Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

Payoff, Mathematically Speaking

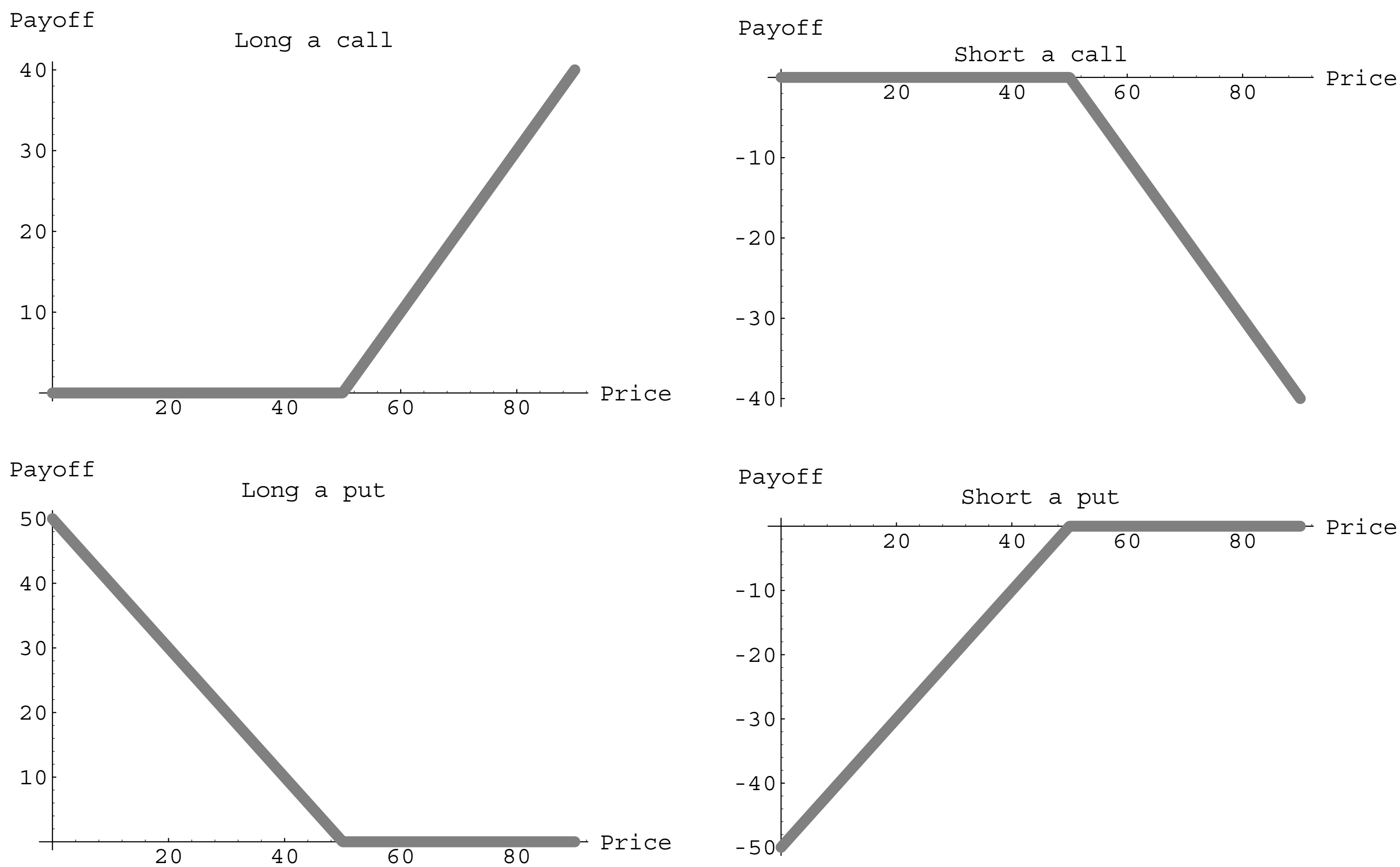
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

the intrinsic value of a call.

- At any time t before the expiration date, we call

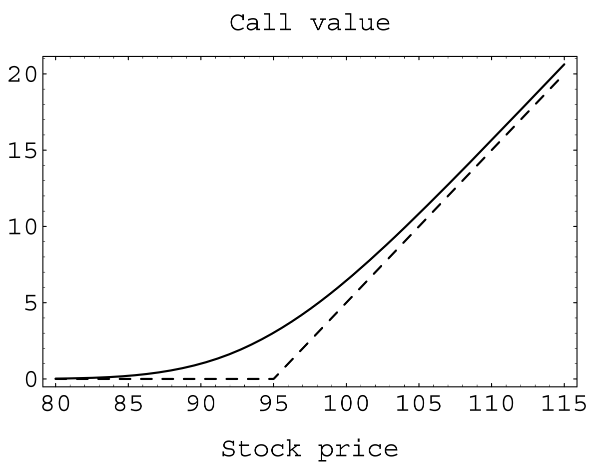
$$\max(0, X - S_t)$$

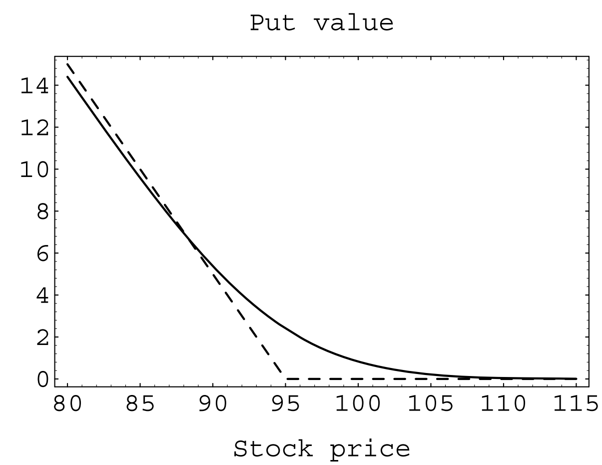
the intrinsic value of a put.

Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time before expiration is a major intellectual breakthrough.

[a] 11% of option holders let in-the-money options expire worthless.


Call value
20
15
10
5
0
80 85 90 95 100 105 110 115
Stock price


Put value
14
12
10
8
6
4
2
0
80 85 90 95 100 105 110 115
Stock price

Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n-for-m stock split, the strike price is only m/n times its previous value, and the number of shares covered by one contract becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- Options are assumed to be unprotected.

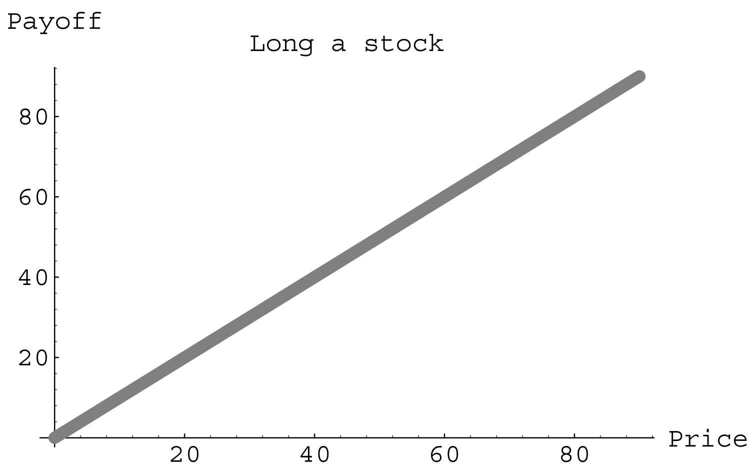
Example

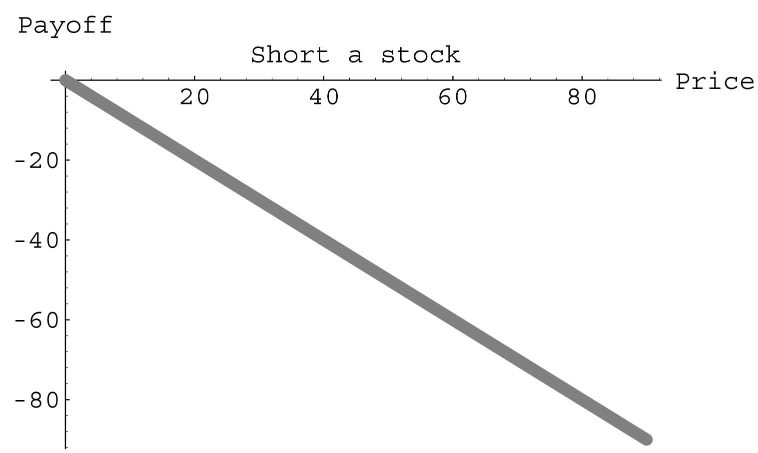
- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.
- Not all assets can be shorted.

Payoff of Stock



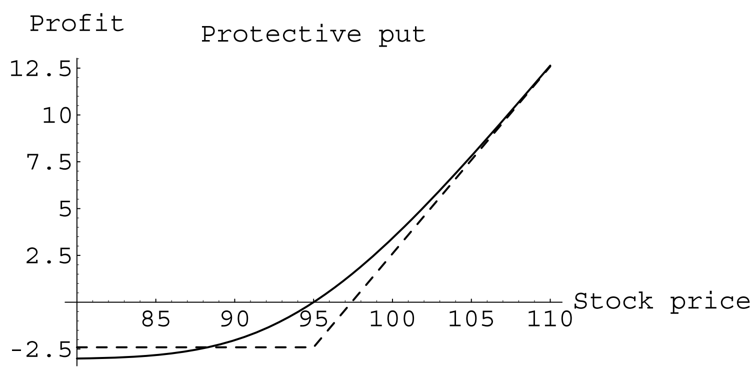


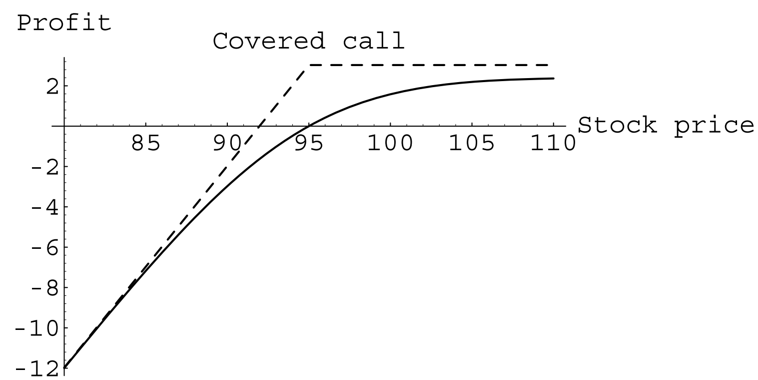
Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Covered call: A long position in stock with a short call.[a]
  - It is "covered" because the stock can be delivered to the buyer of the call if the call is exercised.
- Protective put: A long position in stock with a long put.
- Both strategies break even only if the stock price rises, so they are bullish.

---

[a]A short position has a payoff opposite in sign to that of a long position.





Solid lines are profits of the portfolio one month before maturity assuming the portfolio is set up when $S = 95$ then.

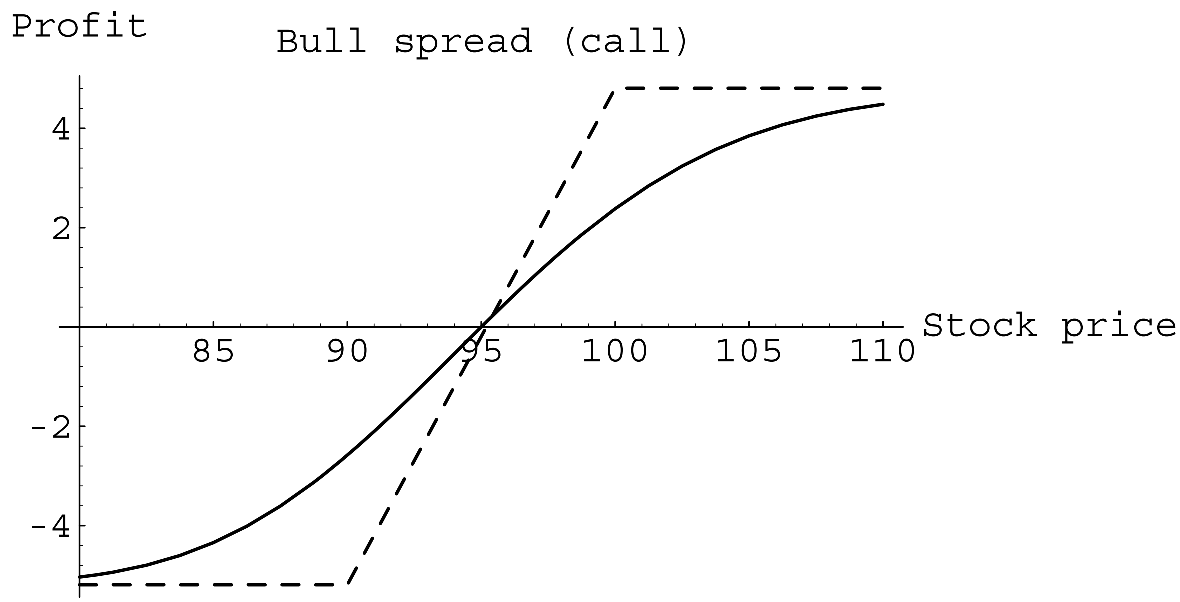
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with

$$X_L < X_M < X_H.$$

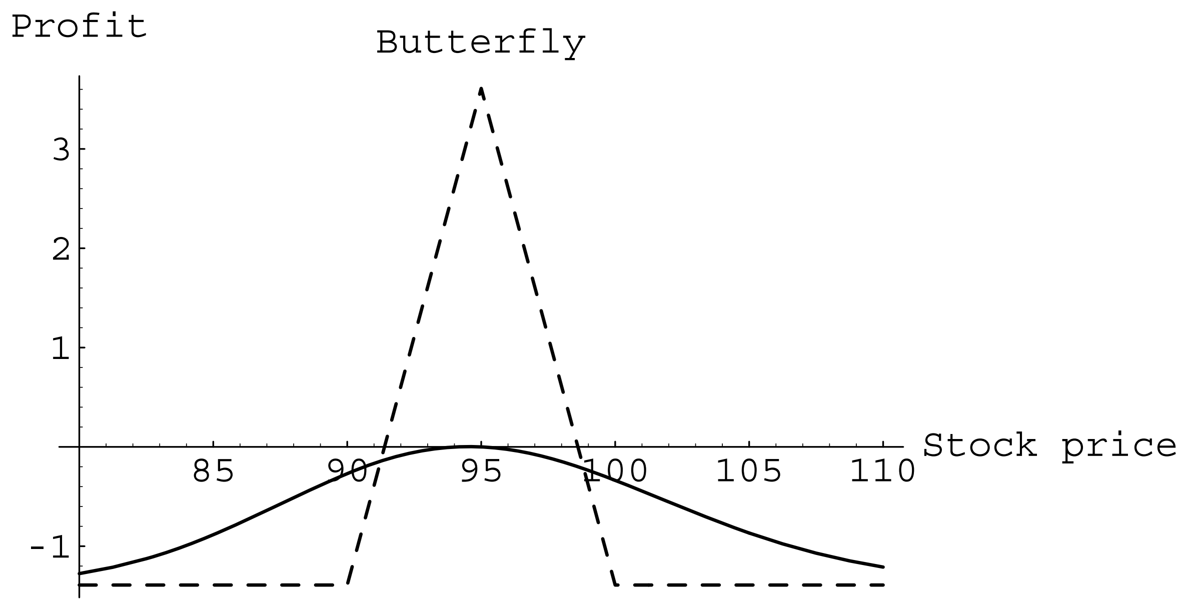
Covered Position: Spread (continued)

- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
    * When both are exercised at expiration.
  - The maximum loss is $C_L - C_H$.
    * When neither is exercised at expiration.
  - If we buy $(X_H - X_L)^{-1}$ units of the bull call spread and $X_H - X_L \to 0$, a (Heaviside) step function emerges as the payoff.


Profit
Bull spread (call)
4
2
-2
-4
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short two X_M calls.


Profit
Butterfly
3
2
1
-1
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- A butterfly spread pays a positive amount at expiration only if the asset price falls between X_L and X_H.
- Take a position in $(X_M - X_L)^{-1}$ units of the butterfly spread.
- When $X_H - X_L \to 0$, it approximates a state contingent claim,[a] which pays \$1 only when the state $S = X_M$ happens.[b]

---

[a] Alternatively, Arrow security.

[b] See Exercise 7.4.5 of the textbook.

Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The (undiscounted) state price equals

$$\frac{\partial^2 C}{\partial X^2}.$$

  - Recall that C is the call's price.[a]
- In fact, the PV of $\partial^2 C/\partial X^2$ is the probability density of the stock price $S_T = X$ at option's maturity.[b]

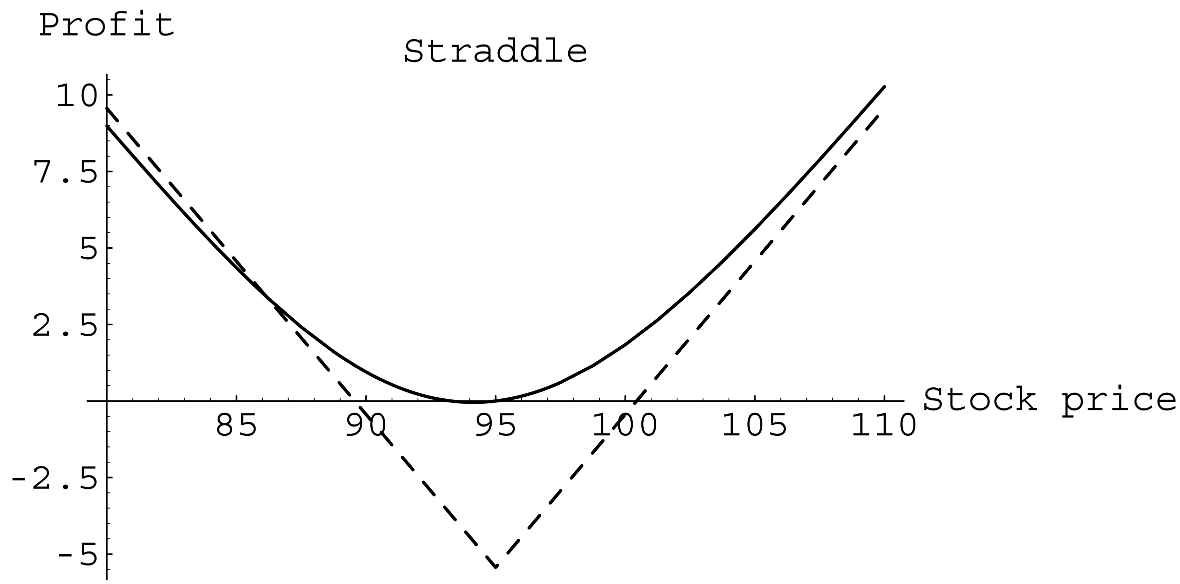
---

[a] One can also use the put (see Exercise 9.3.6 of the textbook).

[b] Breeden and Litzenberger (1978).

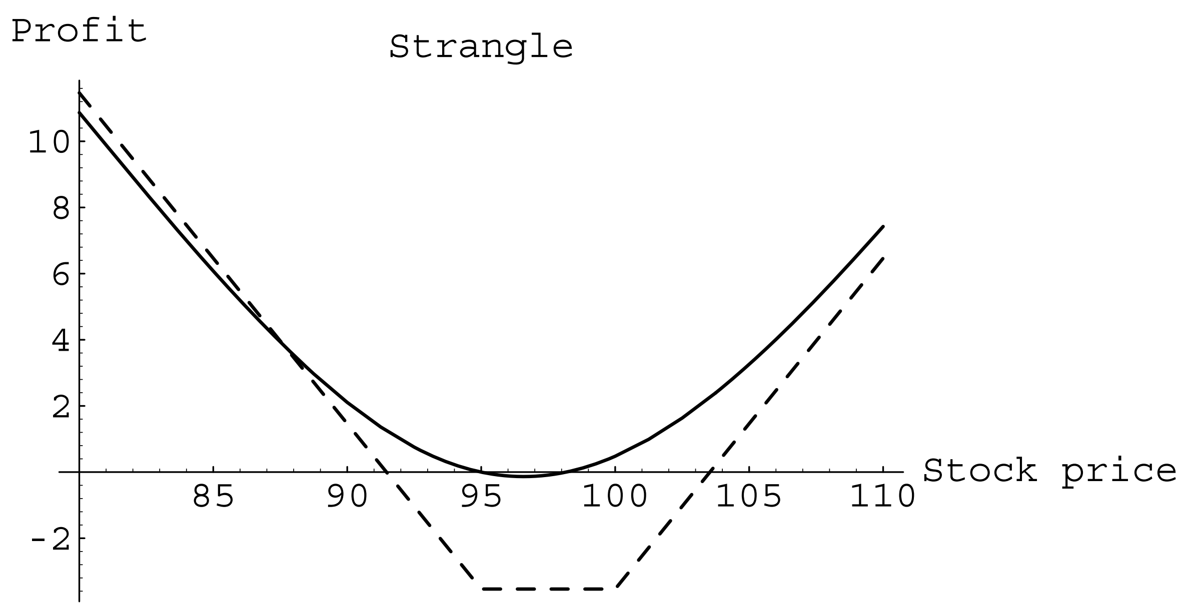
Covered Position: Combination

- A combination consists of options of different types on the same underlying asset.
  - These options must be either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
  - Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
  - Selling a straddle benefits from low volatility.


Profit
Straddle
10
7.5
5
2.5
-2.5
-5
85
90
95
100
105
110
Stock price

Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.



Profit
Strangle
10
8
6
4
2
-2
85
90
95
100
105
110
Stock price

Arbitrage in Option Pricing

All general laws are attended with inconveniences,
when applied to particular cases.
— David Hume (1711–1776)

Arbitrage

- The no-arbitrage principle says there is no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).

Portfolio Dominance Principle

- Consider two portfolios A and B.
- A should be more valuable than B if A's payoff is at least as good as B's under all circumstances and better under some.

Two Simple Corollaries

- A portfolio yielding a zero return in every possible scenario must have a zero PV.
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.
- Two portfolios that yield the same return in every possible scenario must have the same price.[a]

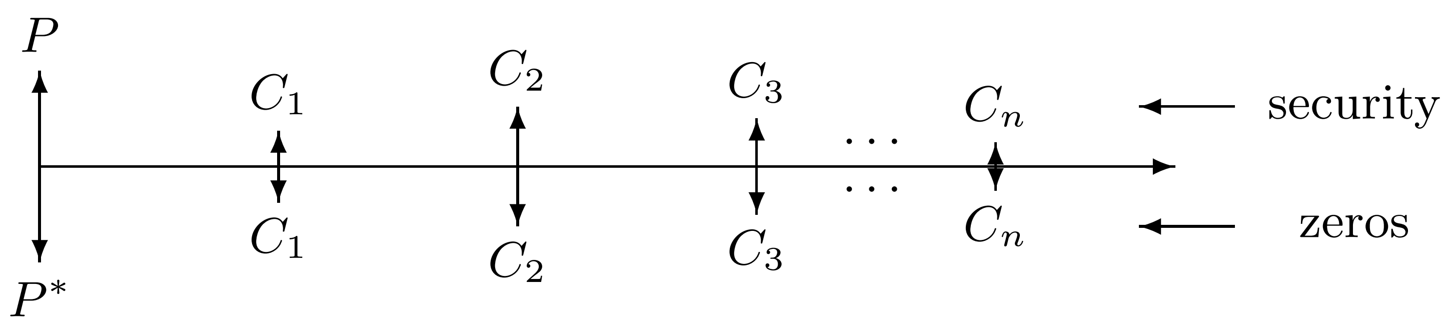
[a] Aristotle, "those who are equal should have everything alike."

The PV Formula (p. 32) Justified

Theorem 1 *For a certain cash flow* $C_1, C_2, \ldots, C_n$,

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Suppose the price $P^* < P$.
- Short the n zeros that match the security's n cash flows.
- The proceeds are P dollars.


P
P^*
C_1
C_1
C_2
C_2
C_3
C_3
$\cdots$
$\cdots$
C_n
C_n
security
zeros

The Proof (concluded)

- Then use P^* of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of $P - P^*$ dollars has been realized now.
- If $P^* > P$, just reverse the trades.

Two More Examples

- An American option cannot be worth less than the intrinsic value.[a]
  - Suppose the opposite is true.
  - So the American option is cheaper than its intrinsic value.
  - For the call: Short the stock and lend X dollars.
  - For the put: Borrow X dollars and buy the stock.
  - In either case, the payoff is the intrinsic value.

---

[a] $\max(0, S_t - X)$ for the call and $\max(0, X - S_t)$ for the put.

Two More Examples (continued)

- (continued)
  - At the same time, buy the option, promptly exercise it, and close the stock position.
    - For the call, call the lent money to exercise it.
    - For the put, deliver the stock and use the received strike price to settle the debt.
  - The cost of buying the option is less than the intrinsic value.
  - So there is an immediate arbitrage profit.

Two More Examples (concluded)

- A put or a call must have a nonnegative value.
  - Suppose otherwise and the option has a negative price.
  - Buy the option for a positive cash flow now.
  - It will end up with a nonnegative amount at expiration.
  - So an arbitrage profit is realized now.

Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.
- Let the current time be time zero.
- $\text{PV}(x)$ stands for the PV of x dollars at expiration.
- Hence $\text{PV}(x) = xd(\tau)$ where τ is the time to expiration.

Put-Call Parity[a]

$$C = P + S - \mathrm{PV}(X). \tag{22}$$

- Consider the portfolio of:
  - One short European call;
  - One long European put;
  - One share of stock;
  - A loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price X and time to expiration, τ.
- The initial cash flow is therefore

$$C - P - S + \mathrm{PV}(X).$$

[a]Castelli (1877).

The Proof (continued)

- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.
- The loan is now X.
- The net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- The net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle (p. 196) implies that the initial investment to set up the portfolio must be nil as well.

Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.
- $S = C - P + \mathrm{PV}(X)$: A stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$.
- $C - P = S - \mathrm{PV}(X)$: A long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

Intrinsic Value

Lemma 2 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- The put-call parity implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

- Recall $C \geq 0$ (p. 202).
- It follows that $C \geq \max(S - X, 0)$, the intrinsic value.
- An American call also cannot be worth less than its intrinsic value (p. 200).

Intrinsic Value (concluded)

A European put on a non-dividend-paying stock may be worth less than its intrinsic value.

Lemma 3 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

- Prove it with the put-call parity.
- Can explain the right figure on p. 173 why $P < X - S$ when S is small.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends.

Theorem 4 (Merton (1973)) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By an exercise in text, $C \geq \max(S - \text{PV}(X), 0)$.
- If the call is exercised, the value is $S - X$.
- But
$$\max(S - \text{PV}(X), 0) \geq S - X.$$

Remarks

- The above theorem does *not* mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock, however.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.[a]

Theorem 5 *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

---

[a]See Theorem 8.4.2 of the textbook.

A General Result[a]

Theorem 6 (Cox and Rubinstein (1985)) *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

Corollary 7 *Any sufficiently well-behaved payoff function can be approximated by a portfolio of calls and puts.*

[a] See Exercise 8.3.6 of the textbook.

Convexity of Option Prices[a]

Lemma 8 *For three otherwise identical calls or puts with strike prices* $X_1 < X_2 < X_3$,

$$\begin{aligned} C_{X_2} &\leq \omega C_{X_1} + (1-\omega)\, C_{X_3} \\ P_{X_2} &\leq \omega P_{X_1} + (1-\omega)\, P_{X_3} \end{aligned}$$

Here

$$\omega \equiv (X_3 - X_2)/(X_3 - X_1).$$

(*Equivalently,* $X_2 = \omega X_1 + (1-\omega)\, X_3$.)

[a] See Lemma 8.5.1 of the textbook.

The Intuition behind Lemma 8[a]

- Set up the following portfolio:

$$\omega C_{X_1} - C_{X_2} + (1-\omega)\, C_{X_3}.$$

- This is a butterfly spread (p. 184).
- It has a nonnegative value as, for any S at maturity,

$$\omega \max(S - X_1, 0) - \max(S - X_2, 0) + (1-\omega)\, \max(S - X_3, 0) \geq 0.$$

- Therefore,

$$\omega C_{X_1} - C_{X_2} + (1-\omega)\, C_{X_3} \geq 0.$$

[a]Contributed by Mr. Cheng, Jen-Chieh (B96703032) on March 17, 2010.

Option on a Portfolio vs. Portfolio of Options

- Consider a portfolio of non-dividend-paying assets with weights ω_i.
- Let C_i denote the price of a European call on asset i with strike price X_i.
- All options expire on the same date.

Option on a Portfolio vs. Portfolio of Options (concluded)

An option on a portfolio is cheaper than a portfolio of options.[a]

Theorem 9 *The call on the portfolio with a strike price*

$$X \equiv \sum_i \omega_i X_i$$

has a value at most

$$\sum_i \omega_i C_i .$$

The same holds for European puts.

[a]See Theorem 8.6.1 of the textbook.

Option Pricing Models

If the world of sense does not fit mathematics,
so much the worse for the world of sense.
— Bertrand Russell (1872–1970)

Black insisted that anything one could do
with a mouse could be done better
with macro redefinitions
of particular keys on the keyboard.
— Emanuel Derman,
My Life as a Quant (2004)

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[a]
  - Known as the Black-Scholes option pricing model.

[a] The results were obtained as early as June 1969.

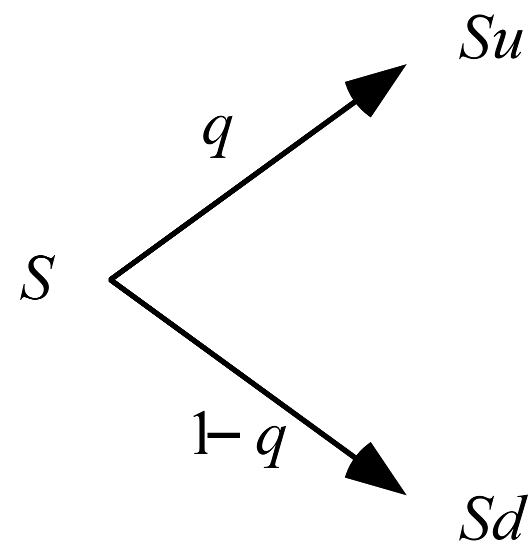
Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1-q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:

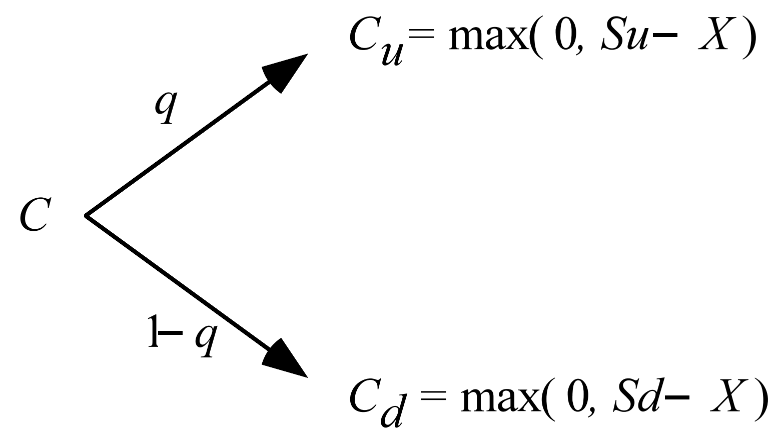
  S, u, d, X, $\hat{r}$, and the number of periods to expiration.


Su
q
S
1 − q
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time 1 if the stock price moves to Su.
- C_d is the call price at time 1 if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is

$$\begin{aligned} &hSu + RB \qquad \text{(up move)},\\ &hSd + RB \qquad \text{(down move)}. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose h and B such that the portfolio *replicates* the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{23}$$

$$B = \frac{uC_d - dC_u}{(u-d)\,R}. \tag{24}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,[a]

$$C = hS + B.$$

- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

[a] Or the replicating portfolio, as it replicates the option.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 4 (p. 210).
- So

$$C = hS + B.$$

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q.[a]
- Hence it is independent of the expected gross return of the stock, $qSu + (1-q)\,Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Then the set of possible stock prices is the same whatever q is.

---

[a]More precisely, not directly dependent on q. Thanks to a lively class discussion on March 16, 2011.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}. \tag{25}$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as

$$pSu + (1 - p)\, Sd = RS.$$

- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate *in a risk-neutral economy.*