Tracking Error Revisited

- Define the dollar gamma as $S^2\Gamma$.
- The change in value of a delta-hedged *long* option position after a duration of Δt is proportional to the dollar gamma.
- It is about

$$(1/2)S^2\Gamma[\,(\Delta S/S)^2 - \sigma^2\Delta t\,].$$

  - $(\Delta S/S)^2$ is called the daily realized variance.

Tracking Error Revisited (continued)

- Let the rebalancing times be $t_1, t_2, \ldots, t_n$.
- Let $\Delta S_i = S_{i+1} - S_i$.
- The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[\left(\frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right].$$

- The tracking error is path dependent.
- It is also known that[a]

$$\sum_{i=0}^{n-1} \left(\frac{\Delta S_i}{S_i} \right)^2 \to \sigma^2 T.$$

[a] Protter (2005).

Tracking Error Revisited (concluded)[a]

- The tracking error ϵ_n *over* n *rebalancing acts* (such as 251,235 on p. 615) has about the same probability of being positive as being negative.
- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[b]
- The root-mean-square tracking error increases with σ at first and then decreases.

---

[a]Bertsimas, Kogan, and Lo (2000).
[b]Grannan and Swindle (1996).

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f / \partial S^2$, helps (theoretically).[a]
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

---

[a]See the numerical example on pp. 231–232 of the text.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

for n_1, n_2, and B.

  – The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 264.
- We will now apply it to price barrier options.

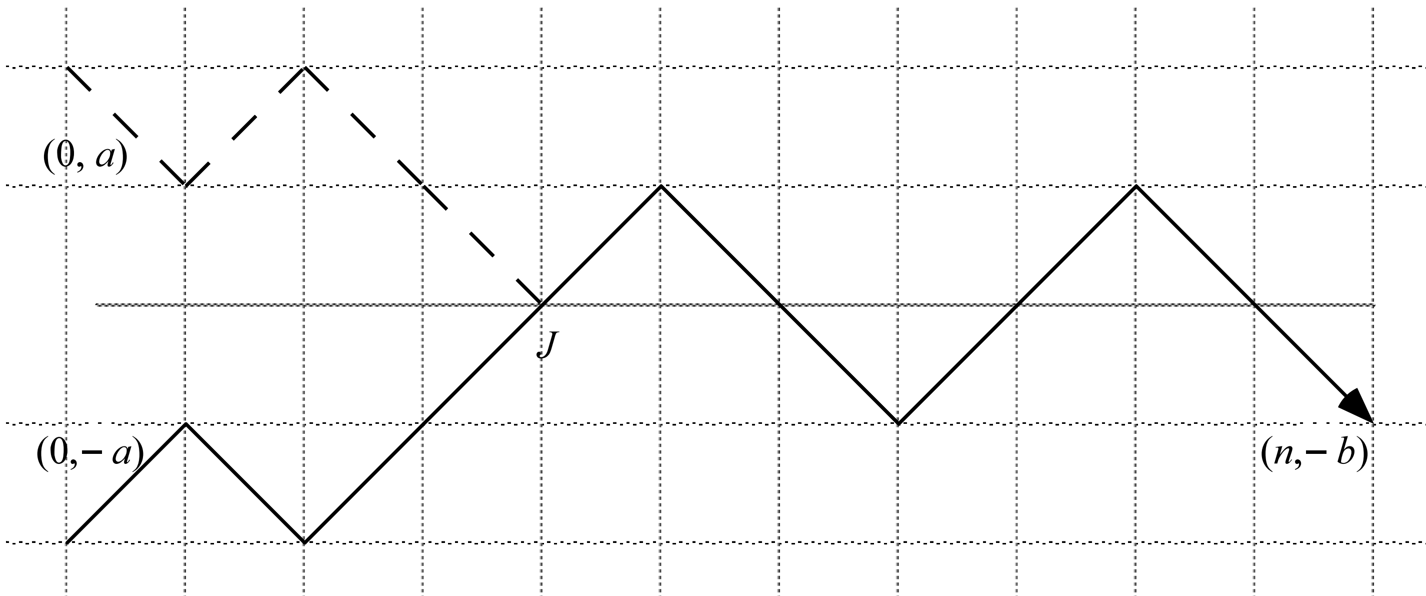
The Reflection Principle[a]

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$$
\begin{array}{lll}
 & (i+1, j+1) & \text{up move } S \to Su \\
(i, j) & & \\
 & (i+1, j-1) & \text{down move } S \to Sd
\end{array}
$$

- How many paths touch the x axis?

[a]André (1887).



(0, a)
J
(0,− a)
(n,− b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.[a]
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} = \binom{n}{\frac{n-\boldsymbol{a}-\boldsymbol{b}}{2}} \tag{68}$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

  - Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

[a] Verify it!

Pricing Barrier Options (Lyuu, 1998)

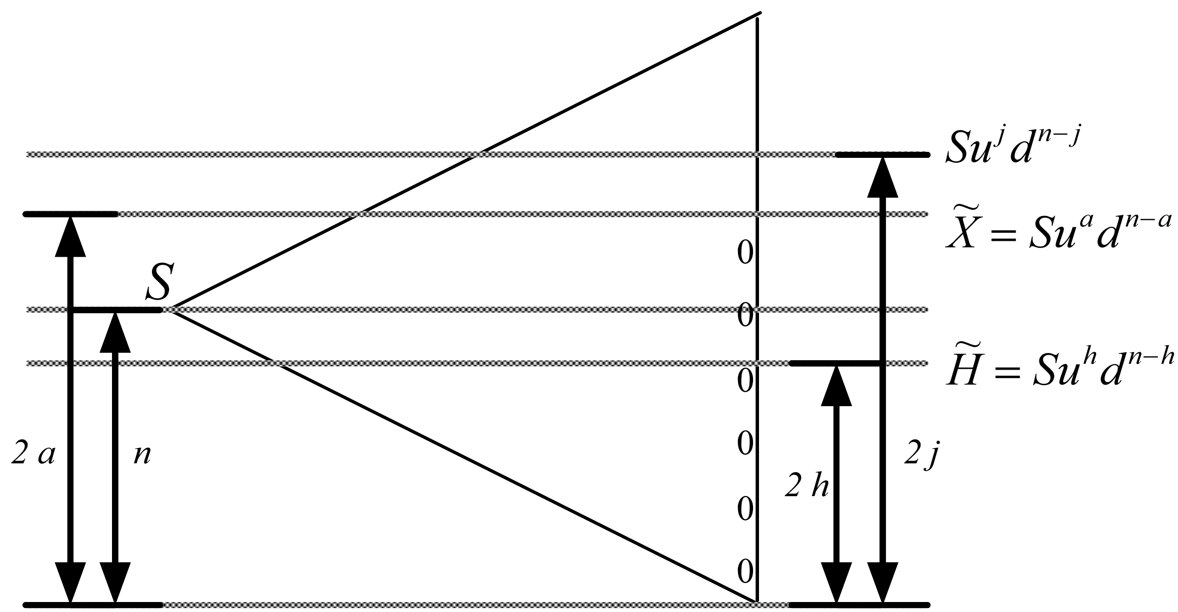
- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

$$
\begin{aligned}
a &\equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\
h &\equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.
\end{aligned}
$$

  - a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to X from above.
  - h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the *terminal* price that is closest to H from below.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  -
$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \qquad (69)$$



$Su^j d^{n-j}$
$\widetilde{X} = Su^a d^{n-a}$
S
$\widetilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j (1-p)^{n-j}$.
- The reflection principle (p. 631) can be applied with

$$\begin{aligned} \boldsymbol{a} &= n - 2h, \\ \boldsymbol{b} &= 2j - 2h, \end{aligned}$$

in Eq. (68) on p. 628 by treating the $\tilde{H}$ line as the x axis.

- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}, \quad j \leq 2h.$$

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X\right)}{R^n}. \tag{70}$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.

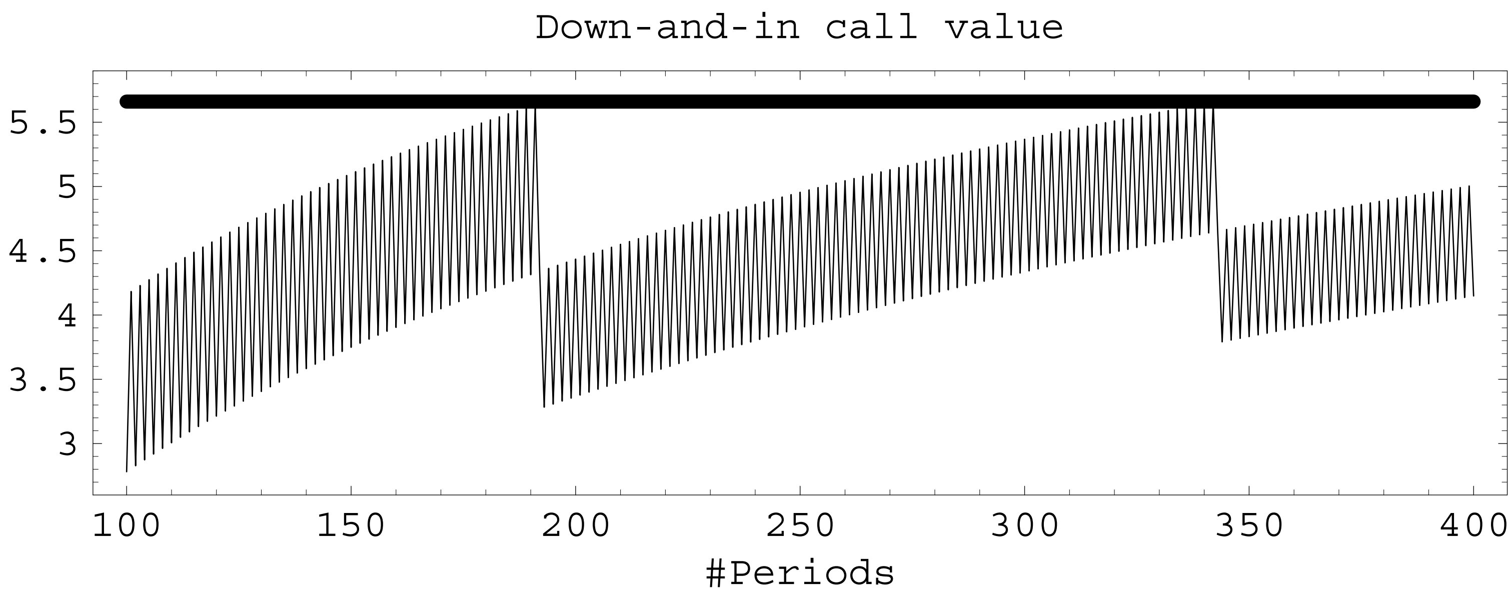
- It yields a linear-time algorithm.[a]

[a] Lyuu (1998).

Convergence of BOPM

- Equation (70) results in the sawtooth-like convergence shown on p. 364 (repeated on next page).
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds between the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)



Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or just above one of the stock price levels, that is,
$$H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$$
for some integer j.
- The preferred n's are thus
$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n+1$ possible *terminal* stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

---

[a] This is because $j = n - 2k$ for some k by Eq. (69) on p. 630. Of course we could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.

Convergence of BOPM (concluded)

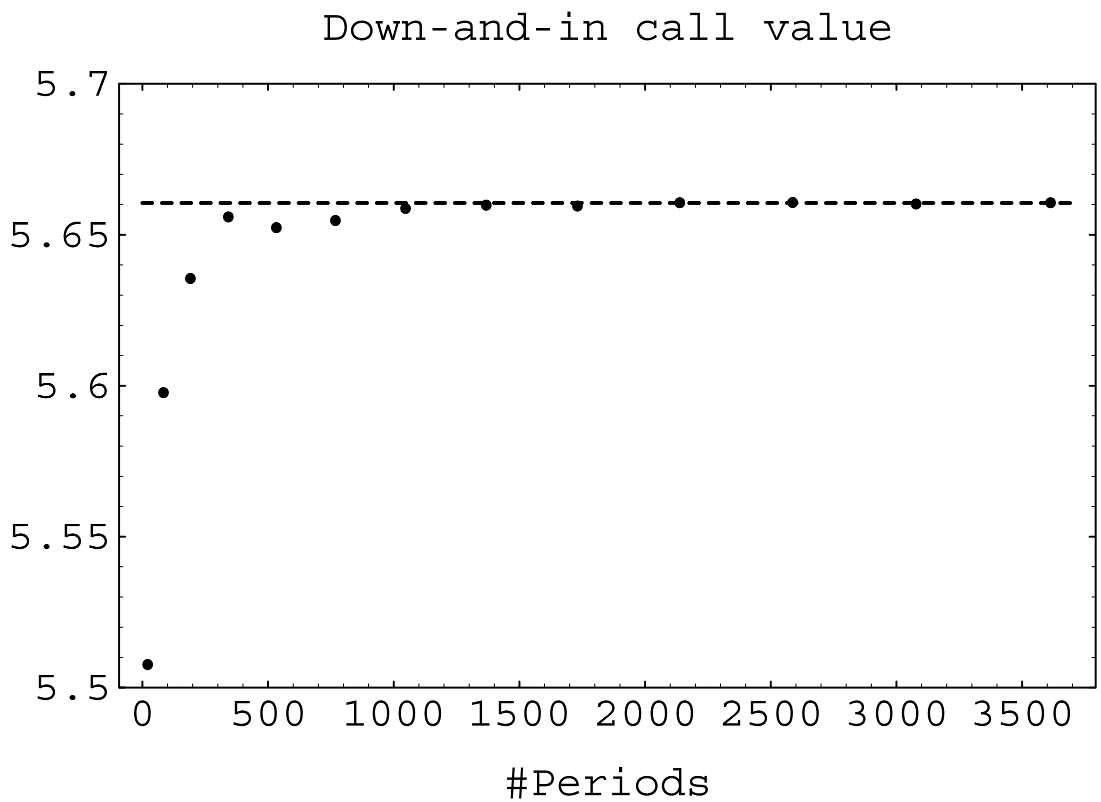
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots,$ where

$$\ell \equiv \left\lfloor \frac{\tau}{\left(\ln(S/H)/(j\sigma)\right)^2} \right\rfloor.$$

- Evaluate pricing formula (70) on p. 633 only with the n's above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0
500
1000
1500
2000
2500
3000
3500
#Periods

Practical Implications

- This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.[a]
- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 641).
- This has profound consequences.[b]

---

[a] Lin (`R95221010`) (2008) and Lin (`R95221010`) and Palmer (2010).
[b] See pp. 654ff.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because
$$n \sim 1/\ln^2(S/H).$$
  - This is called the barrier-too-close problem.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (see p. 643).

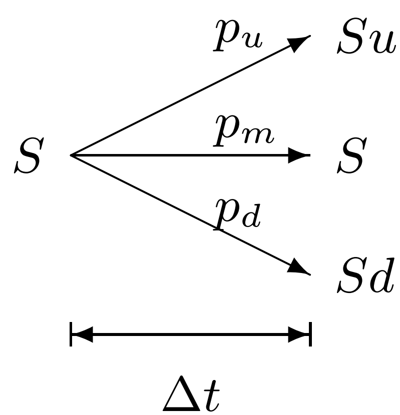
	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

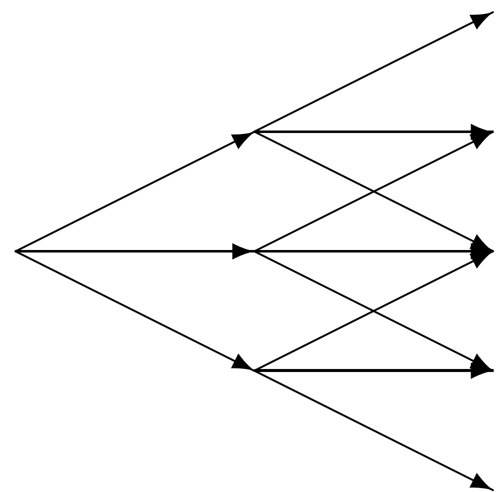
(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion[a]
$$\frac{dS}{S} = r\,dt + \sigma\,dW.$$
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Let the mean and variance of the stock price be SM and S^2V, respectively.

---
[a] Boyle (1988).


p_u
Su
S
p_m
S
p_d
Sd
Δt



Trinomial Tree (continued)

- By Eqs. (21) on p. 160,

$$\begin{aligned} M &\equiv e^{r\Delta t}, \\ V &\equiv M^2(e^{\sigma^2\Delta t}-1). \end{aligned}$$

- Impose the matching of mean and that of variance:

$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &= (p_u u + p_m + (p_d/u))\, S, \\ S^2 V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2. \end{aligned}$$

Trinomial Tree (concluded)

- Use linear algebra to verify that

$$
\begin{aligned}
p_u &= \frac{u\left(V+M^2-M\right)-(M-1)}{(u-1)\left(u^2-1\right)},\\
p_d &= \frac{u^2\left(V+M^2-M\right)-u^3(M-1)}{(u-1)\left(u^2-1\right)}.
\end{aligned}
$$

  - In practice, we must also make sure the probabilities lie between 0 and 1.

- Countless variations.

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$
\begin{aligned}
p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\
p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a]Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- When
$$Se^{-h\lambda\sigma\sqrt{\Delta t}} = H,$$
it takes h down moves to go from S to H, if h is an integer.
- Then
$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}.$$

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- This is easy to achieve by adjusting λ.
- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.[a]
  - Such a λ may not exist for very small n's.
  - This is not hard to check.
- Toward that end, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ to be our h.
- Then let

$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

[a]Why must $\lambda \geq 1$?

Barrier Options Revisited (continued)

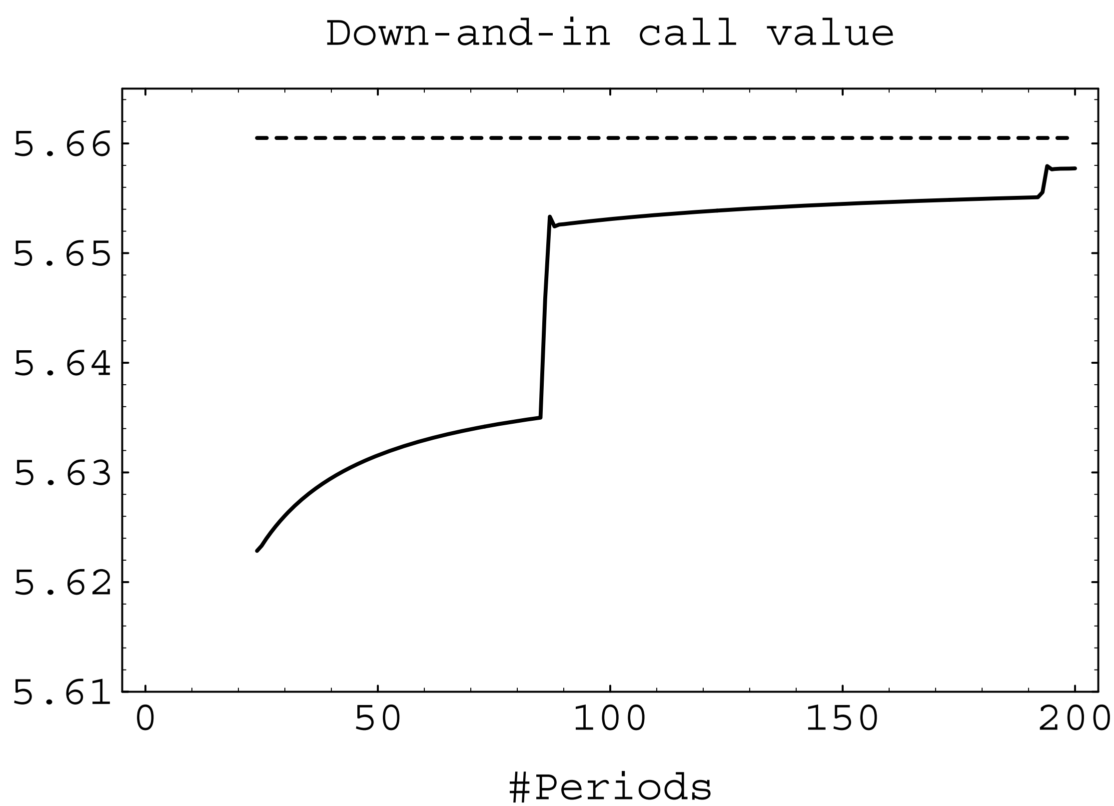
- Alternatively, we can pick
$$h = \left\lfloor \frac{\ln(S/H)}{\sigma\sqrt{\Delta t}} \right\rfloor.$$
- Make sure $h \geq 1$.
- Then let
$$\lambda = \frac{\ln(S/H)}{h\sigma\sqrt{\Delta t}}.$$

Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.
- The following probabilities may be used,

$$
\begin{aligned}
p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\
p_m &= 1 - \frac{1}{\lambda^2}, \\
p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

  - $\mu' \equiv r - \sigma^2/2$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not n.[b]

---

[a] Lyuu (1998).

[b] Patterson and Hennessy (1994).

Algorithms Comparison (continued)

- Pages 364 and 653 seem to show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But does it make the trinomial model better then?

Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 641.
- The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.[a]
  - See pp. 666ff for an alternative solution.
- In fact, the trinomial model also has a linear-time algorithm![b]

---

[a] Lyuu (1998).

[b] Chen (R94922003) (2007).

n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

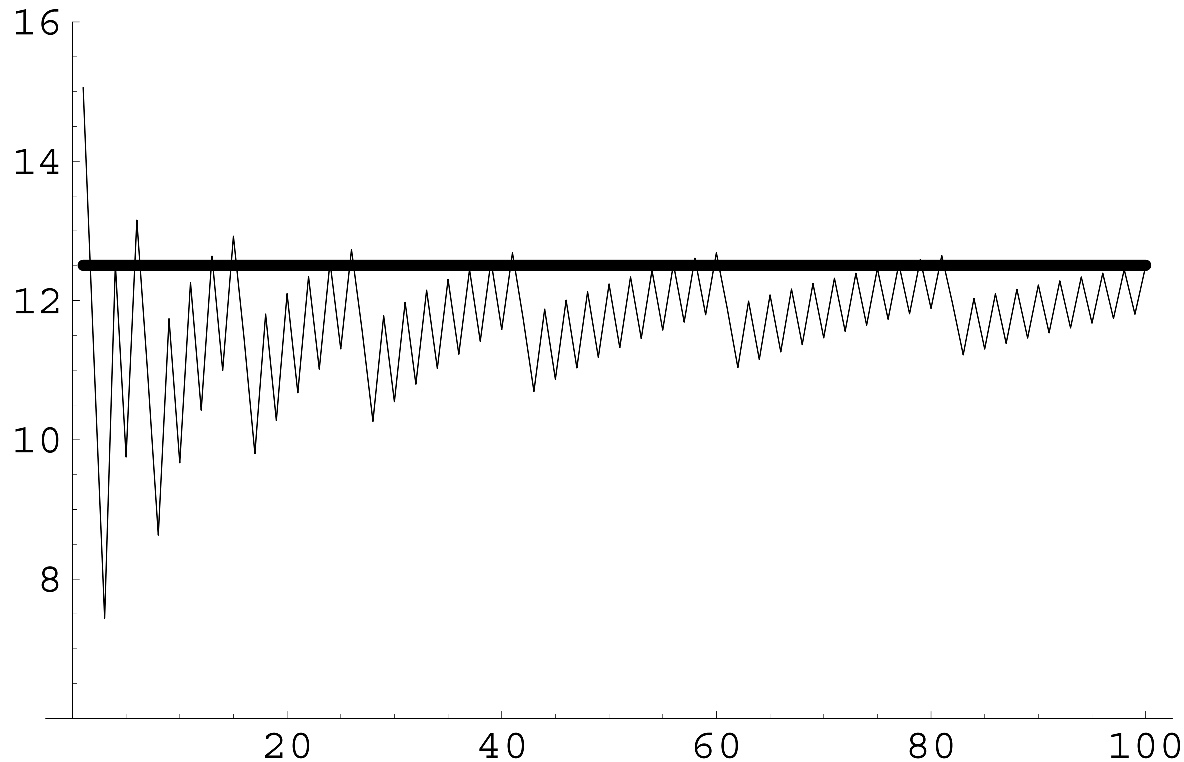
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot on next page).[a]
- The combinatorial method yields a linear-time algorithm (see text).
- This binomial model is $O(1/\sqrt{n})$ convergent in general.[b]

---

[a] Chao (R86526053) (1999); Dai (R86526008, D8852600) and Lyuu (2005).

[b] Gobet (1999).



16
14
12
10
8
20
40
60
80
100

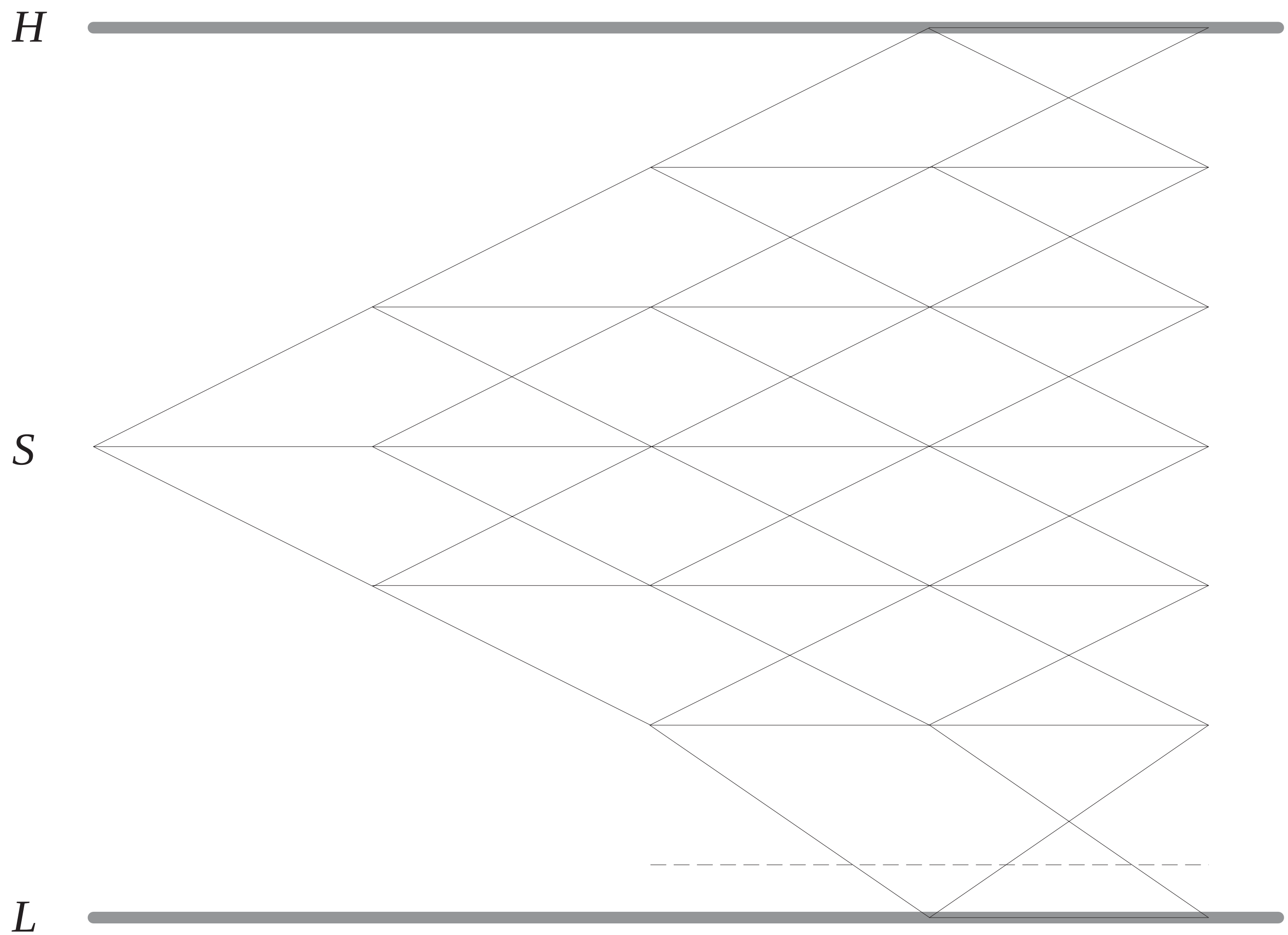
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995).

[b] Hsu (`R7526001`, `D89922012`) and Lyuu (2006). Dai (`R86526008`, `D8852600`) and Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 666ff).



H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.[a]

[a] You probably cannot do the same thing for binomial models (why?). Thanks to a lively discussion on April 25, 2012.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  - The prices between the barriers are

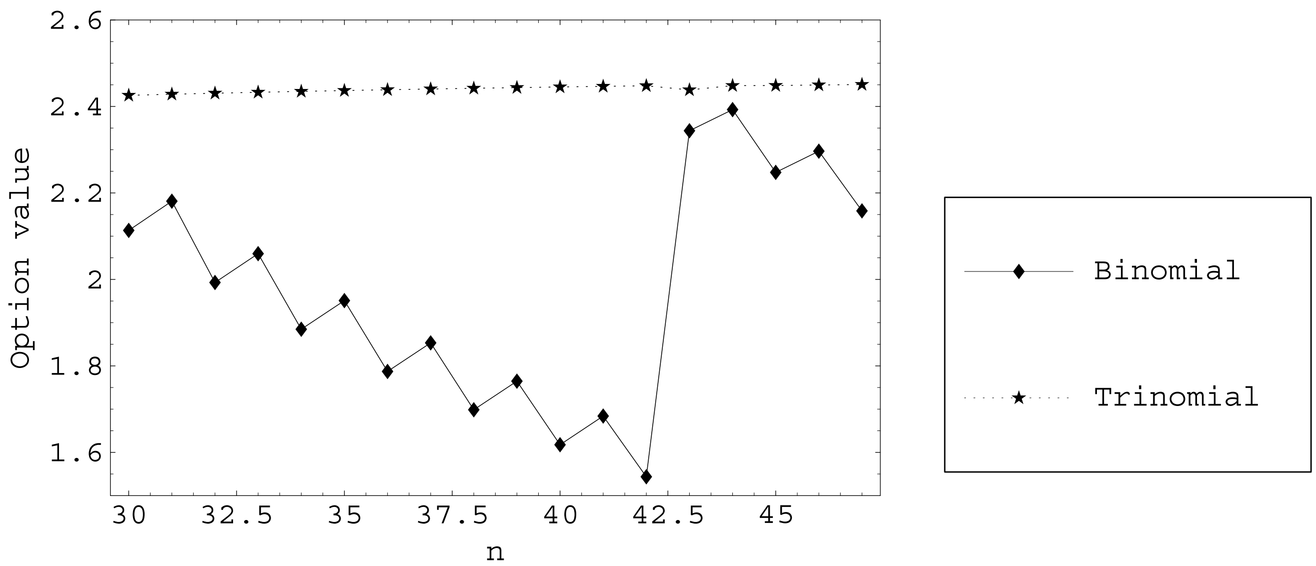
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

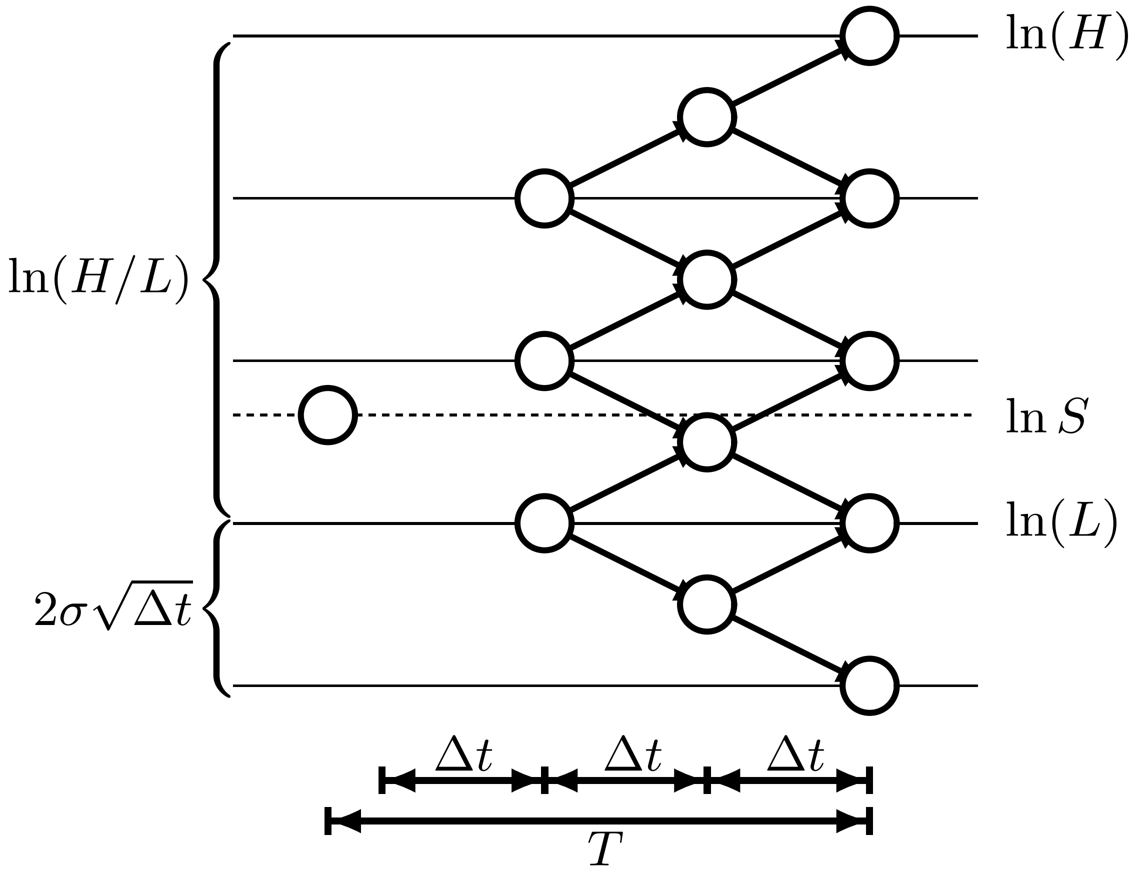
$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



Ideas for Binomial Trees To Handle Two Barriers



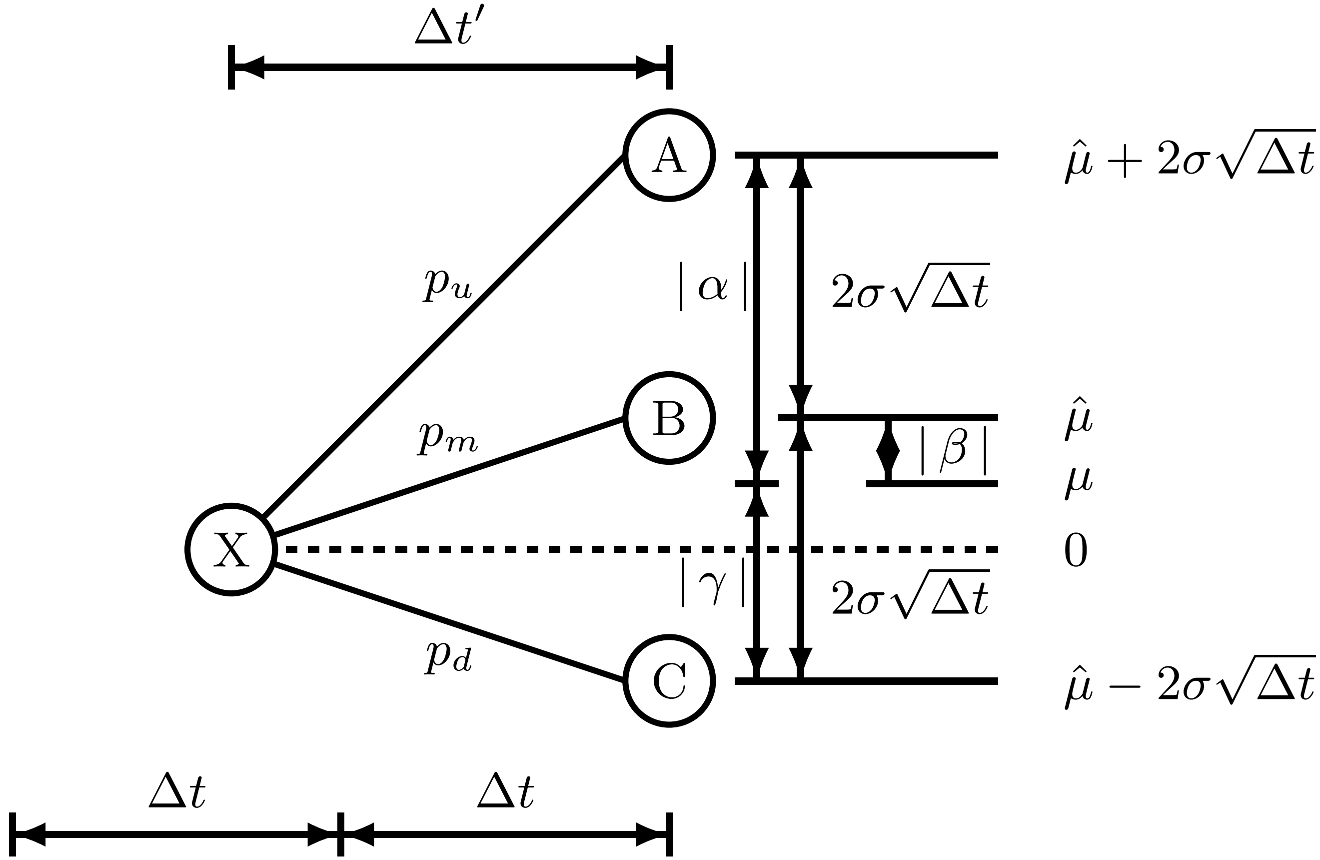
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.[a]
- The resulting tree is called the binomial-trinomial tree.[b]
- Suppose a binomial tree will be built with Δt as the duration of one period.
- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - $\Delta t \leq \Delta t' < 2\Delta t$.

[a] Dai (`R86526008`, `D8852600`) and Lyuu (2006, 2008, 2010).

[b] The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)



The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.
- Let S be the stock price at node X.
- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t + \Delta t'$ equals[a]

$$\mu \equiv \left(r - \sigma^2/2\right) \Delta t'. \tag{71}$$

- Its variance equals

$$\text{Var} \equiv \sigma^2 \Delta t'. \tag{72}$$

- Let node B be the node whose logarithmic return $\hat{\mu} \equiv \ln(s(\text{B})/S)$ is closest to μ among all the nodes on the binomial tree at time $t + \Delta t'$.

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[a]See p. 278.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.
- Review the figure on p. 667 for illustration.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.
- Recall that
$$\hat{\mu} \equiv \ln\left(s(B)/S\right)$$
is the logarithmic return of the middle node B.
- Let α, β, and γ be the differences between μ and the logarithmic returns
$$\ln(s(Z)/S), \quad Z = \mathrm{A}, \mathrm{B}, \mathrm{C},$$
in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}\,, \tag{73}$$

$$\beta \equiv \hat{\mu} - \mu, \tag{74}$$

$$\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}\,. \tag{75}$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \tag{76}$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \text{Var}, \tag{77}$$

$$p_u + p_m + p_d = 1. \tag{78}$$

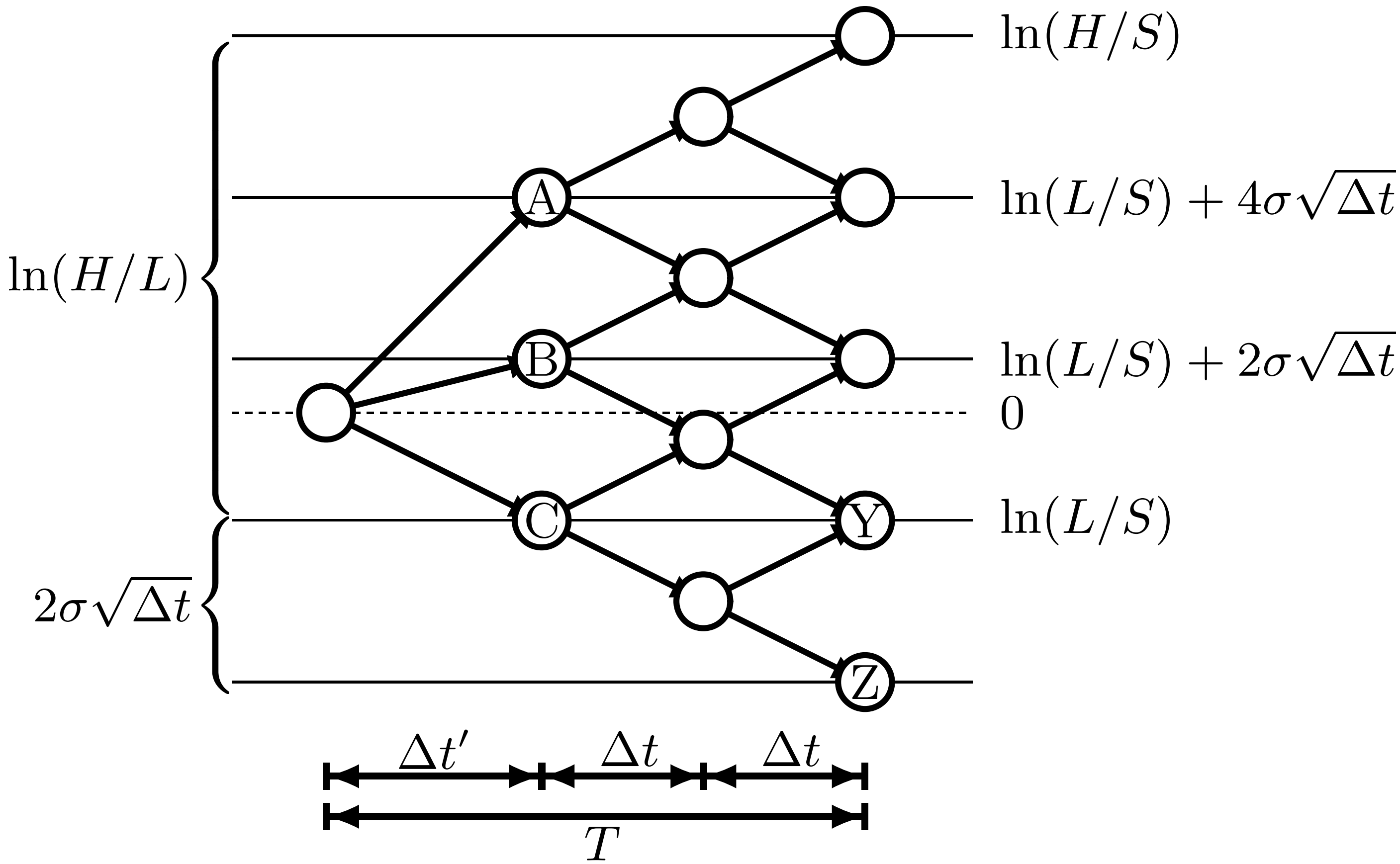
The Binomial-Trinomial Tree (concluded)

- Equation (76) matches the mean (71) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 669.
- Equation (77) matches its variance (72) on p. 669.
- The three probabilities can be proved to lie between 0 and 1.

Pricing Double-Barrier Options

- Consider a double-barrier option with two barriers L and H, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- The idea is to choose a Δt such that
$$\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$$
is a positive integer.
  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 675 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)



Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta\tau \equiv T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.
- So we pick a Δt that is close to, but does not exceed, $\Delta\tau$ and makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ an integer.
- Specifically, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma}\right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier L upward and downward.
- Automatically, a layer coincides with the high barrier H.
- It is unlikely that Δt divides T, however.
- So the position at time 0 and with logarithmic return $\ln(S/S) = 0$ is not occupied by a binomial node to serve as the root node (recall p. 675).

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.
- Between time 0 and time T, the binomial tree spans $T/\Delta t$ periods.
- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.
- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.
- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.
- Calculate the three branching probabilities to them.
- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.
- See the figure on p. 675 for illustration.

Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But a linear-time algorithm exists for double-barrier options on the binomial tree.[a]
- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.
  - That is, nodes A, B, and C on p. 675.
- Then calculate their expected discounted value for the root node.

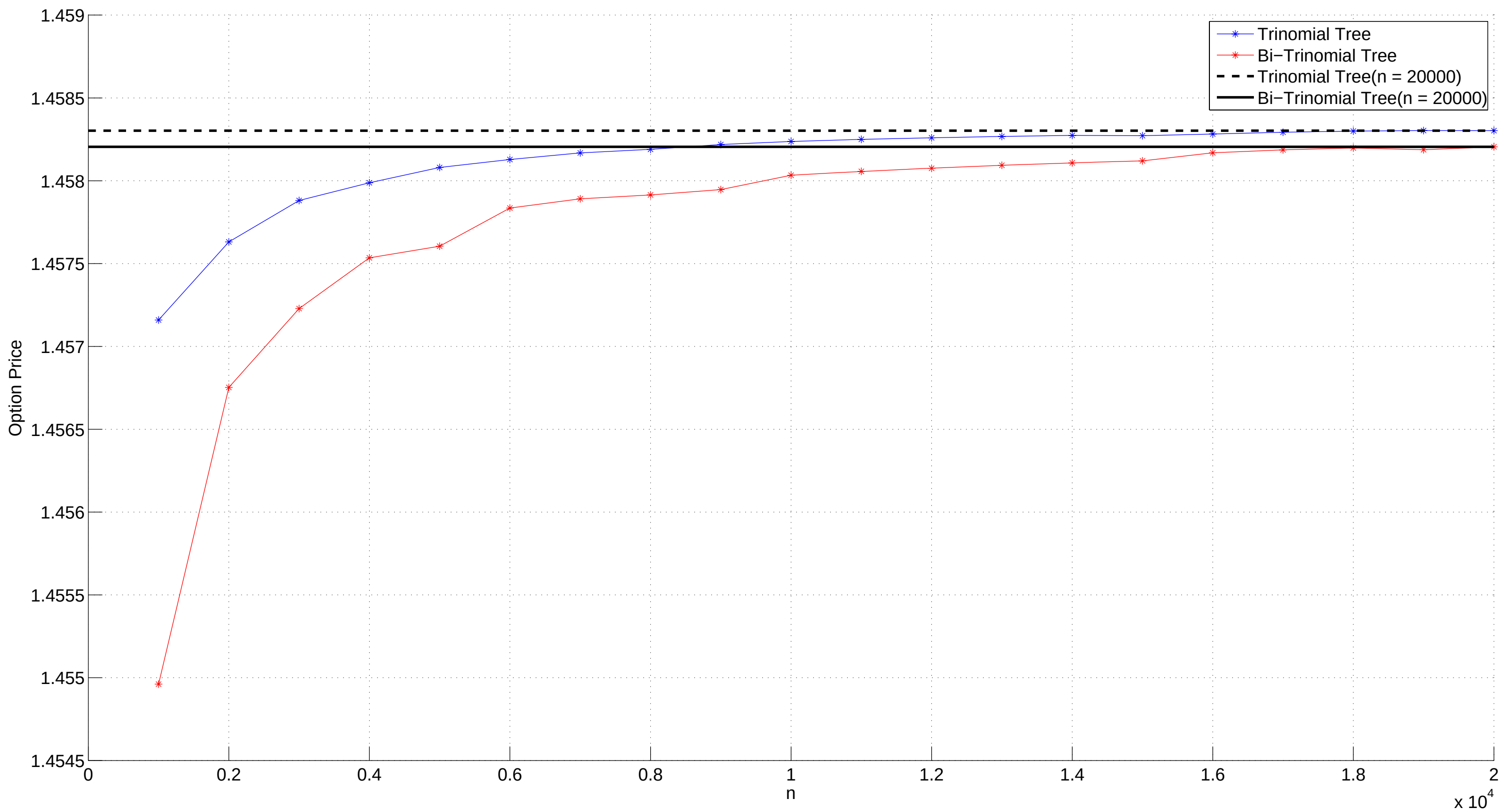
[a]See text; Chao (`R86526053`) (1999); Dai (`R86526008`, `D8852600`) and Lyuu (2008).

Pricing Double-Barrier Options (continued)

- The overall running time is only linear!
- Binomial trees have troubles with pricing barrier options (see p. 364, p. 659, and p. 664).
- Even pit against the trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 682 and p. 683).
- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[a]
- It has an error of $O(1/n^{1-a})$ for any $0 < a < 1$ for double-barrier options.[b]

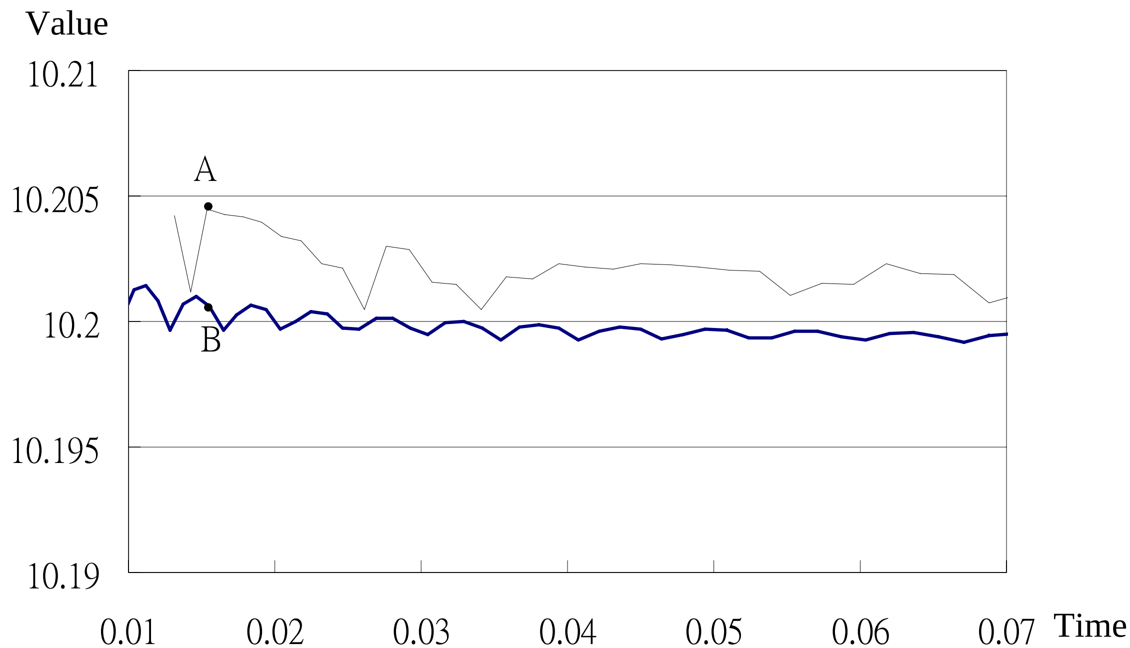
[a]Lyuu and Palmer (2010).
[b]Elisa Appolloni, Gaudenziy, and Zanette (2014).

Pricing Double-Barrier Options[a] (continued)



[a]Generated by Mr. Lin, Ying-Hung (R01723029) on June 6, 2014.

Pricing Double-Barrier Options (concluded)



The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).

The Barrier-Too-Close Problem (p. 642) Revisited

- Our idea solves it.
  - It runs in linear time, unlike an earlier quadratic-time solution with trinomial trees (pp. 649ff).
  - Unlike an earlier solution using combinatorics (p. 633), now the choice of n is not that restricted.
- The handling of single-barrier options is similar.
  - We can build the tree treating S as if it were a second barrier.
  - Alternatively, we can pick $\Delta\tau \equiv T/m$ as our length of a period Δt without any adjustment.

The Barrier-Too-Close Problem Revisited (concluded)

- The earlier trinomial tree is impractical as it needs a very large n when the barrier H is very close to S.[a]
  - It needs at least one *up* move to connect S to H as its middle branch is flat.
  - But when $S \approx H$, that up move must take a very small step, necessitating a small Δt.
- Now the binomial-trinomial tree's middle branch is no longer flat.
- So we can let it connect S to H and the need of a very large n no longer exists!

[a] Recall the table on p. 643.

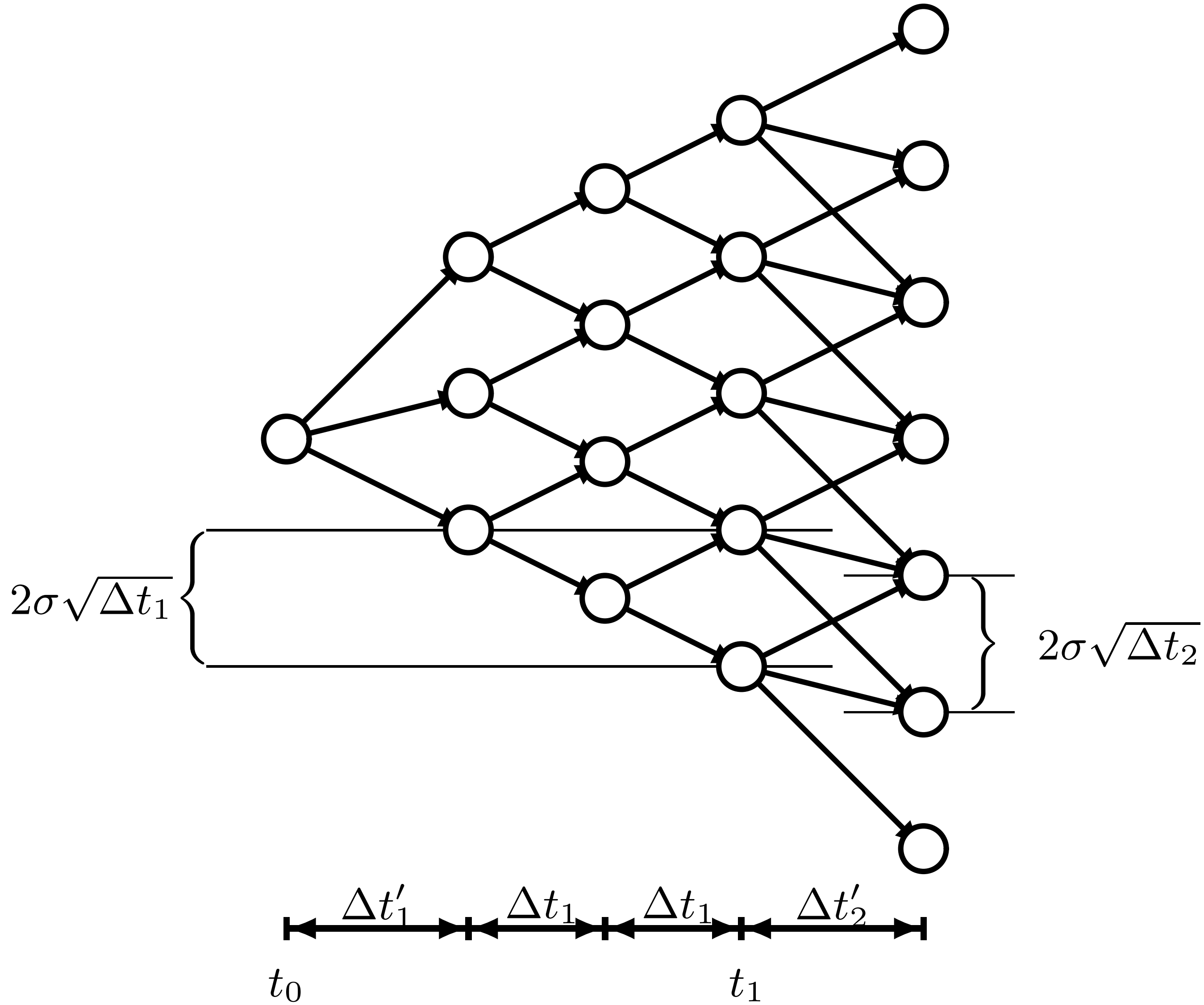
Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- The main difficulty with pricing discrete barrier options lies in matching the monitored times.
- Here is why.
- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$

Pricing Discrete Barrier Options (continued)

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, meaning Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.
- See p. 688.


$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t_1'$
Δt_1
Δt_1
$\Delta t_2'$
t_0
t_1

Pricing Discrete Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]
  1. The one we saw earlier (p. 299) models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

[a]Frishling (2002).

Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., $\alpha(t)S_0 + \beta(t)S_t$), for practitioners.[a]
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But they result in binomial trees that grow exponentially (recall p. 298).
- The binomial-trinomial tree can often avoid this problem.

[a]Henry-Labordère (2009).

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \equiv t/\Delta t$ periods between time 0 and the ex-dividend date.
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D$.
  - Equivalently,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)} \right]^2.$$

Options on a Stock That Pays Known Dividends (continued)

- Build a binomial tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \ldots ,$$

same as the binomial tree.

Options on a Stock That Pays Known Dividends (concluded)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 667.
- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

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[a] Dai (R86526008, D8852600) and Lyuu (2004); Dai (R86526008) (2009).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Option pricing with time-varying volatilities and time-varying barriers.[d]
- Defaultable bond pricing.[e]

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[a] Wu (R96723058) (2009).
[b] Huang (R97922073) (2010).
[c] Huang (R97922081) (2010).
[d] Chou (R97944012) (2010) and Chen (R98922127) (2011).
[e] Dai (R86526008, D8852600), Lyuu, and Wang (F95922018) (2009, 2010, 2014).

General Properties of Trees[a]

- Consider the Ito process,

$$dX = a(X,t)\,dt + \sigma\,dW,$$

  where $a(X,t) = O(1)$ and σ is a constant.

- The mean and volatility of the next move's size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.

- Note that $\sqrt{\Delta t} \gg \Delta t$.

- The tree spacing must be in the order of $\sigma\sqrt{\Delta t}$ if the variance is to be matched.[b]

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[a] Chiu (R98723059) (2012) and Wu (R99922149) (2012).

[b] Lyuu and Wang (F95922018) (2009, 2011) and Lyuu and Wen (D94922003) (2012).