Recurrence Relations

(Difference Equations)

Pure mathematics is the subject in which
we do not know what we are talking about,
or whether what we are saying is true.
— Bertrand Russell (1872–1970)

Recurrence Relations Arise Naturally

- When a problem has a recursive nature, recurrence relations often arise.
  - A problem can be solved by solving 2 subproblems of the same nature.
- When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
  - Sorting, fast Fourier transform, etc.
- Certain combinatorial objects are constructed recursively.

First-Order Linear Homogeneous Recurrence Relations

- Consider the recurrence relation

$$a_{n+1} = da_n,$$

where $n \geq 0$ and d is a constant.

- The **general solution** is given by

$$a_n = Cd^n$$

for any constant C.

  – It satisfies the relation: $Cd^{n+1} = dCd^n$.

- There are infinitely many solutions, one for each choice of C.

First-Order Linear Homogeneous Recurrence Relations (concluded)

- Now suppose we impose the **initial condition** $a_0 = A$.
- Then the (unique) **particular solution** is $a_n = Ad^n$.
  - Because $A = a_0 = Cd^0 = C$.
- Note that $a_n = na_{n-1}$ is *not* a first-order linear homogeneous recurrence relation.
  - Its solution is $n!$ when $a_0 = 1$.

First-Order Linear Nonhomogeneous Recurrence Relations

- Consider the recurrence relation

$$a_{n+1} + da_n = f(n).$$

  - $n \geq 0$.
  - d is a constant.
  - $f(n) : \mathbb{N} \to \mathbb{N}$.

- A general solution no longer exists.

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

- Consider the kth-order recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \tag{72}$$

  where $C_n, C_{n-1}, \ldots, C_{n-k} \in \mathbb{R}$, $C_n \neq 0$, and $C_{n-k} \neq 0$.

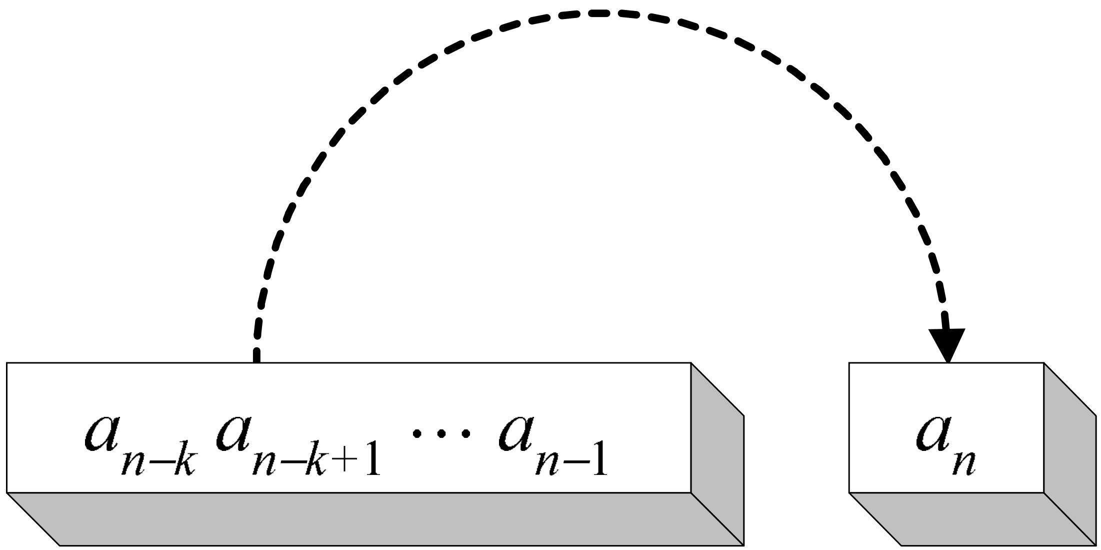
- Add k initial conditions for $a_0, a_1, \ldots, a_{k-1}$.

- Clearly,

$$a_k, a_{k+1}, \ldots$$

  are well-defined.

- Indeed, a_n can be calculated with $O(kn)$ operations.



$a_{n-k}\ a_{n-k+1} \cdots a_{n-1}$
a_n

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients (concluded)

- A solution y for a_n is general if for any particular solution y^*, the undetermined coefficients of y can be found so that y is identical to y^*.
- Any general solution for a_n that satisfies the k initial conditions and Eq. (72) is a particular solution.
- In fact, it is the *unique* particular solution because any solution agreeing at $n = 0, 1, \ldots, k-1$ must agree for all $n \geq 0$.

Conditions for the General Solution

Theorem 71 *Let* $a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}$ *be* k *particular solutions of Eq. (72). If*

$$\begin{vmatrix} a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(k)} \\ a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1}^{(1)} & a_{k-1}^{(2)} & \cdots & a_{k-1}^{(k)} \end{vmatrix} \neq 0, \tag{73}$$

then $a_n = c_1 a_n^{(1)} + c_2 a_n^{(2)} + \cdots + c_k a_n^{(k)}$ *is the general solution, where* $c_1, c_2, \ldots, c_k$ *are arbitrary constants.*[a]

[a] Samuel Goldberg, *Introduction to Difference Equations* (1986).

Fundamental Sets

- The particular solutions of Eq. (72) on p. 522,

$$a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)},$$

  that also satisfy inequality (73) in Theorem 71 (p. 525) are said to form a **fundamental set of solutions**.

- Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots

- Let $r_1, r_2, \ldots, r_k$ be the (characteristic) roots of the **characteristic equation**

$$C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \tag{74}$$

- If $r_1, r_2, \ldots, r_k$ are distinct, then the general solution has the form

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

for constants $c_1, c_2, \ldots, c_k$ determined by the initial conditions.

The Proof

- Assume a_n has the form cr^n for nonzero c and r.
- After substitution into recurrence equation (72) on p. 522, r satisfies characteristic equation (74).
- Let $r_1, r_2, \ldots, r_k$ be the k distinct (nonzero) roots.
- Hence $a_n = r_i^n$ is a solution for $1 \leq i \leq k$.
- Solutions r_i^n form a fundamental set because

The Proof (continued)

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{vmatrix} \neq 0.$$

- The $k \times k$ matrix is called a **Vandermonde matrix**, which is nonsingular whenever $r_1, r_2, \ldots, r_k$ are distinct.[a]

[a]This is a standard result in linear algebra.

The Proof (concluded)

- Hence
$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$
is the general solution.

- The k coefficients $c_1, c_2, \ldots, c_k$ are determined uniquely by the k initial conditions $a_0, a_1, \ldots, a_{k-1}$:
$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}. \quad (75)$$

The Fibonacci Relation

- Consider $a_{n+2} = a_{n+1} + a_n$.
- The initial conditions are $a_0 = 0$ and $a_1 = 1$.[a]
- The characteristic equation is $r^2 - r - 1 = 0$, with two roots $(1 \pm \sqrt{5})/2$.[b]
- The fundamental set is hence

$$\left\{ \left(\frac{1+\sqrt{5}}{2}\right)^n, \left(\frac{1-\sqrt{5}}{2}\right)^n \right\}.$$

[a] Clearly a_n can be calculated with $O(n)$ operations.

[b] The **golden ratio** $(1+\sqrt{5})/2$ has fascinated mathematicians since Pythagoras (570 B.C.–495 B.C.).

The Fibonacci Relation (continued)

- For example, $a_n = (\frac{1+\sqrt{5}}{2})^n$ satisfies the Fibonacci relation, as

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n}.$$

- The general solution is hence

$$a_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{n} + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^{n}. \qquad (76)$$

The Fibonacci Relation (concluded)

- Solve

$$\begin{aligned} 0 &= a_0 = c_1 + c_2 \\ 1 &= a_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} \end{aligned}$$

for $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.

- The particular solution is finally

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n, \qquad (77)$$

known as the **Binet formula**.[a]

[a] So a_n can now be calculated with $O(\log n)$ operations (there is no need to expand $\sqrt{5}$)!

Don't Believe It?

$$
\begin{aligned}
a_2 &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 \\
&= \frac{1}{\sqrt{5}}\frac{1+2\sqrt{5}+5}{4} - \frac{1}{\sqrt{5}}\frac{1-2\sqrt{5}+5}{4} = 1. \\
a_3 &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^3 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^3 \\
&= \frac{1}{\sqrt{5}}\frac{1+3\sqrt{5}+15+5\sqrt{5}}{8} - \frac{1}{\sqrt{5}}\frac{1-3\sqrt{5}+15-5\sqrt{5}}{8} \\
&= 2.
\end{aligned}
$$

Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

$$\begin{aligned} 1 &= a_0 = c_1 + c_2, \\ 2 &= a_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}, \end{aligned}$$

for $c_1 = [\,(1+\sqrt{5}\,)/2\,]^2/\sqrt{5}$ and
$c_2 = -[\,(1-\sqrt{5}\,)/2\,]^2/\sqrt{5}$ to obtain

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+2}. \qquad (78)$$

Initial Conditions (concluded)

- Suppose $a_0 = a_1 = 1$ instead.
- Then solve

$$\begin{aligned} 1 &= a_0 = c_1 + c_2, \\ 1 &= a_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}, \end{aligned}$$

for $c_1 = [\,(1+\sqrt{5}\,)/2\,]/\sqrt{5}$ and $c_2 = -[\,(1-\sqrt{5}\,)/2\,]/\sqrt{5}$ to obtain

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}. \qquad (79)$$

Generating Function for the Fibonacci Numbers

- From $a_{n+2} = a_{n+1} + a_n$, we obtain

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = \sum_{n=0}^{\infty} (a_{n+1} x^{n+2} + a_n x^{n+2}).$$

- Let $f(x)$ be the generating function for $\{ a_n \}_{n=0,1,2,\ldots}$.
- Then

$$f(x) - a_0 - a_1 x = x[\, f(x) - a_0 \,] + x^2 f(x).$$

- Hence

$$f(x) = \frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2}. \tag{80}$$

A Formula for the Fibonacci Numbers

$$a_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-\lceil n/2 \rceil}{\lceil n/2 \rceil - 1}.$$

- From Eq. (80) on p. 537, the generating function is[a]

$$\begin{aligned}
& \frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2} \\
= \quad & \frac{x}{1 - x(1+x)} \\
= \quad & x + x^2(1+x) + x^3(1+x)^2 + \cdots \\
& + x^{n-1}(1+x)^{n-2} + x^n(1+x)^{n-1} + \cdots \\
= \quad & \cdots + \left[\binom{n-\lceil n/2 \rceil}{\lceil n/2 \rceil - 1} + \cdots + \binom{n-2}{1} + \binom{n-1}{0} \right] x^n + \cdots .
\end{aligned}$$

[a]Recall that $a_0 = 0$ and $a_1 = 1$.

Number of Binary Sequences without Consecutive 0s

- Let a_n denote the number of binary sequences of length n without consecutive 0s.
- There are a_{n-1} valid sequences with the nth symbol being 1.
- There are a_{n-2} valid sequences with the nth symbol being 0 because any such sequence must end with 10.
- Hence $a_n = a_{n-1} + a_{n-2}$, a Fibonacci sequence.
- Because $a_1 = 2$ and $a_2 = 3$, we must have $a_0 = 1$ to **retrofit** the Fibonacci sequence.
- The formula is Eq. (78) on p. 535.

Number of Subsets without Consecutive Numbers

- How many subsets of $\{1, 2, \ldots, n\}$ contain no 2 consecutive integers?
- A binary sequences $b_1 b_2 \cdots b_n$ of length n can be interpreted as the set $\{ i : b_i = 0 \} \subseteq \{1, 2, \ldots, n\}$.
- So a subset of $\{1, 2, \ldots, n\}$ without consecutive integers implies a binary sequence without consecutive 0s, and vice versa.
- Hence there are a_n subsets of $\{1, 2, \ldots, n\}$ that contain no 2 consecutive integers, where
  - a_n is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$.

Number of Subsets without Consecutive Numbers (continued)

- From formula (78) on p. 535,

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+2}$$

  is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$.

- The number can also be derived from Eq. (14) on p. 89:

$$\sum_{m=0}^{\lceil n/2 \rceil} \binom{n-m+1}{m} = \binom{n+1}{0} + \binom{n}{1} + \cdots + \binom{n - \lceil n/2 \rceil + 1}{\lceil n/2 \rceil}.$$

Number of Subsets without Consecutive Numbers (concluded)

- Hence, as a bonus,

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}=\sum_{m=0}^{\lceil n/2\rceil}\binom{n-m+1}{m}.$$

Number of Subsets without Cyclically Consecutive Numbers

- How many subsets of $\{1, 2, \ldots, n\}$ contain no 2 consecutive integers when 1 and n are considered consecutive?
- Let a_n be the solution for the problem on p. 540.
- So a_n is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$ (formula appeared in Eq. (78) on p. 535).
- Now assume $n \geq 3$.
- There are a_{n-1} acceptable subsets that do not contain n.

Number of Subsets without Cyclically Consecutive Numbers (continued)

- If n is included, an acceptable subset cannot contain 1 or $n-1$.
- Hence there are a_{n-3} such subsets.
- The total is therefore $L_n \equiv a_{n-1} + a_{n-3}$, the **Lucas number**.[a]
- It can be easily checked that

$$\begin{aligned} L_n &= a_{n-1} + a_{n-3} \\ &= a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \\ &= L_{n-1} + L_{n-2}. \end{aligned}$$

[a] Corrected by Mr. Gong-Ching Lin (B00703082) on May 19, 2012.

Number of Subsets without Cyclically Consecutive Numbers (continued)

- Furthermore, $L_0 = 2$ and $L_1 = 1$.
  - $L_3 = a_2 + a_0 = 3 + 1 = 4$ and
    $L_4 = a_3 + a_1 = 5 + 2 = 7$.
  - So

$$
\begin{aligned}
L_2 &= L_4 - L_3 = 3, \\
L_1 &= L_3 - L_2 = 1, \\
L_0 &= L_2 - L_1 = 2.
\end{aligned}
$$

Number of Subsets without Cyclically Consecutive Numbers (continued)

- The general solution is

$$L_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

by Eq. (76) on p. 532.

- Solve

$$\begin{aligned} 2 &= L_0 = c_1 + c_2, \\ 1 &= L_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}, \end{aligned}$$

for $c_1 = 1$ and $c_2 = 1$.

Number of Subsets without Cyclically Consecutive Numbers (concluded)

- The solution is finally

$$L_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Number of Palindromes Revisited

- A palindrome is a composition for $m \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 102).
- Let a_n denote the number of palindromes for n.
- Clearly, $a_1 = 1$ and $a_2 = 2$.
- Given each palindrome for n, we can do two things.
  - Add 1 to the first and last summands to obtain a palindrome for $n + 2$.
  - Insert summand 1 to the start and end to obtain a palindrome for $n + 2$.
- Hence $a_{n+2} = 2a_n$, $n \geq 1$.

The Proof (continued)

- The characteristic equation $r^2 - 2 = 0$ has two roots $\pm\sqrt{2}$.
- The general solution is hence

$$a_n = c_1(\sqrt{2}\,)^n + c_2(-\sqrt{2}\,)^n.$$

- Solve[a]

$$\begin{aligned} 1 &= a_1 = \sqrt{2}\,(c_1 - c_2), \\ 2 &= a_2 = 2(c_1 + c_2), \end{aligned}$$

for $c_1 = (1 + \frac{1}{\sqrt{2}}\,)/2$ and $c_2 = (1 - \frac{1}{\sqrt{2}}\,)/2$.

[a] This time, we are not retrofitting.

The Proof (concluded)

- The number of palindromes for n therefore equals

$$
\begin{aligned}
a_n &= \frac{1+\frac{1}{\sqrt{2}}}{2}(\sqrt{2})^n + \frac{1-\frac{1}{\sqrt{2}}}{2}(-\sqrt{2})^n \\
&= \begin{cases} \frac{1+\frac{1}{\sqrt{2}}}{2}\, 2^{n/2} + \frac{1-\frac{1}{\sqrt{2}}}{2}\, 2^{n/2}, & \text{if } n \text{ is even,} \\ \frac{1+\frac{1}{\sqrt{2}}}{2}\sqrt{2}\, 2^{(n-1)/2} - \frac{1-\frac{1}{\sqrt{2}}}{2}\sqrt{2}\, 2^{(n-1)/2}, & \text{if } n \text{ is odd,} \end{cases} \\
&= \begin{cases} 2^{n/2}, & \text{if } n \text{ is even,} \\ 2^{(n-1)/2}, & \text{if } n \text{ is odd,} \end{cases} \\
&= 2^{\lfloor n/2 \rfloor}.
\end{aligned}
$$

- This matches Theorem 19 (p. 104).

An Example: A Third-Order Relation

- Consider

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$$

  with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.

- The characteristic equation $2r^3 - r^2 - 2r + 1 = 0$ has three distinct real roots: 1, -1, and 0.5.

- The general solution is

$$\begin{aligned} a_n &= c_1 1^n + c_2(-1)^n + c_3(1/2)^n \\ &= c_1 + c_2(-1)^n + c_3(1/2)^n. \end{aligned}$$

An Example: A Third-Order Relation (concluded)

- Solve the three initial conditions with Eq. (75) on p. 530,

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0.5 \\ 1^2 & (-1)^2 & 0.5^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- The solutions are $c_1 = 2.5$, $c_2 = 1/6$, and $c_3 = -8/3$.

The Case of Complex Roots

- Consider
$$a_n = 2(a_{n-1} - a_{n-2})$$
with $a_0 = 1$ and $a_1 = 2$.
- The characteristic equation $r^2 - 2r + 2 = 0$ has two distinct complex roots $1 \pm i$.
- The general solution is
$$a_n = c_1(1+i)^n + c_2(1-i)^n.$$

The Case of Complex Roots (concluded)

- Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.
- The particular solution becomes[a]

$$\begin{aligned} a_n &= (1+i)^{n-1} + (1-i)^{n-1} \\ &= (\sqrt{2}\,)^n [\,\cos(n\pi/4) + \sin(n\pi/4)\,]. \end{aligned}$$

[a] An equivalent one is $a_n = (\sqrt{2}\,)^{n+1} \cos((n-1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

- Consider the recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0,$$

where $C_n, C_{n-1}, \ldots$ are real constants, $C_n \neq 0$, $C_{n-k} \neq 0$.

- Let r be a characteristic root of **multiplicity** m, where $2 \leq m \leq k$, of the characteristic equation

$$f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0.$$

- The general solution that involves r has the form

$$(A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1})\, r^n, \qquad (81)$$

with $A_0, A_1, \ldots, A_{m-1}$ are constants to be determined.

The Proof

- If $f(x)$ has a root r of multiplicity m, then $f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$.
- Because $r \neq 0$ is a root of multiplicity m, it is easy to check that

$$
\begin{aligned}
0 &= r^{n-k} f(r), \\
0 &= r(r^{n-k} f(r))', \\
0 &= r(r(r^{n-k} f(r))')', \\
&\vdots \\
0 &= \overbrace{r(\cdots r(r(}^{m-1} r^{n-k} f(r) \overbrace{)')' \cdots)'}^{m-1}.
\end{aligned}
$$

The Proof (continued)

- Note that we differentiate and then multiply by r before iterating.
- These give

$$
\begin{aligned}
0 &= C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k}, \\
0 &= C_n n r^n + C_{n-1}(n-1)\, r^{n-1} + \cdots + C_{n-k}(n-k)\, r^{n-k}, \\
0 &= C_n n^2 r^n + C_{n-1}(n-1)^2 r^{n-1} + \cdots + C_{n-k}(n-k)^2 r^{n-k}, \\
&\vdots
\end{aligned}
$$

The Proof (continued)

- Now, $a_n = n^k r^n$, $0 \leq k \leq m-1$, is indeed a solution because the kth row above says

$$\begin{aligned} & 0 \\ = \;& C_n n^k r^n + C_{n-1}(n-1)^k r^{n-1} + \cdots + C_{n-k}(n-k)^k r^{n-k} \\ = \;& C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k}. \end{aligned}$$

The Proof (continued)

- From Eq. (73) on p. 525, $r^n, nr^n, n^2 r^n, \ldots, n^{m-1} r^n$ form a fundamental set if[a]

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ r & r & \cdots & r \\ r^2 & 2r^2 & \cdots & 2^{m-1} r^2 \\ \vdots & \vdots & \ddots & \vdots \\ r^{m-1} & (m-1)\, r^{m-1} & \cdots & (m-1)^{m-1} r^{m-1} \end{vmatrix} \neq 0.$$

- But it is a Vandermonde matrix in disguise.

[a]The ith row sets $n = i - 1$, $i = 1, 2, \ldots, m$.

The Proof (concluded)

- In fact, after deleting the first row and column, the determinant equals

$$(m-1)!\, r^{1+2+\cdots+(m-1)}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (m-1) & \cdots & (m-1)^{m-2} \end{vmatrix} \neq 0.$$

Nonhomogeneous Recurrence Relations

- Consider

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \tag{82}$$

- If $a_n = a_{n-1} + f(n)$, then the solution is $a_n = a_0 + \sum_{i=1}^{n} f(i)$.
  - A closed-form formula exists if one for $\sum_{i=1}^{n} f(i)$ does.
- In general, no failure-free methods exist except for specific $f(n)$s.
  - Consult pp. 441–2 of the textbook (4th ed.).

Examples ($c, c_1, c_2, \ldots$ Are Arbitrary Constants)

$a_{n+1} - a_n = 0$	$a_n = c$
$a_{n+1} - a_n = 1$	$a_n = n + c$
$a_{n+1} - a_n = n$	$a_n = n(n-1)/2 + c$
$a_{n+2} - 3a_{n+1} + 2a_n = 0$	$a_n = c_1 + c_2 2^n$
$a_{n+2} - 3a_{n+1} + 2a_n = 1$	$a_n = c_1 + c_2 2^n - n$
$a_{n+2} - a_n = 0$	$a_n = c_1 + c_2(-1)^n$
$a_{n+1} = a_n/(1 + a_n)$	$a_n = c/(1 + cn)$

Trial and Error

- Consider $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- Calculations show that $a_2 = 4$ and $a_3 = 12$.
- Conjecture:

$$a_n = n2^{n-1}. \tag{83}$$

- Verify that, indeed,

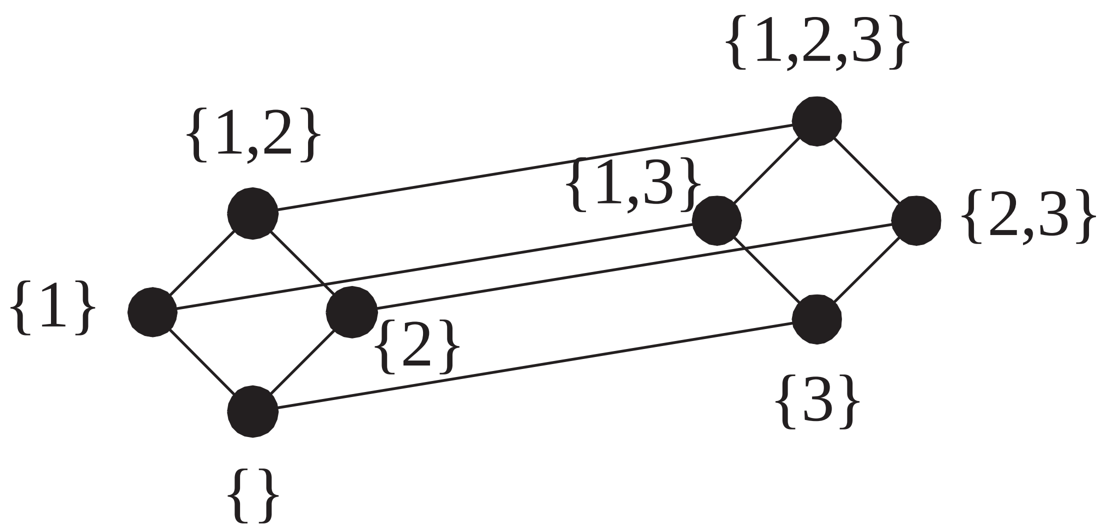
$$(n+1)\,2^n = 2(n2^{n-1}) + 2^n,$$

and $a_1 = 1$.

Application: Number of Edges of a Hasse Diagram

- Let a_n be the number of edges of the Hasse diagram for the partial order $(2^{\{1,2,\ldots,n\}}, \subseteq)$.
- Consider the Hasse diagrams H_1 for $(2^{\{1,2,\ldots,n\}}, \subseteq)$ and H_2 for $(\{T \cup \{n+1\} : T \subseteq \{1,2,\ldots,n\}\}, \subseteq)$.
  - H_1 and H_2 are "isomorphic."
- The Hasse diagram for $(2^{\{1,2,\ldots,n+1\}}, \subseteq)$ is constructed by adding an edge from each node T of H_1 to node $T \cup \{n+1\}$ of H_2.
- Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- The desired number has been solved in Eq. (83) on p. 563.

Illustration with $(2^{\{1,2,3\}}, \subseteq)$



Trial and Error Again

- Consider $a_{n+1} - A a_n = B$.
- Calculations show that

$$\begin{aligned} a_1 &= A a_0 + B, \\ a_2 &= A a_1 + B = A^2 a_0 + B(A+1), \\ a_3 &= A a_2 + B = A^3 a_0 + B(A^2 + A + 1). \end{aligned}$$

- Conjecture (easily verified by substitution):

$$a_n = \begin{cases} A^n a_0 + B\,\frac{A^n - 1}{A - 1}, & \text{if } A \neq 1 \\ a_0 + Bn, & \text{if } A = 1 \end{cases}. \qquad (84)$$

Financial Application: Compound Interest[a]

- Consider $a_{n+1} = (1 + r)\, a_n$.
  - Deposit grows at a period interest rate of $r > 0$.
  - The initial deposit is a_0 dollars.
- The solution is obviously

$$a_n = (1 + r)^n a_0.$$

- The deposit therefore grows exponentially with time.

[a] "In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect," wrote Joseph Alois Schumpeter (1883–1950) in *Capitalism, Socialism and Democracy* (1942).

Financial Application: Amortization

- Consider $a_{n+1} = (1 + r)\, a_n - M$.
  - The initial loan amount is a_0 dollars.
  - The monthly payment is M dollars.
  - The outstanding loan principal after the nth payment is a_{n+1}.
- By Eq. (84) on p. 566, the solution is

$$a_n = (1 + r)^n a_0 - M\, \frac{(1 + r)^n - 1}{r}.$$

The Proof (concluded)

- What is the unique monthly payment M for the loan to be closed after k months?
- Set $a_k = 0$ to obtain

$$a_k = (1+r)^k a_0 - M\,\frac{(1+r)^k - 1}{r} = 0.$$

- Hence

$$M = \frac{(1+r)^k a_0 r}{(1+r)^k - 1}.$$

- This is standard calculation for home mortgages and annuities.[a]

[a] Lyuu (2002).

Trial and Error a Third Time

- Consider the more general $a_{n+1} - A a_n = B C^n$.
- Calculations show that

$$\begin{aligned} a_1 &= A a_0 + B, \\ a_2 &= A a_1 + BC = A^2 a_0 + B(A + C), \\ a_3 &= A a_2 + BC^2 = A^3 a_0 + B(A^2 + AC + C^2). \end{aligned}$$

- Conjecture (easily verified by substitution):

$$a_n = \begin{cases} A^n a_0 + B \frac{A^n - C^n}{A - C} & \text{if } A \neq C \\ A^n a_0 + B A^{n-1} n & \text{if } A = C \end{cases}. \qquad (85)$$

Application: Runs of Binary Strings

- A run is a maximal consecutive list of identical objects (p. 106).
  - Binary string "`0 0 1 1 1 0`" has 3 runs.
- Let r_n denote the total number of runs determined by the 2^n binary strings of length n.
- First, $r_1 = 2$.
  - Each of "`0`" and "`1`" has 1 run.
- In general, suppose we append a bit to an $(n-1)$-bit string $b_1 b_2 \cdots b_{n-1}$ to make $b_1 b_2 \cdots b_{n-1} b_n$.

The Proof (continued)

- For those with $b_{n-1} = b_n$ (i.e., the last 2 bits are identical), the total number of runs does not change.
  - The total number of runs remains r_{n-1}.
- For those with $b_{n-1} \neq b_n$ (i.e., the last 2 bits are distinct), the total number of runs *increases* by 1 for *each* $(n-1)$-bit string.
  - There are 2^{n-1} of them.
  - So the total number of runs becomes $r_{n-1} + 2^{n-1}$.
- Hence

$$r_n = 2r_{n-1} + 2^{n-1}, n \geq 2.$$

The Proof (concluded)

- By Eq. (85) on p. 570,
$$r_n = 2^n r_0 + 2^{n-1} n.$$
- To make sure that $r_1 = 2$, it is easy to see that $r_0 = 1/2$.
- Hence
$$r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1}(n+1).$$
  - The recurrence is identical to that for the number of edges of a Hasse diagram (p. 564) except for the initial condition.
  - Its solution appeared in Eq. (83) on p. 563, $a_n = n2^{n-1}$.

Method of Undetermined Coefficients

- Recall Eq. (82) on p. 561, repeated below:

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \qquad (86)$$

- Let $a_n^{(h)}$ denote the general solution of the associated *homogeneous* relation (with $f(n) = 0$).
- Let $a_n^{(p)}$ denote a particular solution of the *nonhomogeneous* relation.
- Then

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

- All the entries in the table on p. 562 fit the claim.

Conditions for the General Solution

Similar to Theorem 71 (p. 525), we have the following theorem.

Theorem 72 *Let* $a_n^{(p)}$ *be any particular solution of the nonhomogeneous recurrence relation Eq. (86) on p. 574. Let*

$$a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}$$

be the general solution of its homogeneous version as specified in Theorem 71. Then $a_n^{(h)} + a_n^{(p)}$ *is the general solution of Eq. (86) on p. 574.*

Solution Techniques

- Typically, one finds the general solution of its homogeneous version $a_n^{(h)}$ first.
- Then one finds a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation Eq. (86) on p. 574.
- Make sure $a_n^{(p)}$ is "independent" of $a_n^{(h)}$.
- Finally, use the initial conditions to nail down the coefficients of $a_n^{(h)}$.
- Output $a_n^{(h)} + a_n^{(p)}$.

$a_{n+1} - Aa_n = B$ Revisited

- Recall that the general solution is $a_n^{(h)} = cA^n$.
- A particular solution is

$$a_n^{(p)} = \begin{cases} B/(1-A) & \text{if } A \neq 1 \\ Bn & \text{if } A = 1 \end{cases}.$$

- So $a_n = cA^n + a_n^{(p)}$.
- In particular,

$$c = a_0 - a_0^{(p)} = \begin{cases} a_0 - B/(1-A) & \text{if } A \neq 1 \\ a_0 & \text{if } A = 1 \end{cases}.$$

$a_{n+1} - Aa_n = B$ Revisited (concluded)

- The solution matches Eq. (84) on p. 566.
- We can rewrite the solution as

$$a_n = \begin{cases} A^n [\, a_0 - a_n^{(p)} \,] + a_n^{(p)}, & \text{if } A \neq 1 \\ a_0 + a_n^{(p)}, & \text{if } A = 1 \end{cases} . \tag{87}$$

Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 7^n$ with $a_0 = 2$

- $a_n^{(h)} = c \times 3^n$, because the characteristic equation has the nonzero root 3.
- We propose $a_n^{(p)} = a \times 7^n$.
- Place $a \times 7^n$ into the relation to obtain $a \times 7^n - 3a \times 7^{n-1} = 5 \times 7^n$.
- Hence $a = 35/4$ and $a_n^{(p)} = (35/4) \times 7^n = (5/4) \times 7^{n+1}$.
- The general solution is $a_n = c \times 3^n + (5/4) \times 7^{n+1}$.
- Now, $c = -27/4$ because $a_0 = 2 = c + (5/4) \times 7$.
- So the solution is $a_n = -(27/4) \times 3^n + (5/4) \times 7^{n+1}$.

Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 3^n$ with $a_0 = 2$

- As before, $a_n^{(h)} = c \times 3^n$.
- But this time $a_n^{(h)}$ and $f(n) = 5 \times 3^n$ are *not* "independent."
- So propose $a_n^{(p)} = an \times 3^n$.
- Plug $an \times 3^n$ into the relation to obtain $an \times 3^n - 3a(n-1) \times 3^{n-1} = 5 \times 3^n$.
- Hence $a = 5$ and $a_n^{(p)} = 5n \times 3^n$.
- The general solution is $a_n = c \times 3^n + 5n \times 3^n$.
- Finally we find that $c = 2$ with use of $a_0 = 2$.

Nonhomogeneous $a_{n+1} - 2a_n = n + 1$ with $a_0 = 4$

- From Eq. (84) on p. 566, $a_n^{(h)} = c \times 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute this particular solution into the relation to yield
$$a(n+1) + b - 2(an + b) = n + 1.$$
- Rearrange the above to obtain
$$(-a-1)\, n + (a - b - 1) = 0.$$
- This holds for all n if $a = -1$ and $b = -2$.

The Proof (concluded)

- Hence $a_n^{(p)} = -n - 2$.
- The general solution is

$$a_n = c \times 2^n - n - 2.$$

- Use the initial condition

$$4 = a_0 = c - 2$$

  to obtain $c = 6$.
- The solution to the complete relation is

$$a_n = 6 \times 2^n - n - 2.$$

Nonhomogeneous $a_{n+1} - a_n = 2n + 3$ with $a_0 = 1$

- This equation is very similar to the previous one: $a_{n+1} - 2a_n = n + 1$.
- First, $a_n^{(h)} = d \times 1^n = d$.
- If one guesses $a_n^{(p)} = an + b$ as before, then

$$a_{n+1} - a_n = a(n+1) + b - an - b = a,$$

  which cannot be right.[a]
- So we guess $a_n^{(p)} = an^2 + bn + c$.

---

[a] Contributed by Mr. Yen-Chieh Sung (B01902011) on June 17, 2013.

The Proof (continued)

- Substitute this particular solution into the relation to yield

$$a(n+1)^2 + b(n+1) + c - (an^2 + bn + c) = 2n + 3.$$

- Simplify the above to obtain

$$2an + (a+b) = 2n + 3.$$

- Hence $a = 1$ and $b = 2$.
- Hence $a_n^{(p)} = n^2 + 2n + c$.
- The general solution is $a_n = n^2 + 2n + c$.[a]

[a] We merge d into c.

The Proof (concluded)

- Use the initial condition

$$1 = a_0 = c$$

to obtain $c = 1$.

- The solution to the complete relation is

$$a_n = n^2 + 2n + 1 = (n+1)^2.$$

- It is very different from the solution to the previous example: $a_n = 6 \times 2^n - n - 2$.

Nonhomogeneous $a_{n+2} - 3a_{n+1} + 2a_n = 2$ with $a_0 = 0$ and $a_1 = 2$

- The characteristic equation $r^2 - 3r + 2 = 0$ has roots 2 and 1.
- So $a_n^{(h)} = c_1 1^n + c_2 2^n = c_1 + c_2 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute $a_n^{(p)}$ into the relation to yield
$$a(n+2) + b - 3[\, a(n+1) + b\,] + 2(an + b) = 2.$$
- Rearrange the above to obtain $a = -2$.
- Hence $a_n^{(p)} = -2n + b$.

The Proof (concluded)

- The general solution is now $a_n = c_1 + c_2 2^n - 2n$.[a]
- Use the initial conditions

$$\begin{aligned} 0 = a_0 &= c_1 + c_2, \\ 2 = a_1 &= c_1 + 2c_2 - 2. \end{aligned}$$

  to obtain $c_1 = -4$ and $c_2 = 4$.
- The solution to the complete relation is

$$a_n = -4 + 2^{n+2} - 2n.$$

[a] We merge b into c_1.