Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a]See a counterexample in the text.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1+S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j-i$ periods where $j>i$.
- Will have $[\,1+S(i)\,]^i[\,1+S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j-i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

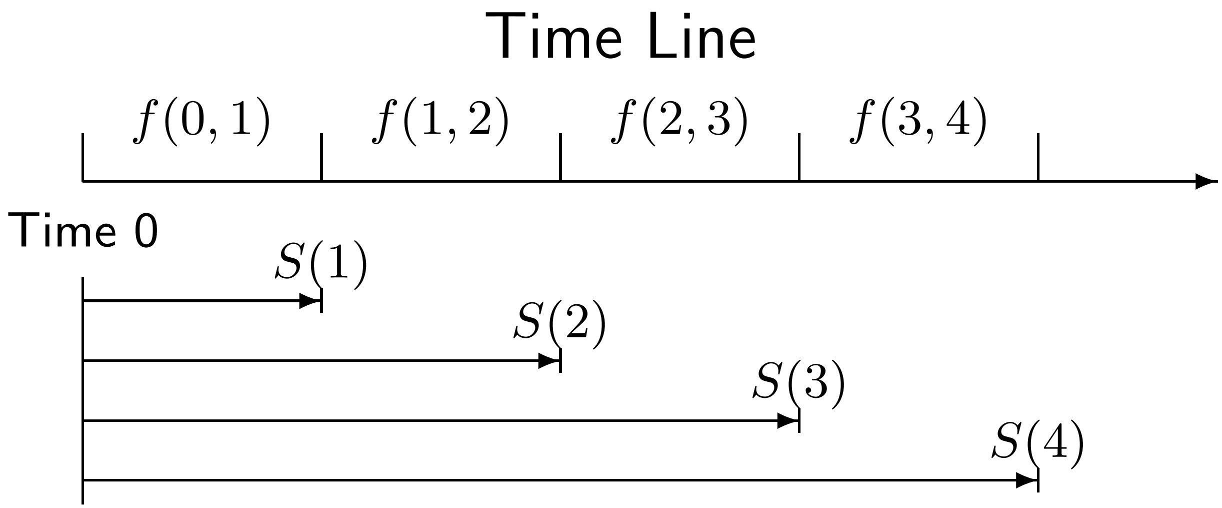
- When $S(i, j)$ equals

$$f(i, j) \equiv \left[\frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \tag{15}$$

  we will end up with $[\,1 + S(j)\,]^j$ dollars again.

- By definition, $f(0, j) = S(j)$.

- $f(i, j)$ is called the (implied) forward rates.
  - More precisely, the $(j - i)$-period forward rate i periods from now.


Time Line
$f(0,1)$
$f(1,2)$
$f(2,3)$
$f(3,4)$
Time 0
$S(1)$
$S(2)$
$S(3)$
$S(4)$

Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.
- $f(i,i+1)$ are called the instantaneous forward rates or one-period forward rates.

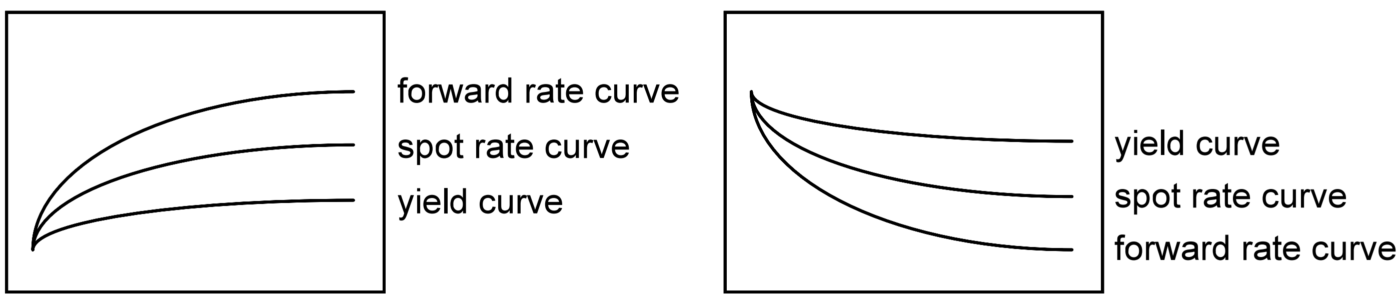
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
yield curve
spot rate curve
forward rate curve

(a) (b)

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive

$$[\,1 + S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,].$$

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

$$S(n) = \{[\,1 + S(1)\,][\,1 + f(1,2)\,] \cdots [\,1 + f(n-1,n)\,]\}^{1/n} - 1. \tag{16}$$

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

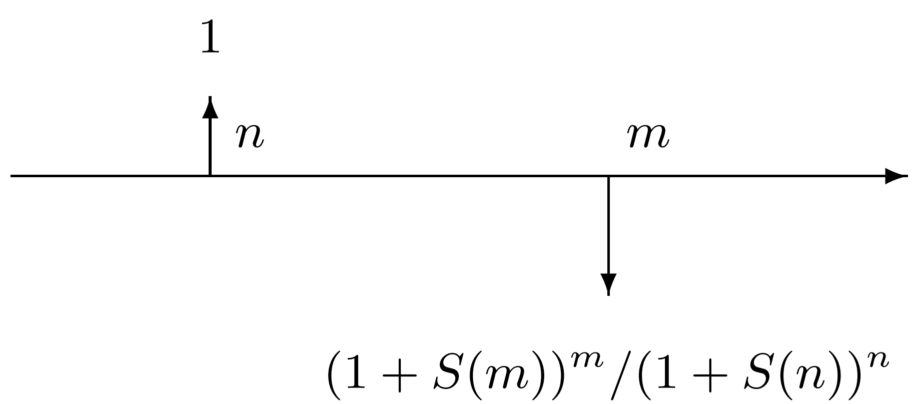
$$f(T, T+1) = \frac{d(T)}{d(T+1)} - 1.$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$ dollars.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow: $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.

Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the rate $f(n, m)$ between times n and m.



Forward Contracts

- We had generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to
  - Borrow money at time n in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

- Can the spot rate curve be an arbitrary curve?[a]

---

[a] Contributed by Mr. Dai, Tian-Shyr (`R86526008`, `D88526006`) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

---

[a] Compare it with Eq. (16) on p. 129.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}.$$

- The one-period forward rate:

$$f(j, j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- Now,

$$f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$

- So $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$ (i.e., a normal spot rate curve).
- If $S(T) < -T(\partial S/\partial T)$, then $f(T) < 0$.[a]

[a] Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\, S(a, b)\,]. \tag{17}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (15) on p. 123.
  - So $E[\,S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So
$$(1+S(2))^2 = (1+S(1))\, E[\,1+S(1,2)\,] \tag{18}$$
because of the equivalency between buying a two-period bond and rolling over one-period bonds.
- After rearrangement,
$$\frac{1}{E[\,1+S(1,2)\,]} = \frac{1+S(1)}{(1+S(2))^2}.$$

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1 + S(2))^{-2}$ dollars and sells it after one period.
  - The expected return[a] is

$$E[\,(1 + S(1,2))^{-1}\,]/(1 + S(2))^{-2}.$$

  - Strategy two buys a one-period bond with a return of $1 + S(1)$.

[a] More precisely, the one-plus return.

A "Bad" Expectations Theory (concluded)

- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2} = E\left[\frac{1}{1+S(1,2)}\right].$$

- Combine this with Eq. (18) on p. 139 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use $g(x) \equiv (1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \quad \text{for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration in Practice

- To handle more general types of spot rate curve changes, define a vector $[\, c_1, c_2, \ldots, c_n\,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\,1, 1, \ldots, 1\,]$.
  - Twist: $[\,1, 1, \ldots, 1, -1, \ldots, -1\,]$, $[\,1.8\%, 1.6\%, 1.4\%, 1\%, 0\%, -1\%, -1.4\%, \ldots]$, etc.
  - ....
- At least one c_i should be 1 as the reference point.

Duration in Practice (concluded)

- Let

$$P(y) \equiv \sum_i C_i / (1 + S(i) + y c_i)^i$$

be the price associated with the cash flow $C_1, C_2, \ldots$.

- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y) - P(-\Delta y)}{P(0)\Delta y}.$$

- Modified duration equals the above when

$$[\, c_1, c_2, \ldots, c_n \,] = [\, 1, 1, \ldots, 1 \,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli
(1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \equiv E\,[\,(X - \mu_X)(Y - \mu_Y)\,],$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \equiv \sqrt{\text{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \equiv \frac{\text{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

[a]Paul Wilmott (2009), "the correlations between financial quantities are notoriously unstable."

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\, X_i, X_j \,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\, X_i \,]$$

when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation $E[\, X \mid I \,]$ is the expected value of X conditional on I; it is a random variable.
- The law of iterated conditional expectations:

$$E[\, X \,] = E[\, E[\, X \mid I \,] \,].$$

- If I_2 contains at least as much information as I_1, then

$$E[\, X \mid I_1 \,] = E[\, E[\, X \mid I_2 \,] \mid I_1 \,]. \tag{19}$$

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$

- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the following distribution function

$$\text{Prob}[\, X \leq z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is

$$\theta_X(t) \equiv E[\, e^{tX} \,].$$

- The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{20}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\, X_i, X_j\,]\right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$.[a]

- X_i are said to have a multivariate normal distribution.

[a] Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1\,]$ so that

$$\text{Prob}[\,X \leq x\,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \equiv \sqrt{-2(\ln \omega)/\omega}\,.$$

---

[a]As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0,1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i\right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

---

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- So the general formula is

$$\frac{(\sum_{i=1}^{n} \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

---

[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$\begin{aligned} U &\equiv aX_1, \\ V &\equiv \rho U + \sqrt{1-\rho^2}\, aX_2. \end{aligned}$$

Generation of Bivariate Normal Distributions (concluded)

- U and V are the desired random variables with

$$\begin{aligned} \text{Var}[\,U\,] &= \text{Var}[\,V\,] = a^2, \\ \text{Cov}[\,U, V\,] &= \rho a^2. \end{aligned}$$

- Note that the mapping between (X_1, X_2) and (U, V) is one-to-one.

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu+\sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \tag{21}$$

respectively.

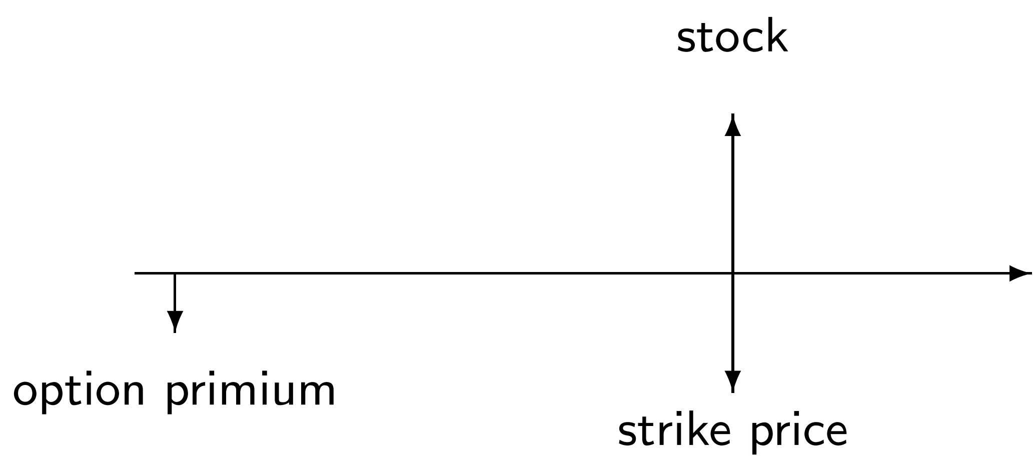
  – They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.

Option Basics

The shift toward options as
the center of gravity of finance [. . .]
— Merton H. Miller (1923–2000)

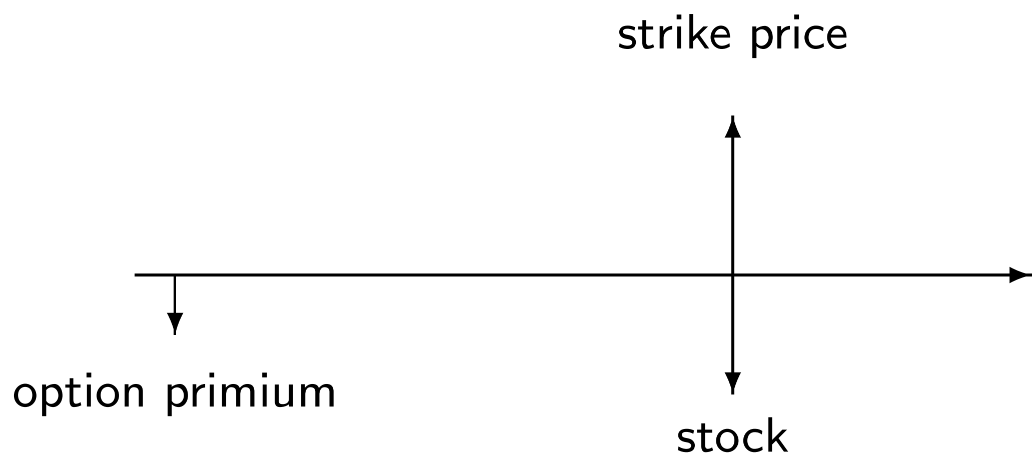
Calls and Puts

- A call gives its holder the right to buy a number of the underlying asset by paying a strike price.



Calls and Puts (continued)

- A put gives its holder the right to sell a number of the underlying asset for the strike price.



Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.
- How to price options?
  - It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.