0.14573.
- The equation

$$x \times 0.000105173 + (1 - x) \times 0.0001227 = 0.000105609$$

is satisfied by $x = 0.9751$.

- So the option for the down state is approximated by

$$x \times 0 + (1 - x) \times 0.14573 = 0.00362.$$

Numerical Examples (continued)

- The up move leads to the state with option price 1.05240.
- The middle move leads to the state with option price 0.48366.
- The option price at node $(1, 0)$ is finally calculated as

$$0.4775 \times 1.05240 + 0.0400 \times 0.48366 + 0.4825 \times 0.00362 = 0.52360.$$

Numerical Examples (continued)

- A variance following an interpolated variance may exceed the maximum variance or be exceeded by the minimum variance.
- When this happens, the option price corresponding to the maximum or minimum variance will be used during backward induction.[a]

---

[a] Cakici and Topyan (2000).

Numerical Examples (concluded)

- But an interpolated variance may choose a branch that goes into a node that is *not* reached in forward induction.[a]
- In this case, the algorithm fails.
- The Ritchken-Trevor algorithm does not have this problem as all interpolated variances are involved in the forward-induction phase.
- It may be hard to calculate the implied β_1 and β_2 from option prices.[b]

---

[a] Lyuu and Wu (`R90723065`) (2005).

[b] Chang (`R93922034`) (2006).

Complexities of GARCH Models[a]

- The Ritchken-Trevor algorithm explodes exponentially if n is big enough (p. 857).
- The mean-tracking tree of Lyuu and Wu (2005) makes sure explosion does not happen if n is not too large.[b]
- The next page summarizes the situations for many GARCH option pricing models.
  - Our earlier treatment is for NGARCH only.

---

[a] Lyuu and Wu (`R90723065`) (2003, 2005).

[b] Similar to, but earlier than, the binomial-trinomial tree on pp. 667ff.

Complexities of GARCH Models (concluded)[a]

Model	Explosion	Non-explosion
NGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda + c)^2 \leq 1$
LGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda)^2 \leq 1$
AGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda)^2 \leq 1$
GJR-GARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + (\beta_2 + \beta_3)(\sqrt{n} + \lambda)^2 \leq 1$
TS-GARCH	$\beta_1 + \beta_2 \sqrt{n} > 1$	$\beta_1 + \beta_2(\lambda + \sqrt{n}) \leq 1$
TGARCH	$\beta_1 + \beta_2 \sqrt{n} > 1$	$\beta_1 + (\beta_2 + \beta_3)(\lambda + \sqrt{n}) \leq 1$
Heston-Nandi	$\beta_1 + \beta_2(c - \frac{1}{2})^2 > 1$ & $c \leq \frac{1}{2}$	$\beta_1 + \beta_2 c^2 \leq 1$
VGARCH	$\beta_1 + (\beta_2/4) > 1$	$\beta_1 \leq 1$

---

[a] Chen (`R95723051`) (2008); Chen (`R95723051`), Lyuu, and Wen (`D94922003`) (2012).

Introduction to Term Structure Modeling

The fox often ran to the hole
by which they had come in,
to find out if his body was still thin enough
to slip through it.
— *Grimm's Fairy Tales*

And the worst thing you can have
is models and spreadsheets.
— Warren Buffet, May 3, 2008

Outline

- Use the binomial interest rate tree to model stochastic term structure.
  - Illustrates the basic ideas underlying future models.
  - Applications are generic in that pricing and hedging methodologies can be easily adapted to other models.
- Although the idea is similar to the earlier one used in option pricing, the current task is more complicated.
  - The evolution of an entire term structure, not just a single stock price, is to be modeled.
  - Interest rates of various maturities cannot evolve arbitrarily, or arbitrage profits may occur.

Issues

- A stochastic interest rate model performs two tasks.
  - Provides a stochastic process that defines future term structures without arbitrage profits.
  - "Consistent" with the observed term structures.

History

- Methodology founded by Merton (1970).
- Modern interest rate modeling is often traced to 1977 when Vasicek and Cox, Ingersoll, and Ross developed simultaneously their influential models.
- Early models have fitting problems because they may not price today's benchmark bonds correctly.
- An alternative approach pioneered by Ho and Lee (1986) makes fitting the market yield curve mandatory.
- Models based on such a paradigm are called (somewhat misleadingly) arbitrage-free or no-arbitrage models.

Binomial Interest Rate Tree

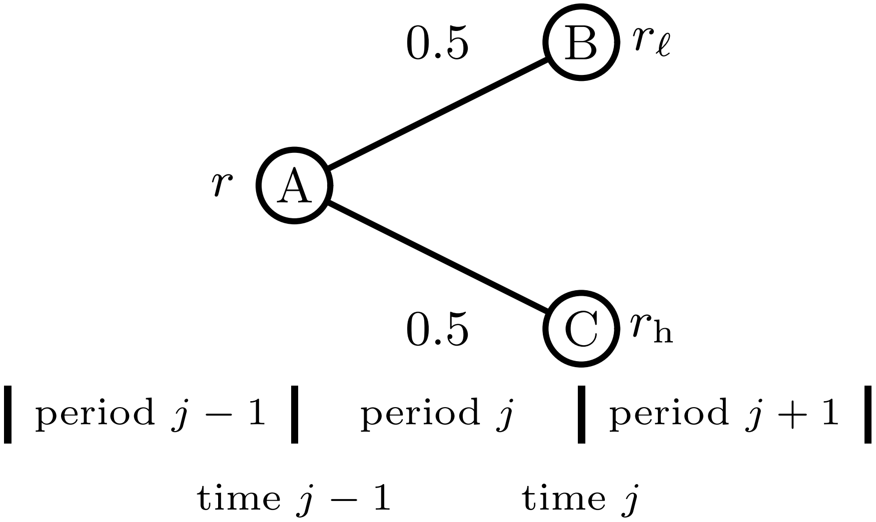
- Goal is to construct a no-arbitrage interest rate tree consistent with the yields and/or yield volatilities of zero-coupon bonds of all maturities.
  - This procedure is called calibration.[a]
- Pick a binomial tree model in which the logarithm of the future short rate obeys the binomial distribution.
  - Exactly like the CRR tree.
- The limiting distribution of the short rate at any future time is hence lognormal.

---

[a] Derman (2004), "complexity without calibration is pointless."

Binomial Interest Rate Tree (continued)

- A binomial tree of future short rates is constructed.
- Every short rate is followed by two short rates in the following period (p. 902).
- In the figure on p. 902, node A coincides with the start of period j during which the short rate r is in effect.
- At the conclusion of period j, a new short rate goes into effect for period $j + 1$.


0.5
B
r_ℓ
r
A
0.5
C
$r_{\rm h}$
period $j-1$
period j
period $j+1$
time $j-1$
time j

Binomial Interest Rate Tree (continued)

- This may take one of two possible values:
  - r_ℓ: the "low" short-rate outcome at node B.
  - r_h: the "high" short-rate outcome at node C.
- Each branch has a 50% chance of occurring in a risk-neutral economy.
- We require that the paths combine as the binomial process unfolds.
- This model can be traced to Salomon Brothers.[a]

---

[a] Tuckman (2002).

Binomial Interest Rate Tree (continued)

- The short rate r can go to r_{h} and r_{ℓ} with equal risk-neutral probability $1/2$ in a period of length Δt.
- Hence the volatility of $\ln r$ after Δt time is
$$\sigma = \frac{1}{2}\,\frac{1}{\sqrt{\Delta t}}\,\ln\left(\frac{r_{\mathrm{h}}}{r_{\ell}}\right) \tag{107}$$
(see Exercise 23.2.3 in text).
- Above, σ is annualized, whereas r_{ℓ} and r_{h} are period based.

Binomial Interest Rate Tree (continued)

- Note that
$$\frac{r_{\mathrm{h}}}{r_\ell} = e^{2\sigma\sqrt{\Delta t}}.$$
- Thus greater volatility, hence uncertainty, leads to larger r_{h}/r_ℓ and wider ranges of possible short rates.
- The ratio r_{h}/r_ℓ may depend on time if the volatility is a function of time.
- Note that r_{h}/r_ℓ has nothing to do with the current short rate r if σ is independent of r.

Binomial Interest Rate Tree (continued)

- In general there are j possible rates[a] in period j,

  $$r_j,\, r_j v_j,\, r_j v_j^2,\, \ldots,\, r_j v_j^{j-1},$$

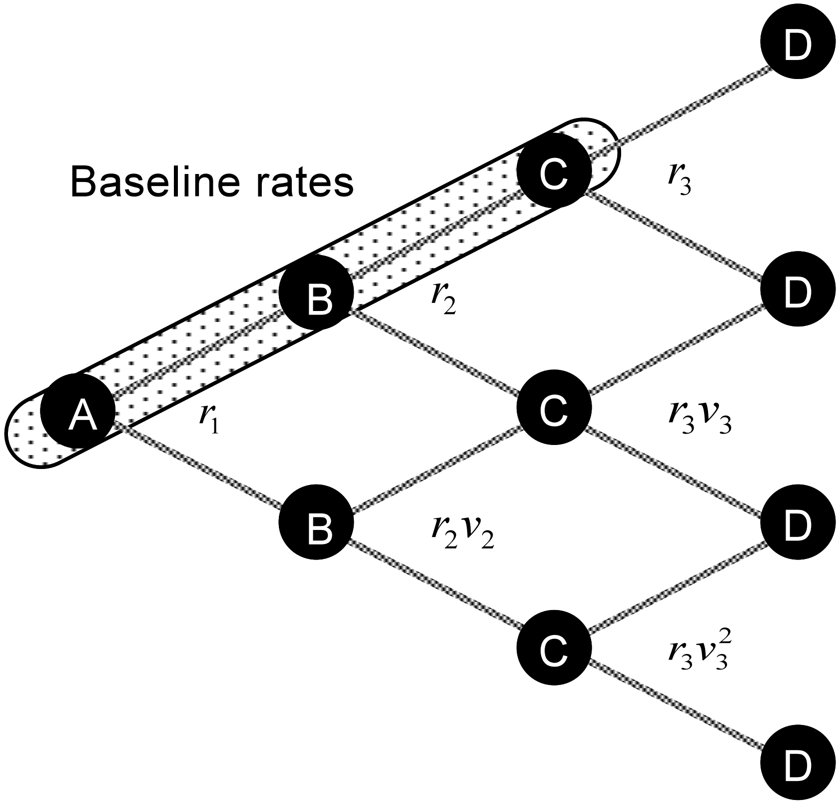
  where

  $$v_j \equiv e^{2\sigma_j \sqrt{\Delta t}} \tag{108}$$

  is the multiplicative ratio for the rates in period j (see figure on next page).
- We shall call r_j the baseline rates.
- The subscript j in σ_j is meant to emphasize that the short rate volatility may be time dependent.

---

[a] Not $j+1$.


Baseline rates
A
B
B
C
C
C
D
D
D
D
r_1
r_2
$r_2 v_2$
r_3
$r_3 v_3$
$r_3 v_3^2$

Binomial Interest Rate Tree (concluded)

- In the limit, the short rate follows the following process,

$$r(t) = \mu(t)\, e^{\sigma(t)\, W(t)}, \tag{109}$$

in which the (percent) short rate volatility $\sigma(t)$ is a deterministic function of time.

- The expected value of $r(t)$ equals $\mu(t)\, e^{\sigma(t)^2 (t/2)}$.
- Hence a declining short rate volatility is usually imposed to preclude the short rate from assuming implausibly high values.
- Incidentally, this is how the binomial interest rate tree achieves mean reversion.

Memory Issues

- Path independency: The term structure at any node is independent of the path taken to reach it.
- So only the baseline rates r_i and the multiplicative ratios v_i need to be stored in computer memory.
- This takes up only $O(n)$ space.[a]
- Storing the whole tree would take up $O(n^2)$ space.
  - Daily interest rate movements for 30 years require roughly $(30 \times 365)^2/2 \approx 6 \times 10^7$ double-precision floating-point numbers (half a gigabyte!).

---

[a] Throughout, n denotes the depth of the tree.

Set Things in Motion

- The abstract process is now in place.
- We need the annualized rates of return of the riskless bonds that make up the benchmark yield curve and their volatilities.
- In the U.S., for example, the on-the-run yield curve obtained by the most recently issued Treasury securities may be used as the benchmark curve.

Set Things in Motion (concluded)

- The term structure of (yield) volatilities[a] can be estimated from:
  - Historical data (historical volatility).
  - Or interest rate option prices such as cap prices (implied volatility).
- The binomial tree should be found that is consistent with both term structures.
- Here we focus on the term structure of interest rates.

---

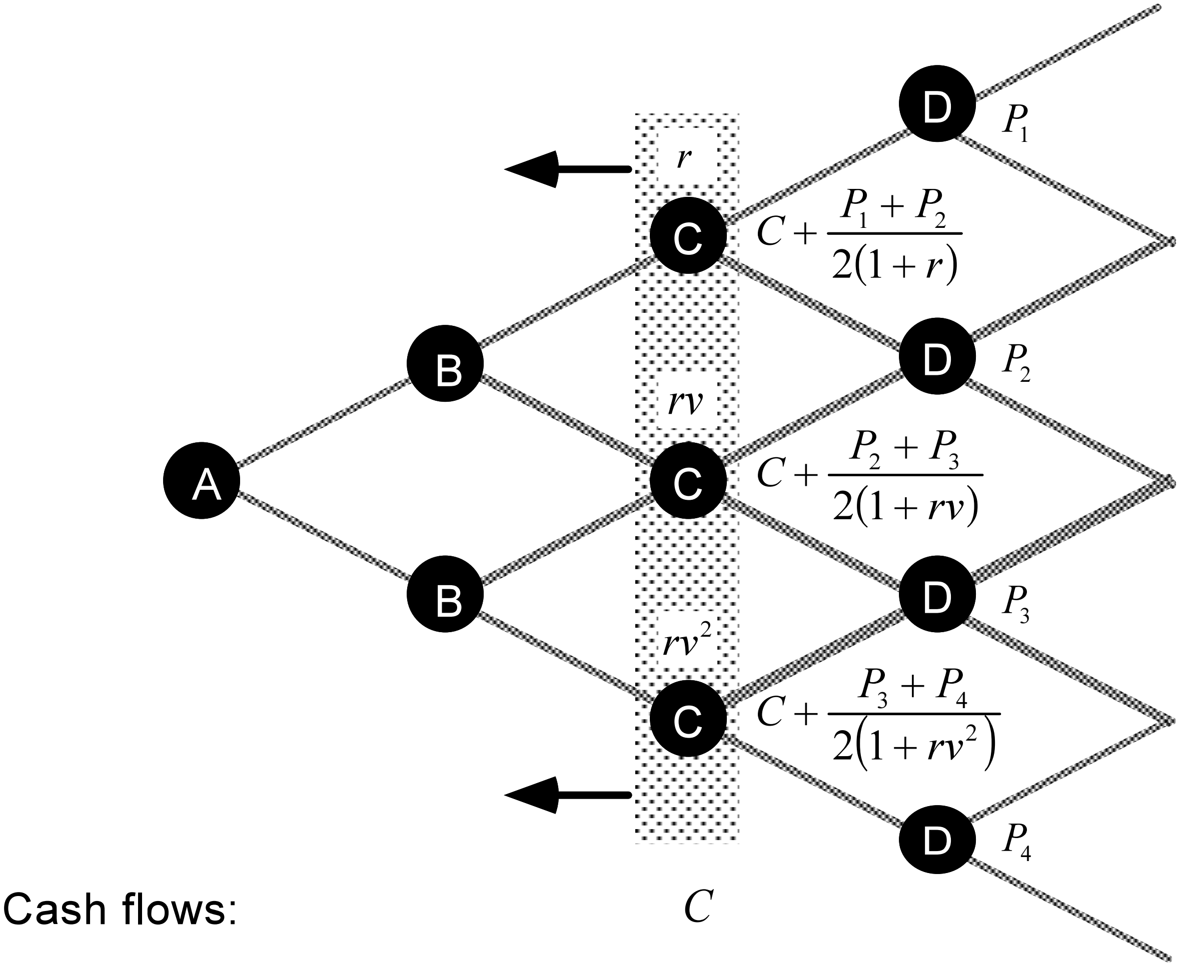
[a] Or simply the volatility (term) structure.

Model Term Structures

- The model price is computed by backward induction.
- Refer back to the figure on p. 902.
- Given that the values at nodes B and C are P_{B} and P_{C}, respectively, the value at node A is then

$$\frac{P_{\mathrm{B}} + P_{\mathrm{C}}}{2(1+r)} + \textsf{cash flow at node } \mathrm{A}.$$

- We compute the values column by column without explicitly expanding the binomial interest rate tree (see next page).
- This takes $O(n^2)$ time and $O(n)$ space.


D
P_1
r
C
$C+\frac{P_1+P_2}{2(1+r)}$
B
D
P_2
rv
A
C
$C+\frac{P_2+P_3}{2(1+rv)}$
B
D
P_3
rv^2
C
$C+\frac{P_3+P_4}{2\left(1+rv^2\right)}$
D
P_4
Cash flows:
C

Term Structure Dynamics

- An n-period zero-coupon bond's price can be computed by assigning \$1 to every node at period n and then applying backward induction.
- Repeating this step for $n = 1, 2, \ldots$, one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics.
  - Taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.

Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
  - This was called calibration (the reverse of pricing).
- Assume the short rate volatility is such that

$$v \equiv \frac{r_{\rm h}}{r_\ell} = 1.5,$$

independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let f_j denote the forward rate in period j.
- This forward rate can be derived from the market discount function via

$$f_j = \frac{d(j)}{d(j+1)} - 1$$

(see Exercise 5.6.3 in text).

An Approximate Calibration Scheme (continued)

- Since the ith short rate $r_j v_j^{i-1}$, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left(\frac{2}{1+v_j} \right)^{j-1} f_j. \tag{110}$$

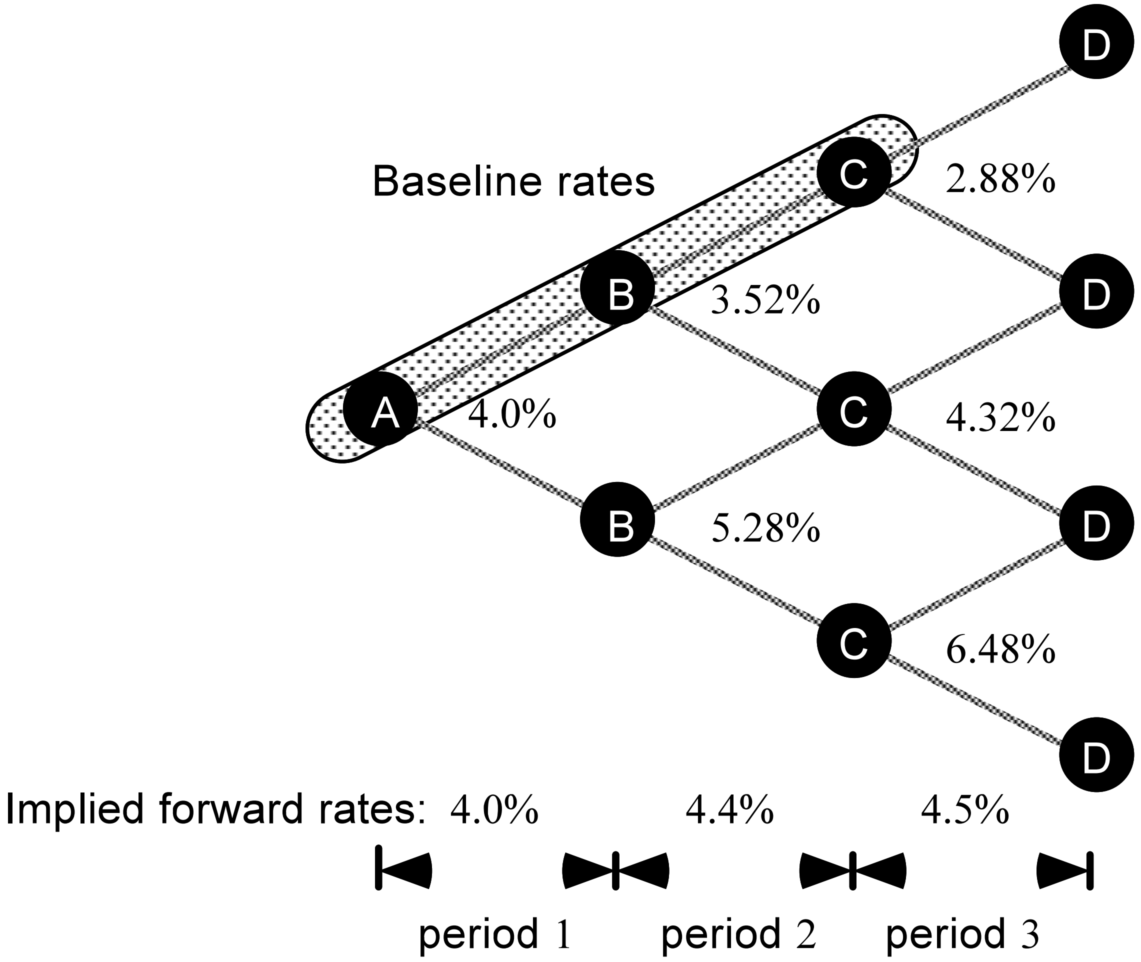
- This binomial interest rate tree is trivial to set up, in $O(n)$ time.

An Approximate Calibration Scheme (continued)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

  $$\frac{1}{4} \times \frac{1}{1.04} \times \left(\frac{1}{1.0352} \times \left(\frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left(\frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

  or 0.88155, which exceeds discount factor 0.88135.
- The tree is thus *not* calibrated.


D
Baseline rates
C
2.88%
B
3.52%
D
A
4.0%
C
4.32%
B
5.28%
D
C
6.48%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree overprices the bonds (see Exercise 23.2.4 in text).
- Suppose we replace the baseline rates r_j by $r_j v_j$.
- Then the resulting tree underprices the bonds.[a]
- The true baseline rates are thus bounded between r_j and $r_j v_j$.

---

[a] Lyuu and Wang (`F95922018`) (2009, 2011).

Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the m-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate r_m for period m.
- A root-finding method is applied to solve $f(r_m) = P$ for r_m given the zero's price P and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in $O(n^3)$ time.

Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.[a]
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price, the Arrow-Debreu price, or Green's function.
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 0 to time n.

---

[a] Jamshidian (1991).

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time j and there are $j+1$ nodes.
  - The unknown baseline rate for period j is $r \equiv r_j$.
  - The multiplicative ratio is $v \equiv v_j$.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at *earlier* time $j-1$, corresponding to rates $r, rv, \ldots, rv^{j-1}$ for period j.
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.
- We want to find r based on $P_1, P_2, \ldots, P_j$ and the price of the j-period zero-coupon bond.

Binomial Interest Rate Tree Calibration (continued)

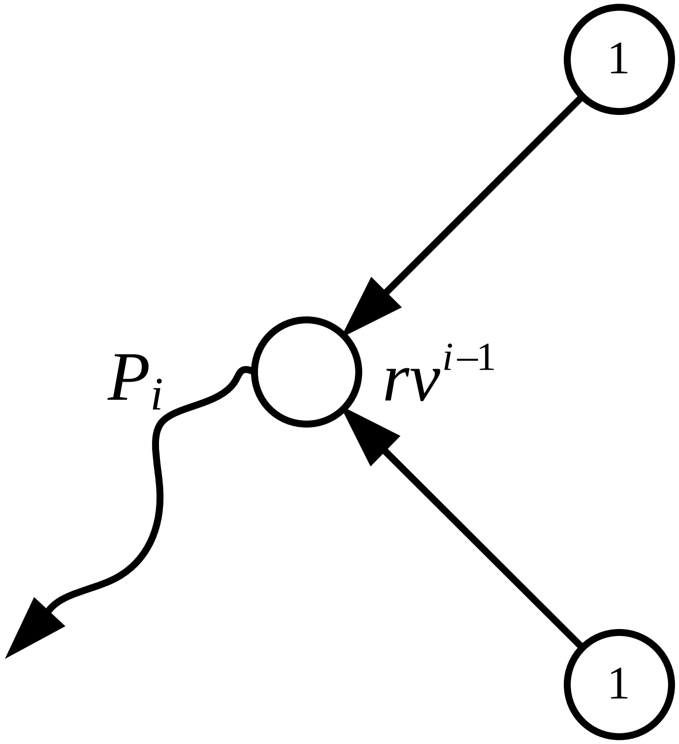
- One dollar at time j has a known market value of $1/[\,1 + S(j)\,]^j$, where $S(j)$ is the j-period spot rate.
- Alternatively, this dollar has a present value of

  $$g(r) \equiv \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}$$

  (see next plot).
- So we solve

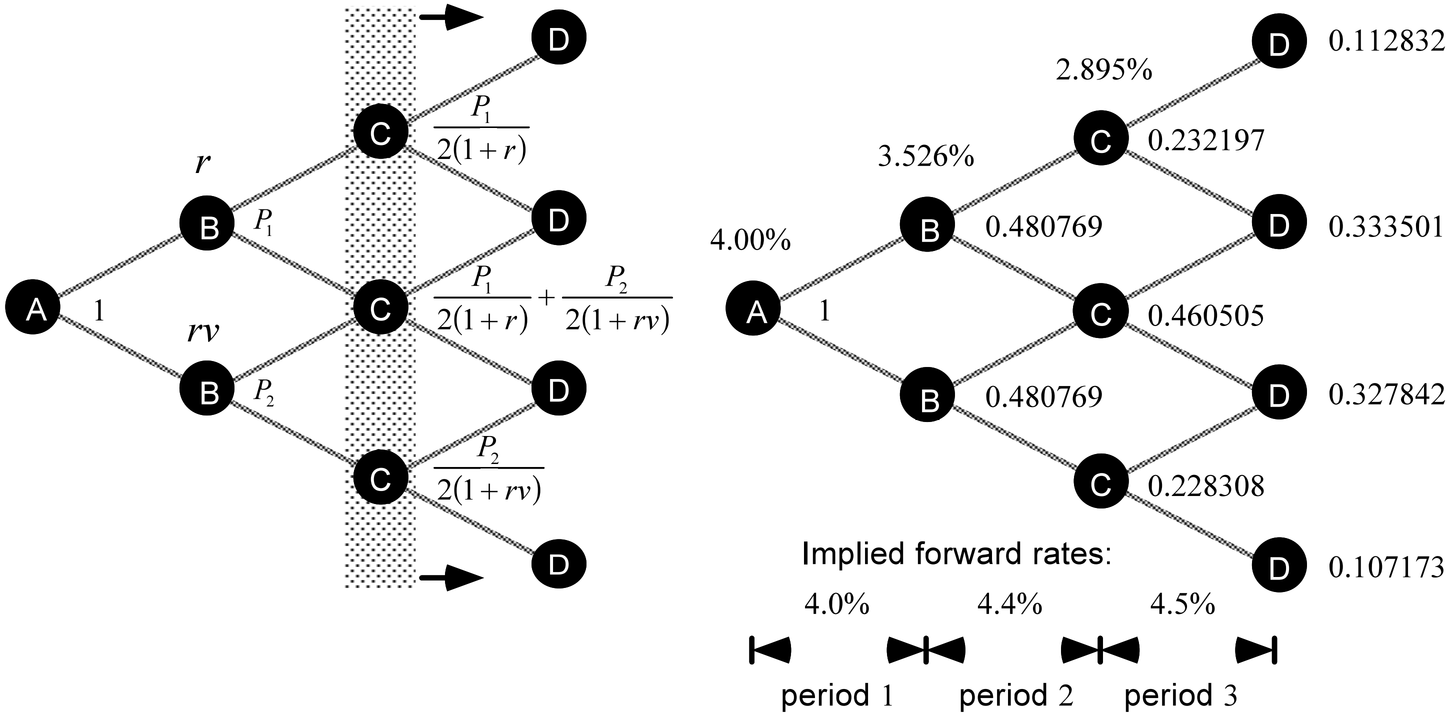
  $$g(r) = \frac{1}{[\,1 + S(j)\,]^j} \tag{111}$$

  for r.

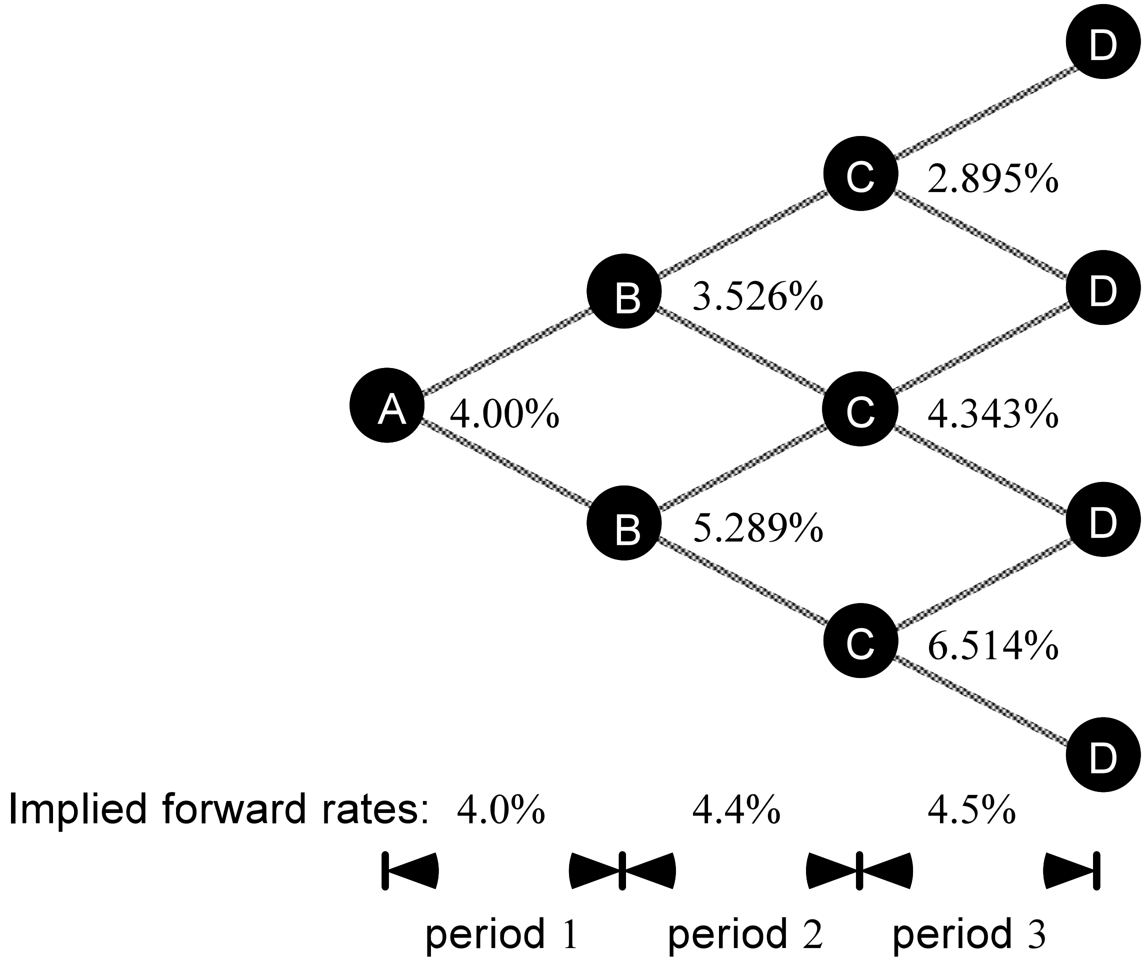

1
P_i
rv^{i-1}
1

Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for r is guaranteed.
- The state prices at time j can now be calculated (see panel (a) next page).
- We call a tree with these state prices a binomial state price tree (see panel (b) next page).
- The calibrated tree is depicted on p. 928.


A
B
C
D
1
r
rv
P_1
P_2
$\frac{P_1}{2(1+r)}$
$\frac{P_1}{2(1+r)} + \frac{P_2}{2(1+rv)}$
$\frac{P_2}{2(1+rv)}$
4.00%
3.526%
2.895%
0.480769
0.480769
0.232197
0.460505
0.228308
0.112832
0.333501
0.327842
0.107173
Implied forward rates:
4.0%
4.4%
4.5%
period 1
period 2
period 3

(a) (b)


D
C
2.895%
B
3.526%
D
A
4.00%
C
4.343%
B
5.289%
D
C
6.514%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the r in Eq. (111) on p. 924 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The total running time is $O(n^2)$, as each root-finding routine consumes $O(j)$ time.
- With a good initial guess,[a] the Newton-Raphson method converges in only a few steps.[b]

---

[a] Such as the $r_j = (\frac{2}{1+v_j})^{j-1} f_j$ on p. 917.

[b] Lyuu (1999).

A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, r_2, satisfies

$$\frac{0.480769}{1+r_2}+\frac{0.480769}{1+1.5\times r_2}=0.92101.$$

- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in panel (b) on p. 927 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

A Numerical Example (concluded)

- The baseline rate for the third period, r_3, satisfies

$$\frac{0.232197}{1+r_3}+\frac{0.460505}{1+1.5\times r_3}+\frac{0.228308}{1+(1.5)^2\times r_3}=0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 918 using the new rates:

$$\frac{1}{4}\times\frac{1}{1.04}\times[\frac{1}{1.03526}\times(\frac{1}{1.02895}+\frac{1}{1.04343})+\frac{1}{1.05289}\times(\frac{1}{1.04343}+\frac{1}{1.06514})],$$

  which equals 0.88135, an exact match.

- The tree on p. 928 prices without bias the benchmark securities.

Spread of Nonbenchmark Bonds

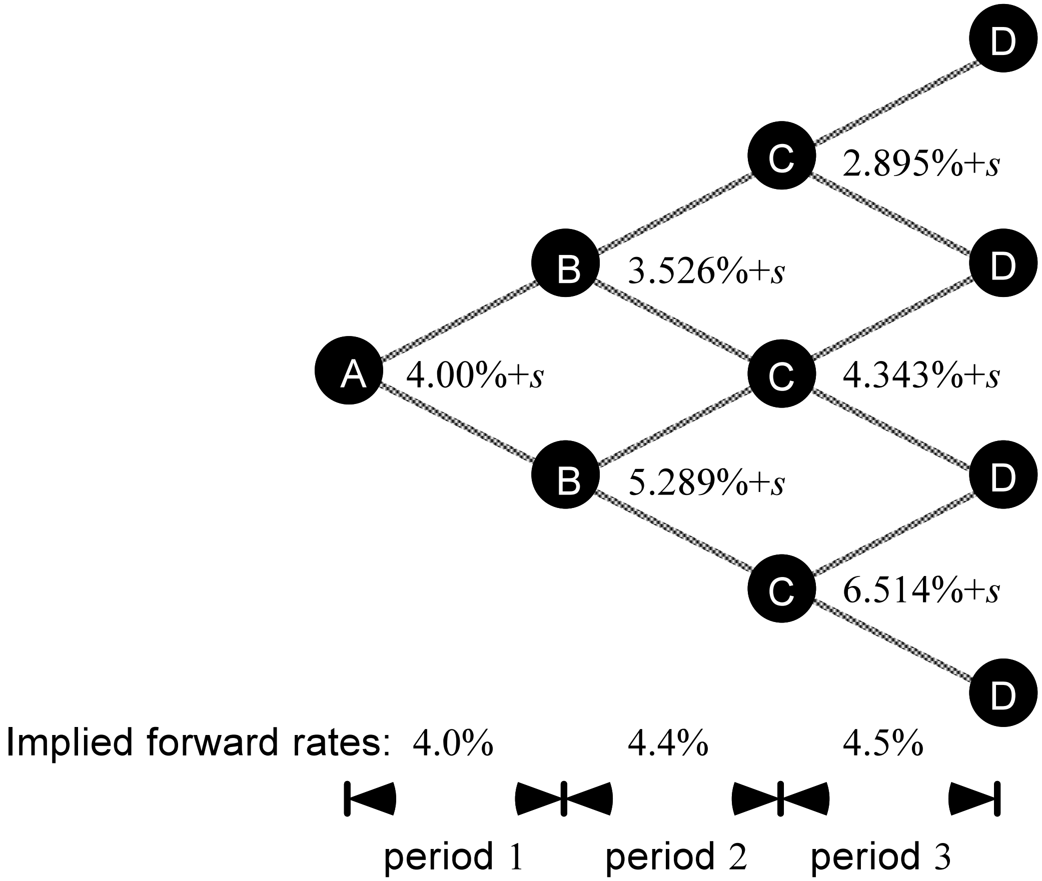
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 934.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \right.$$
$$\left. \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.


D
C
2.895%+s
B
3.526%+s
D
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- We will employ the Newton-Raphson root-finding method to solve
$$p(s) - P = 0$$
for s.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

Spread of Nonbenchmark Bonds (continued)

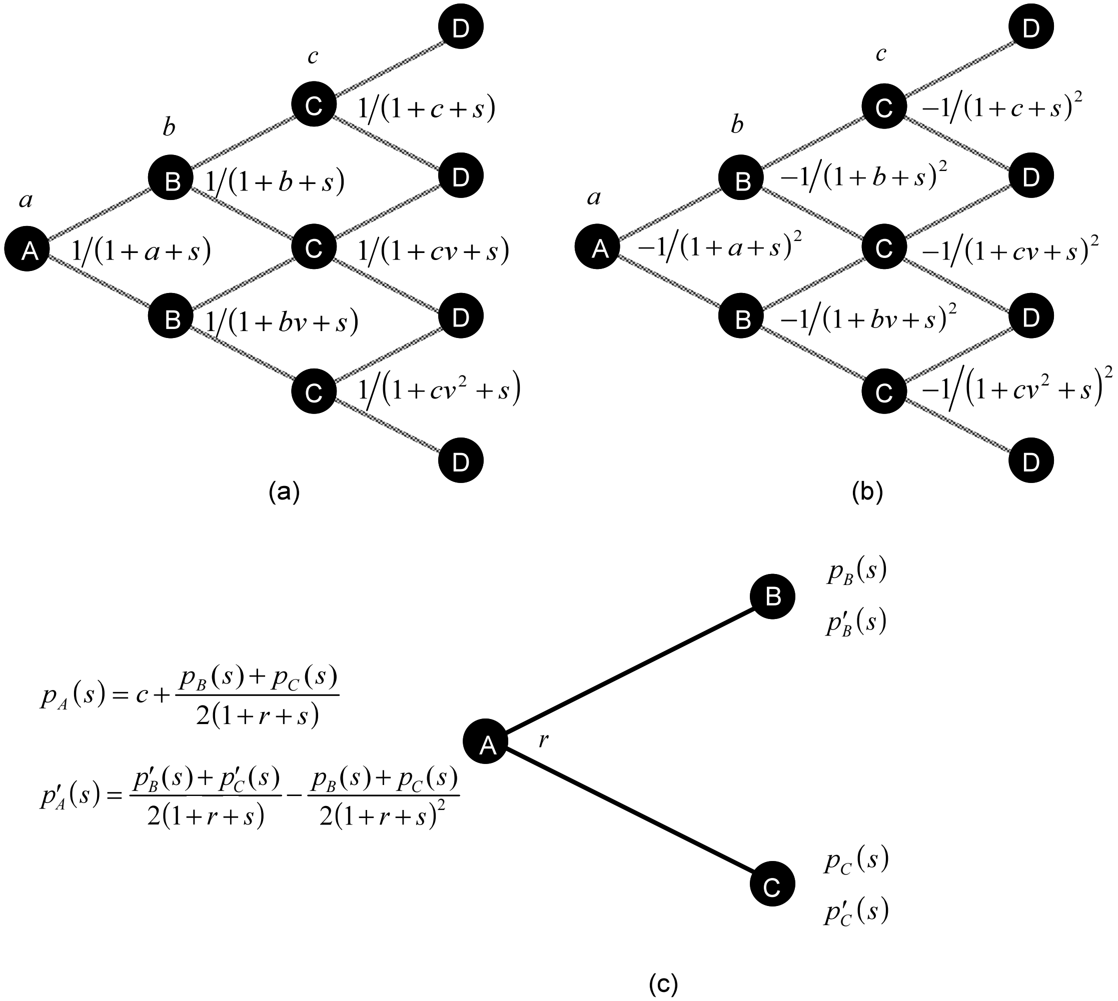
- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1+r+s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1+r+s)^2}. \tag{112}$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- When A is a terminal node, simply use the payoff function for $p_{\mathrm{A}}(s)$.[a]

---

[a] Contributed by Mr. Chou, Ming-Hsin (`R02723073`) on May 28, 2014.



$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1+r+s)}$$

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1+r+s)} - \frac{p_B(s) + p_C(s)}{2(1+r+s)^2}$$

Spread of Nonbenchmark Bonds (continued)

- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 938).
- This is called the differential tree method.[a]
  - Similar ideas can be found in automatic differentiation (AD)[b] and backpropagation[c] in artificial neural networks.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n)$.

---

[a] Lyuu (1999).

[b] Rall (1981).

[c] Rumelhart, Hinton, and Williams (1986).

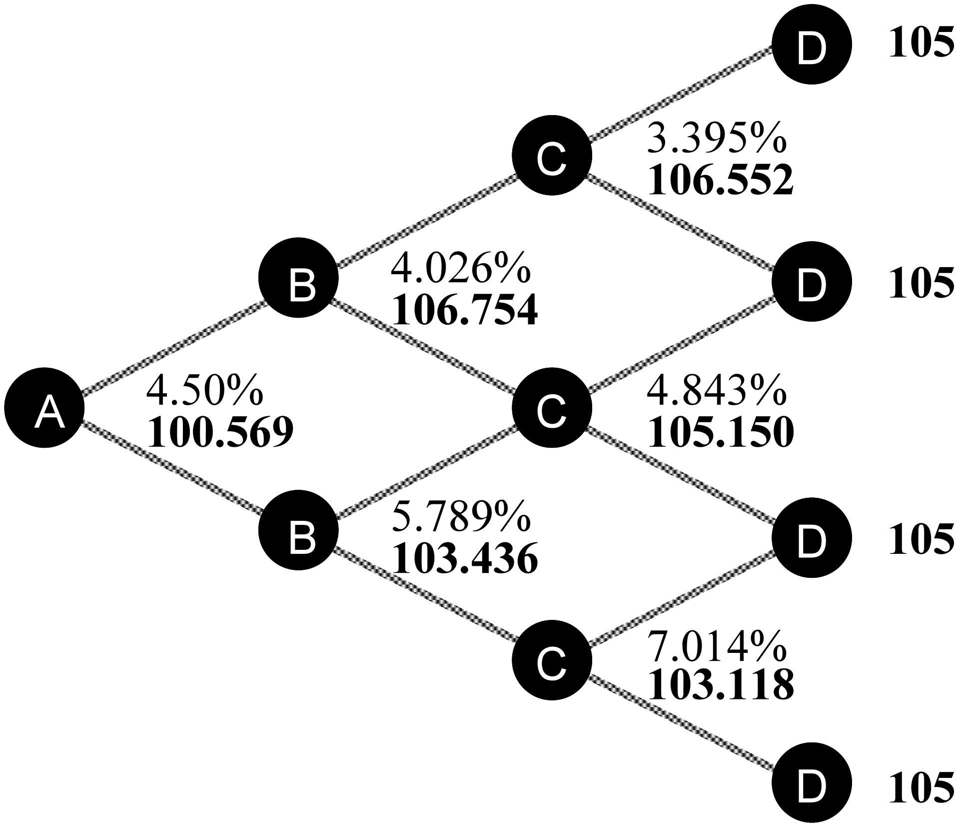
Spread of Nonbenchmark Bonds (continued)

Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 942).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 118) and static spread (p. 119) of the nonbenchmark bond over an otherwise identical benchmark bond.



Cash flows:	5	5	105

More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)[a]

American call			**American put**		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

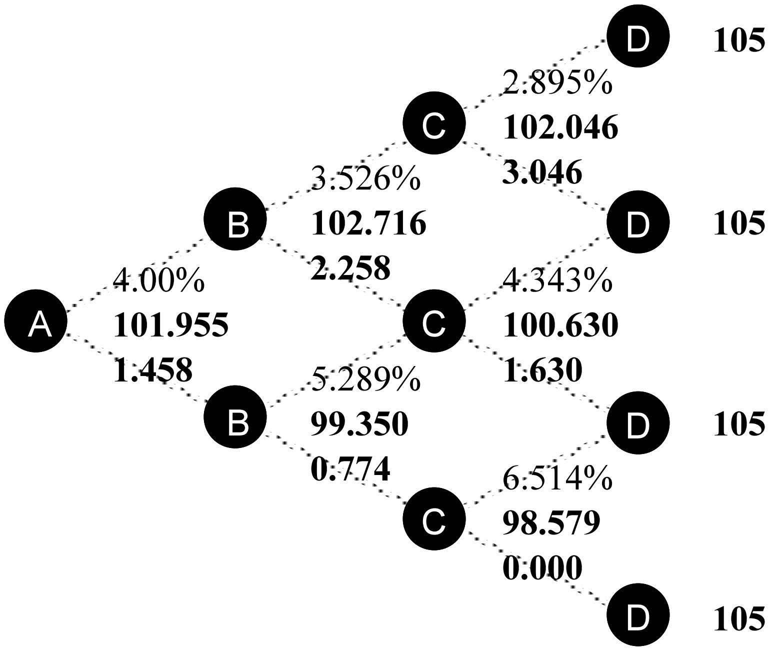
Intel 166MHz Pentium, running on Microsoft Windows 95.

---

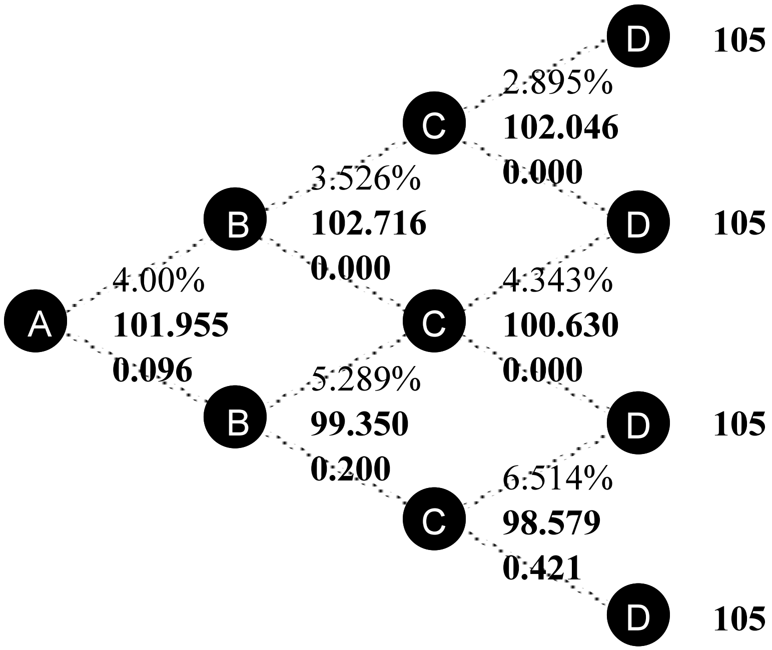
[a] Lyuu (1999).

Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 945 the three-year Treasury's price minus the $5 interest at year 2 could be $102.046, $100.630, or $98.579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of $3.046 and $1.630, and out of the money in the third scenario.


D
105
C
2.895%
102.046
3.046
B
3.526%
102.716
2.258
D
105
A
4.00%
101.955
1.458
C
4.343%
100.630
1.630
B
5.289%
99.350
0.774
D
105
C
6.514%
98.579
0.000
D
105

(a)


D
105
C
2.895%
102.046
0.000
B
3.526%
102.716
0.000
D
105
A
4.00%
101.955
0.096
C
4.343%
100.630
0.000
B
5.289%
99.350
0.200
D
105
C
6.514%
98.579
0.421
D
105

(b)

Fixed-Income Options (continued)

- The option value is calculated to be $1.458 on p. 945(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth $98.579 without the accrued interest.
- The option value is computed to be $0.096 on p. 945(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\text{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\text{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \text{PV}(I) - \text{PV}(X).$$

Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above P_{h} and P_{ℓ} denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_{ℓ} denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 945 as examples.
- Their deltas are

$$\begin{aligned} \frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441, \\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059, \end{aligned}$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_ℓ, respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as

$$\frac{1}{2} \ln\left(\frac{y_{\mathrm{h}}}{y_\ell}\right). \tag{113}$$

Volatility Term Structures (continued)

- For example, based on the tree on p. 928, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2}\ln\left(\frac{0.05289}{0.03526}\right)=20.273\%.$$

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

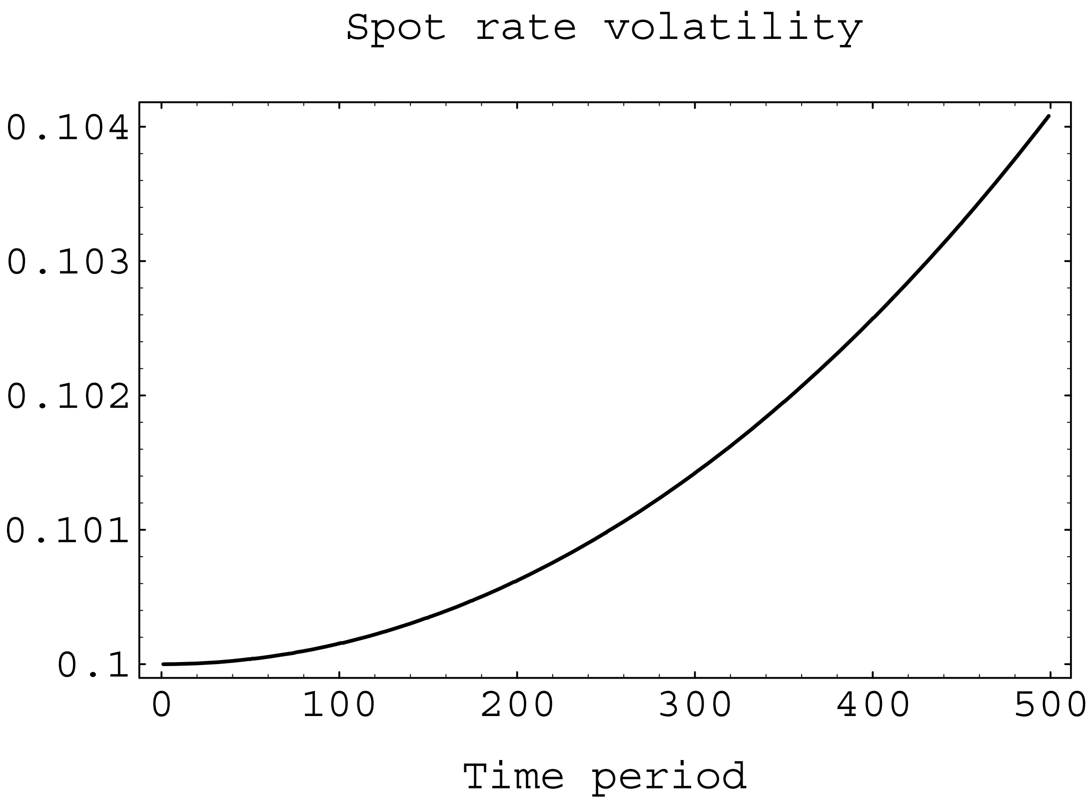
$$\frac{1}{2} \ln \left(\frac{0.053531}{0.0357} \right) = 20.256\%,$$

slightly less than the one-year yield volatility.

- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.[a]
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

---

[a] The relation is reversed for price volatilities (duration).



Short rate volatility given flat %10 volatility term structure.

Volatility Term Structures (concluded)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (108) on p. 906—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.[a]

---

[a] Black, Derman, and Toy (1990).