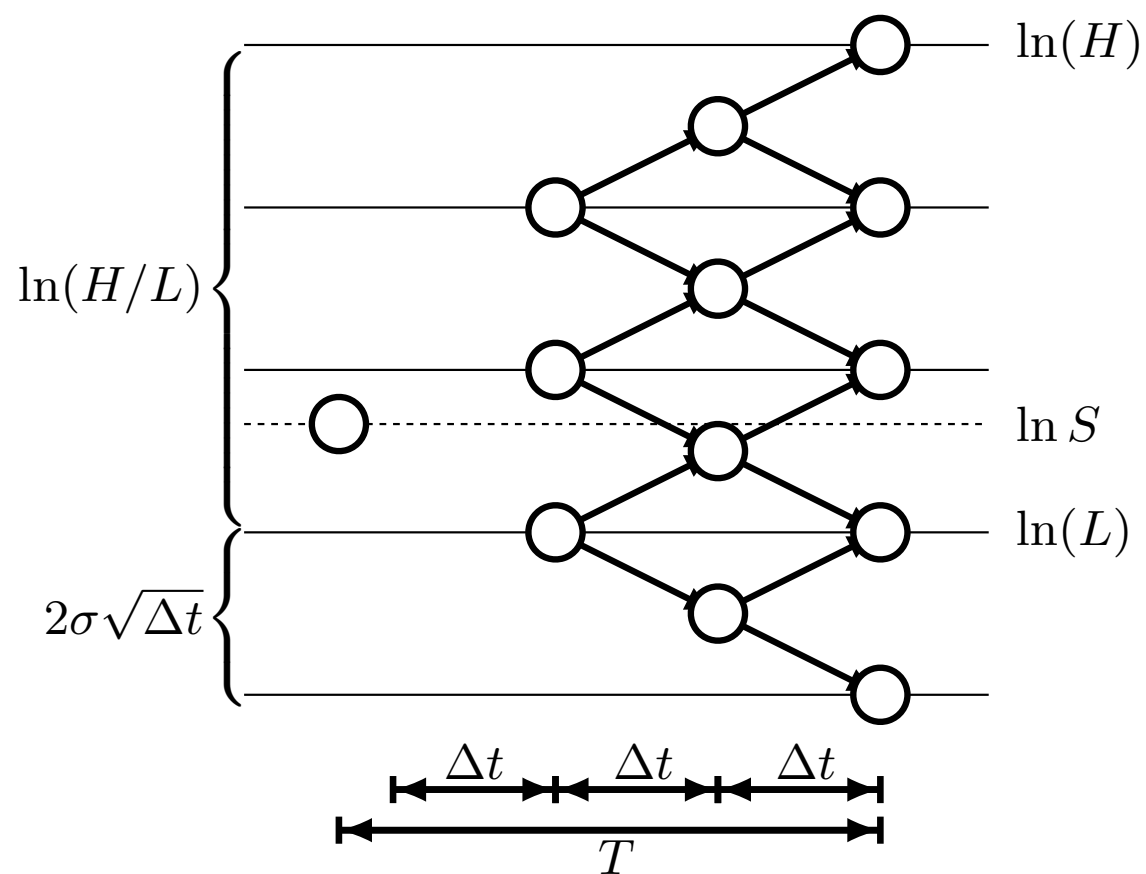# Ideas for Binomial Trees To Handle Two Barriers
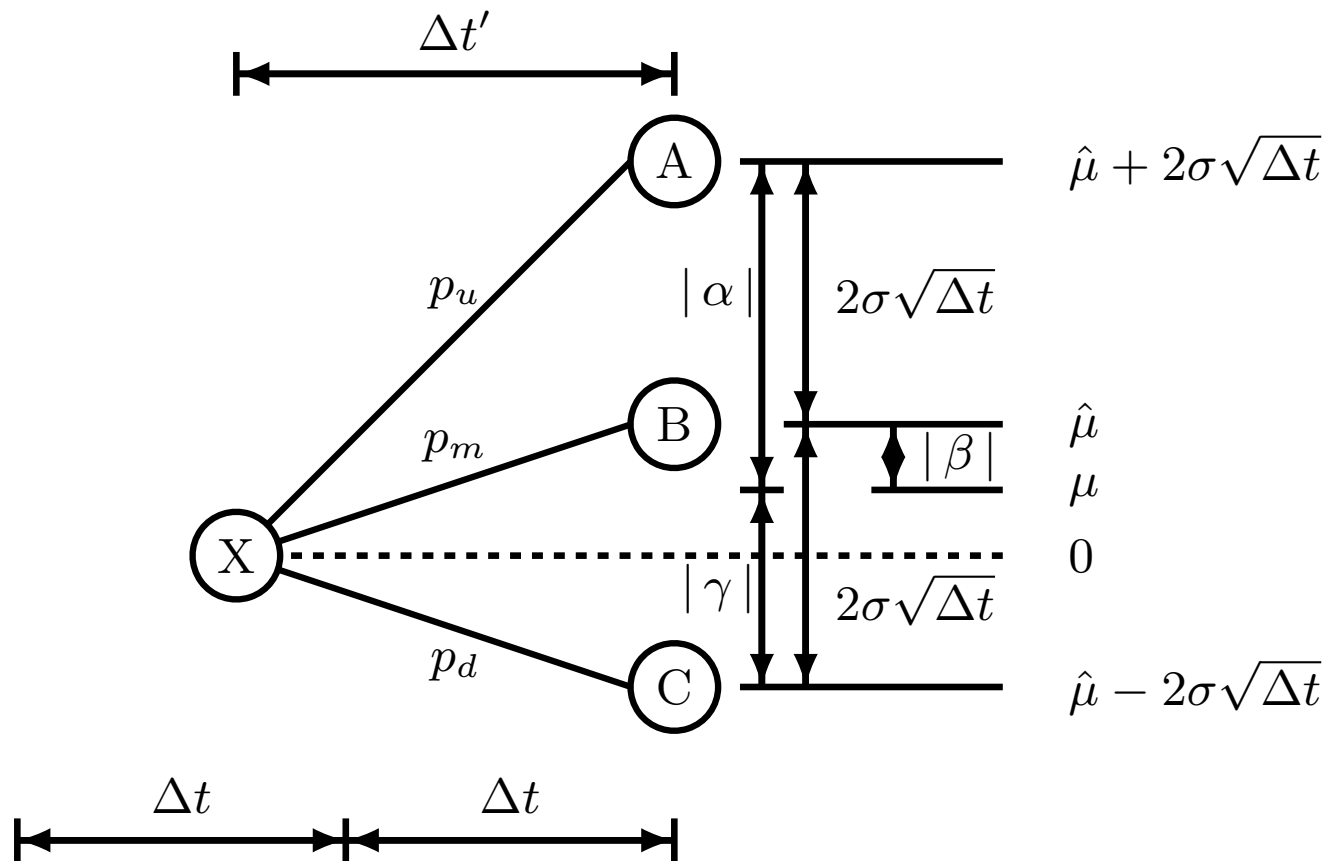
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.[a]

- The resulting tree is called the binomial-trinomial tree.[b]

- Suppose a binomial tree will be built with Δt as the duration of one period.

- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.

  - $\Delta t \leq \Delta t' < 2\Delta t$.

---

[a]Dai (`R86526008`, `D8852600`) and Lyuu (2006, 2008, 2010).
[b]The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)

The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:

  1. The mean and variance of the stock price are matched.

  2. The branching probabilities are between 0 and 1.

- Let S be the stock price at node X.

- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t + \Delta t'$ equals[a]

$$\mu \equiv \left(r - \sigma^2/2 \right) \Delta t'. \tag{69}$$

- Its variance equals

$$\mathrm{Var} \equiv \sigma^2 \Delta t'. \tag{70}$$

- Let node B be the node whose logarithmic return $\hat{\mu} \equiv \ln(s(\mathrm{B})/S)$ is closest to μ among all the nodes on the binomial tree at time $t + \Delta t'$.

---

[a]See p. 275.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.

- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.

- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.

- Review the figure on p. 674 for illustration.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.

- Recall that

$$\hat{\mu} \equiv \ln\left(s(B)/S\right)$$

is the logarithmic return of the middle node B.

- Let α, β, and γ be the differences between μ and the logarithmic returns

$$\ln(s(Z)/S), \quad Z = A, B, C,$$

in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}, \qquad (71)$$

$$\beta \equiv \hat{\mu} - \mu, \qquad (72)$$

$$\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}. \qquad (73)$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \qquad (74)$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \text{Var}, \qquad (75)$$

$$p_u + p_m + p_d = 1. \qquad (76)$$

The Binomial-Trinomial Tree (concluded)

- Equation (74) matches the mean (69) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 676.

- Equation (75) matches its variance (70) on p. 676.

- The three probabilities can be proved to lie between 0 and 1.
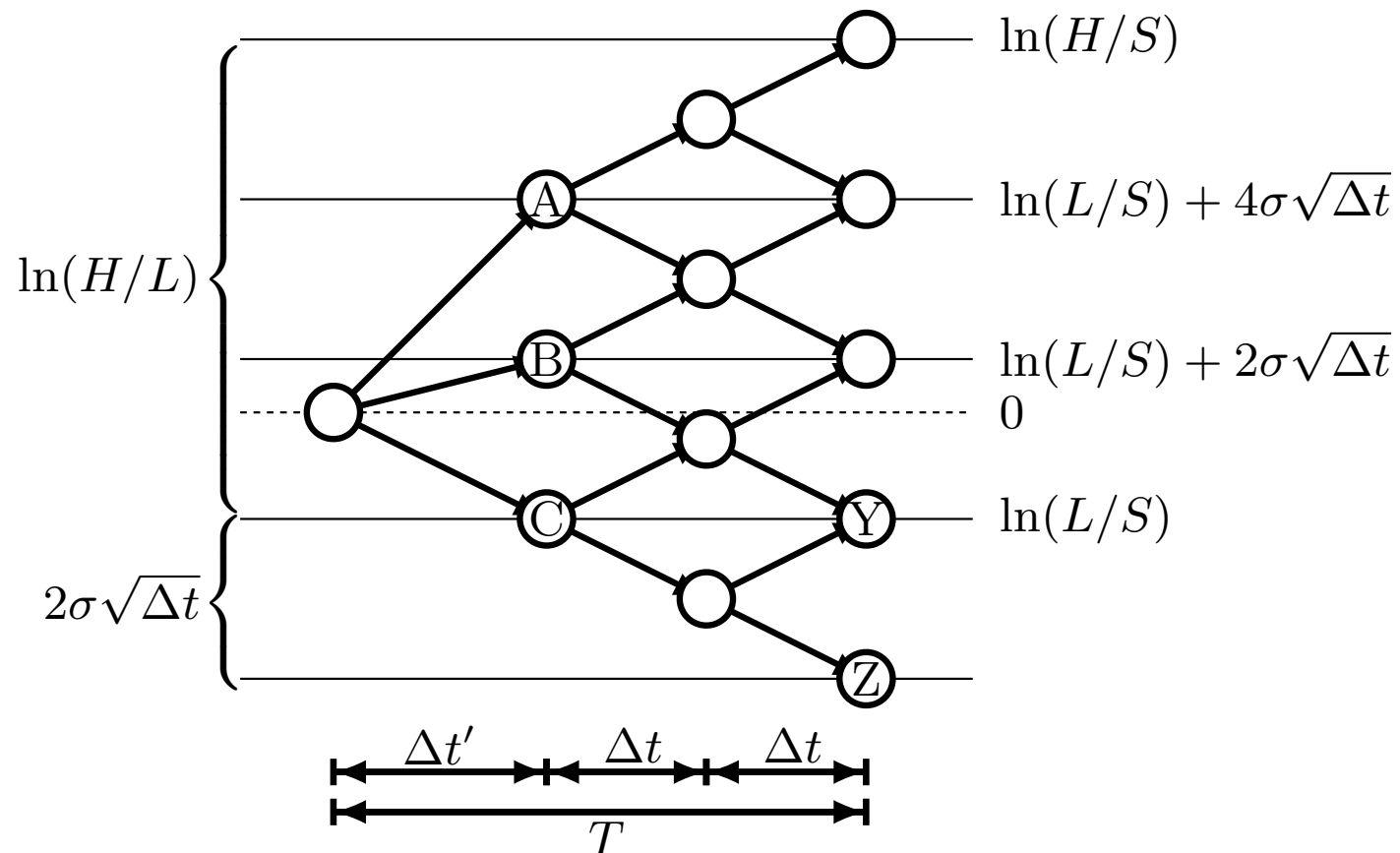
Pricing Double-Barrier Options

- Consider a double-barrier option with two barriers L and H, where $L < S < H$.

- We need to make each barrier coincide with a layer of the binomial tree for better convergence.

- This means choosing a Δt such that

$$\kappa \equiv \frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$$

  is a positive integer.

  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 682 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)

Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.

- Suppose we pick $\Delta\tau \equiv T/m$ for the length of each period.

- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.

- So we pick a Δt that is close to, but does not exceed, $\Delta\tau$ and makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ an integer.

- Specifically, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma} \right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.

- Start with the binomial part.

- Lay out the nodes from the low barrier L upward and downward.

- Automatically, a layer coincides with the high barrier H.

- It is unlikely that Δt divides T, however.

- So the position at time 0 and with logarithmic return $\ln(S/S) = 0$ is not occupied by a binomial node to serve as the root node (recall p. 682).

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.

- Between time 0 and time T, the binomial tree spans $T/\Delta t$ periods.

- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.

- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.

- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.

- Calculate the three branching probabilities to them.

- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.

- See the figure on p. 682 for illustration.

Pricing Double-Barrier Options (continued)

- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.

- That takes quadratic time.

- But we know a linear-time algorithm exists for double-barrier options on the binomial tree (see text).

- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.

  - That is, nodes A, B, and C on p. 682.

- Then calculate their expected discounted value for the root node.

- The overall running time is only linear!

Pricing Double-Barrier Options (continued)

- Binomial trees have troubles with pricing barrier options (see p. 375, p. 666, and p. 671).

- Even pit against the much better trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 689).

- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[a]

- It has an error of $O(1/n^{1-a})$ for any $0 < a < 1$ for double-barrier options.[b]

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[a]Lyuu and Palmer (2010).
[b]Elisa Appolloni, Gaudenziy, and Zanette (2014).

Pricing Double-Barrier Options (concluded)
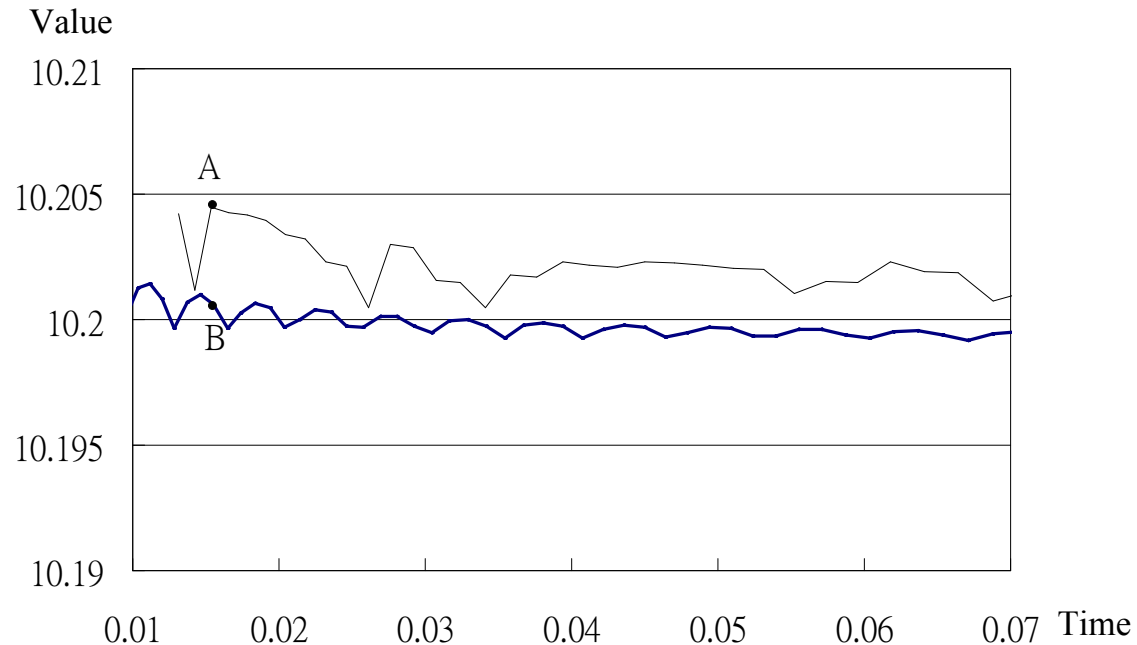
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).
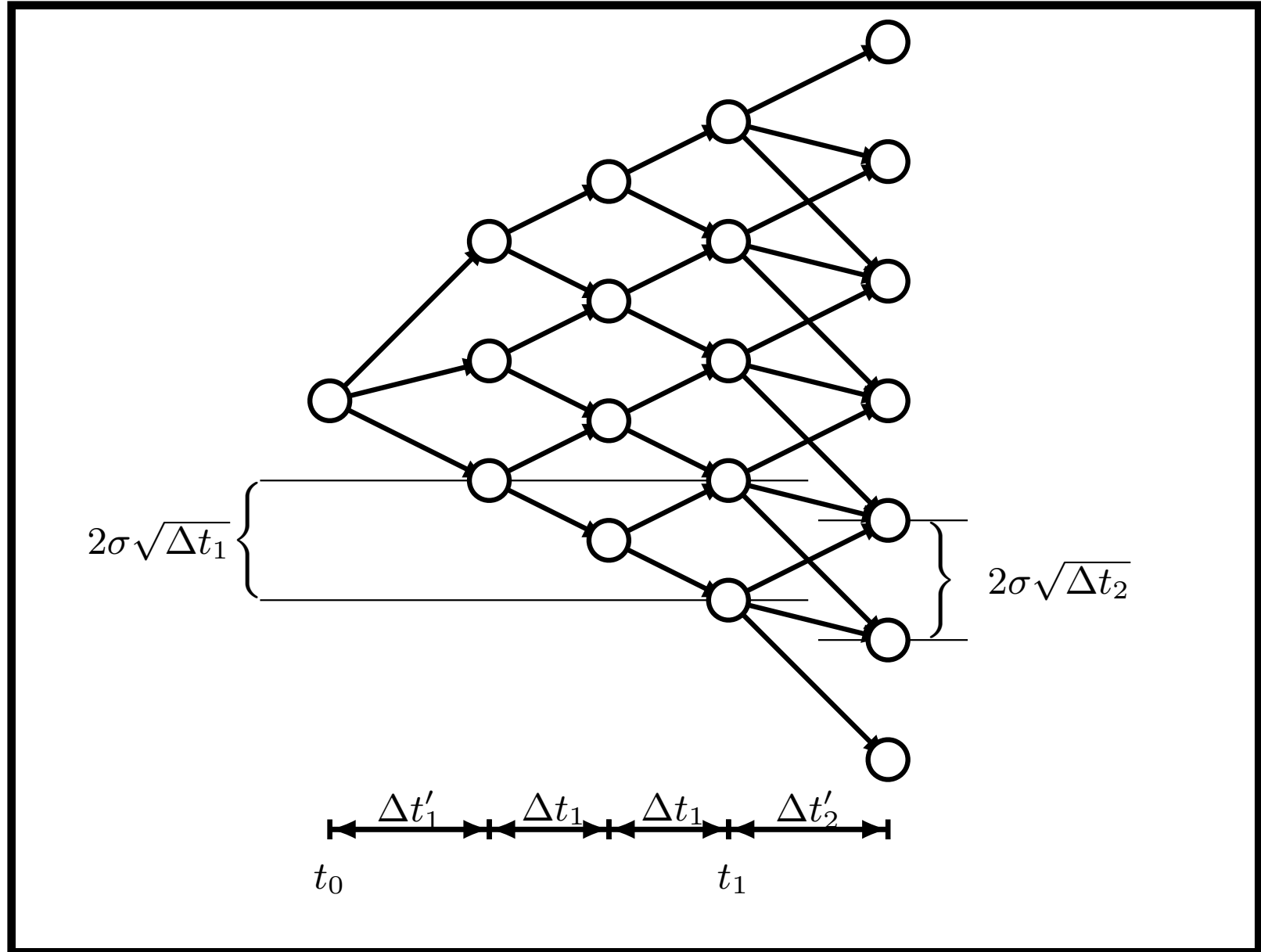
Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.

- They are more common than the continuously monitored versions.

- The main difficulty with pricing discrete barrier options lies in matching the monitored times.

- Here is why.

- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$

Pricing Discrete Barrier Options (continued)

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, meaning Δt divides t_i for all i.

- The binomial-trinomial tree can handle discrete options with ease, however.

- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.

-

$2\sigma\sqrt{\Delta t_1}$

$2\sigma\sqrt{\Delta t_2}$

$\Delta t_1'$ Δt_1 Δt_1 $\Delta t_2'$

t_0 t_1

Pricing Discrete Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for
  option pricing with known dividends.[a]

  1. The one we saw earlier (p. 292) models the stock
     price minus the present value of the anticipated
     dividends as following geometric Brownian motion.

  2. One can also model the stock price plus the forward
     values of the dividends as following geometric
     Brownian motion.

  ---
  [a]Frishling (2002).

Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:

  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.

  - The dividend is part cash and part yield (i.e., $\alpha(t)S_0 + \beta(t)S_t$), for practitioners.[a]

- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.

- But they result in binomial trees that grow exponentially (recall p. 291).

- The binomial-trinomial tree can often avoid this problem.

---

[a]Henry-Labordère (2009).

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.

- So there are $m \equiv t/\Delta t$ periods between time 0 and the ex-dividend date.

- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D$.

  - Equivalently,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)}\right]^2.$$

Options on a Stock That Pays Known Dividends (continued)

- Build a binomial tree from time 0 to time t as before.

- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.

- Assume the top node's price equals S'.

  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.

- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \ldots,$$

same as the binomial tree.

Options on a Stock That Pays Known Dividends (concluded)

- For each node at time t below the top node, we build the trinomial connection.

- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 674.

- Hence the construction can be completed.

- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.

- The resulting tree is called the stair tree.[a]

---

[a]Dai (`R86526008`, `D8852600`) and Lyuu (2004); Dai (`R86526008`) (2009).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]

- Option pricing with stochastic volatilities.[b]

- Efficient Parisian option pricing.[c]

- Option pricing with time-varying volatilities and time-varying barriers.[d]

- Defaultable bond pricing.[e]

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[a]Wu (R96723058) (2009).
[b]Huang (R97922073) (2010).
[c]Huang (R97922081) (2010).
[d]Chou (R97944012) (2010) and Chen (R98922127) (2011).
[e]Dai (R86526008, D8852600), Lyuu, and Wang (F95922018) (2009, 2010, 2014).

General Properties of Trees[a]

- Consider the Ito process,

$$dX = a(X, t)\, dt + \sigma\, dW,$$

 where $a(X, t) = O(1)$ and σ is a constant.

- The mean and volatility of the next move's size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.

- Note that $\sqrt{\Delta t} \gg \Delta t$.

- The tree spacing must be in the order of $\sigma\sqrt{\Delta t}$ if the variance is to be matched.[b]

---

[a]Chiu (`R98723059`) (2012) and Wu (`R99922149`) (2012).
[b]Lyuu and Wang (`F95922018`) (2009, 2011) and Lyuu and Wen (`D94922003`) (2012).

General Properties of Trees (concluded)

- It can also be proved that either B is a tree node or both A and C are tree nodes.
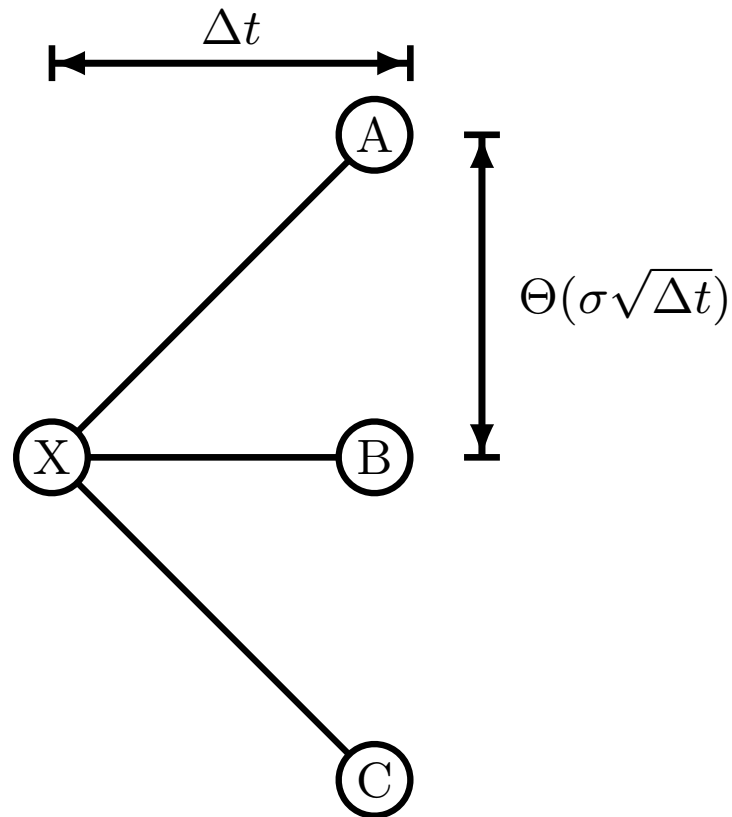
Merton's Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model's assumptions.

- Stochastic volatility and jump processes have been proposed to address this problem.

- Merton's jump-diffusion model is our focus.[a]

---

[a]Merton (1976).

Merton's Jump-Diffusion Model (continued)

- This model superimposes a jump component on a diffusion component.

- The diffusion component is the familiar geometric Brownian motion.

- The jump component is composed of lognormal jumps driven by a Poisson process.

  – It models the sudden changes in the stock price because of the arrival of important new information.

Merton's Jump-Diffusion Model (continued)

- Let S_t be the stock price at time t.

- The risk-neutral jump-diffusion process for the stock price follows

$$\frac{dS_t}{S_t} = (r - \lambda \bar{k}) \, dt + \sigma \, dW_t + k \, dq_t. \qquad (77)$$

- Above, σ denotes the volatility of the diffusion component.

Merton's Jump-Diffusion Model (continued)

- The jump event is governed by a compound Poisson process q_t with intensity λ, where k denotes the magnitude of the *random* jump.

  - The distribution of k obeys

  $$\ln(1 + k) \sim N\left(\gamma, \delta^2\right)$$

  with mean $\bar{k} \equiv E(k) = e^{\gamma + \delta^2/2} - 1.$

- The model with $\lambda = 0$ reduces to the Black-Scholes model.

Merton's Jump-Diffusion Model (continued)

- The solution to Eq. (77) on p. 704 is

$$S_t = S_0 e^{(r - \lambda \bar{k} - \sigma^2/2)\, t + \sigma W_t} U(n(t)), \qquad (78)$$

where

$$U(n(t)) = \prod_{i=0}^{n(t)} (1 + k_i).$$

- k_i is the magnitude of the ith jump with $\ln(1 + k_i) \sim N(\gamma, \delta^2)$.

- $k_0 = 0$.

- $n(t)$ is a Poisson process with intensity λ.

Merton's Jump-Diffusion Model (concluded)

- Recall that $n(t)$ denotes the number of jumps that occur up to time t.

- As $k > -1$, stock prices will stay positive.

- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.

Tree for Merton's Jump-Diffusion Model[a]

- Define the S-logarithmic return of the stock price S' as

$$\ln(S'/S).$$

- Define the logarithmic distance between stock prices S' and S as

$$|\ln(S') - \ln(S)| = |\ln(S'/S)|.$$

---

[a]Dai (`R86526008`, `D8852600`), Wang (`F95922018`), Lyuu, and Liu (2010).

Tree for Merton's Jump-Diffusion Model (continued)

- Take the logarithm of Eq. (78) on p. 706:

$$M_t \equiv \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t, \tag{79}$$

  where

$$X_t \equiv \left(r - \lambda\bar{k} - \sigma^2/2\right) t + \sigma W_t, \tag{80}$$

$$Y_t \equiv \sum_{i=0}^{n(t)} \ln\left(1 + k_i\right). \tag{81}$$

- It decomposes the S_0-logarithmic return of S_t into the diffusion component X_t and the jump component Y_t.

Tree for Merton's Jump-Diffusion Model (continued)

- Motivated by decomposition (79) on p. 709, the tree construction divides each period into a diffusion phase followed by a jump phase.

- In the diffusion phase, X_t is approximated by the BOPM.

- Hence X_t can make an up move to $X_t + \sigma\sqrt{\Delta t}$ with probability p_u or a down move to $X_t - \sigma\sqrt{\Delta t}$ with probability p_d.
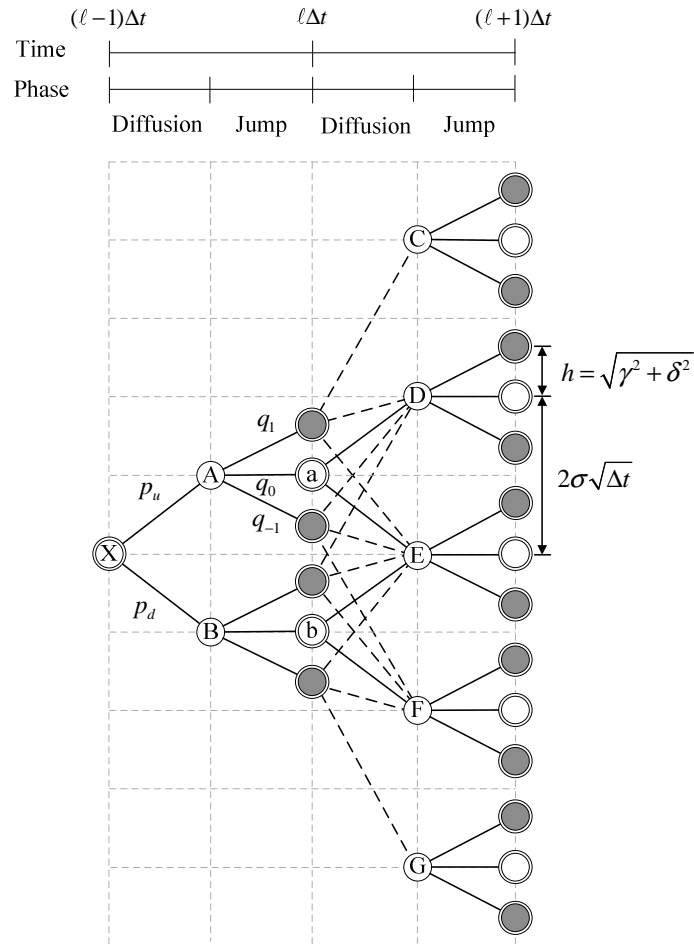
Tree for Merton's Jump-Diffusion Model (continued)

- According to BOPM,

$$
\begin{aligned}
p_u &= \frac{e^{\mu \Delta t} - d}{u - d}, \\
p_d &= 1 - p_u,
\end{aligned}
$$

  except that $\mu = r - \lambda \bar{k}$ here.

- The diffusion component gives rise to diffusion nodes.

- They are spaced at $2\sigma\sqrt{\Delta t}$ apart such as the white nodes A, B, C, D, E, F, and G on p. 712.

White nodes are diffusion nodes. Gray nodes are jump nodes. In the diffusion phase, the solid black lines denote the binomial structure of BOPM, whereas the dashed lines denote the trinomial structure. Here m is set to one here for simplicity. Only the double-circled nodes will remain after the construction. Note that a and b are diffusion nodes because no jump occurs ($j = 0$ in Eq. (82)) in the jump phase.

Tree for Merton's Jump-Diffusion Model (continued)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from *each* diffusion node to $2m$ jump nodes that match the first $2m$ moments of the lognormal jump.

- The m jump nodes above the diffusion node are spaced at h apart.

- The same holds for the m jump nodes below the diffusion node.

- The gray nodes at time $\ell \Delta t$ on p. 712 are jump nodes.

Tree for Merton's Jump-Diffusion Model (continued)

- Overall, the random variable M_t is approximated as follows.

- Following M_t, the S_0-logarithmic returns of the $2 \times (2m+1)$ nodes at time $t + \Delta t$ are

$$M_{t+\Delta t} = M_t + c\sigma\sqrt{\Delta t} + jh, \quad c = \pm 1, j = 0, \pm 1, \pm 2, \ldots, \pm m. \tag{82}$$

  - $c \in \{-1, 1\}$ denotes the up or down move of the stock price driven by the diffusion component.
  - j denotes the number of jumps above or below the diffusion nodes, driven by the jump component.

Tree for Merton's Jump-Diffusion Model (continued)

- The magnitude of the basic jump size is set to

$$h = \sqrt{\gamma^2 + \delta^2} \,. \tag{83}$$

- In summary, the stock prices at the end of the period are

$$Sue^{jh}, Sde^{jh}, -m \le j \le m.$$

- Only these $2 \times (2m + 1)$ nodes at the end of the jump phase will be retained for the tree; the nodes at the end of the diffusion phase will not.

Tree for Merton's Jump-Diffusion Model (continued)

- The probabilities remain to be calculated.

- Let μ_i' be the ith moment of $Y_{t+\Delta t} - Y_t$.

- The probabilities for the jump phase are

$$q_j, \quad -m \le j \le m.$$

Tree for Merton's Jump-Diffusion Model (continued)

- They must match the first $2m$ moments of the jump component, i.e.,

$$
\begin{aligned}
\sum_{j=-m}^{m} (jh)^i q_j \\
= \quad & \mu'_i \\
\equiv \quad & E\left[\left(\sum_{w=0}^{n(\Delta t)} \ln(1 + k_w) \right)^i \right], \ i = 1, 2, \ldots, 2m. \quad (84)
\end{aligned}
$$

Tree for Merton's Jump-Diffusion Model (continued)

- For sufficiently small Δt, the approximation $\mu'_i \approx \kappa_i$ is accurate as $\mu'_1 = \kappa_1$ and $\mu'_i = \kappa_i + O((\Delta t)^2)$, $i = 2, 3, \ldots$, where[a]

$$
\begin{aligned}
\kappa_1 &= \lambda \Delta t (\gamma), \\
\kappa_2 &= \lambda \Delta t \left(\gamma^2 + \delta^2 \right), \\
\kappa_3 &= \lambda \Delta t \left(\gamma^3 + 3\gamma\delta^2 \right), \\
\kappa_4 &= \lambda \Delta t \left(\gamma^4 + 6\gamma^2\delta^2 + 3\delta^4 \right), \\
\kappa_5 &= \lambda \Delta t \left(\gamma^5 + 10\gamma^3\delta^2 + 15\gamma\delta^4 \right), \\
\kappa_6 &= \lambda \Delta t \left(\gamma^6 + 15\gamma^4\delta^2 + 45\gamma^2\delta^4 + 15\delta^6 \right), \\
\kappa_7 &= \lambda \Delta t \left(\gamma^7 + 21\gamma^5\delta^2 + 105\gamma^3\delta^4 + 105\gamma\delta^6 \right), \\
&\vdots
\end{aligned}
$$

---

[a]Stuart and Ord (1994).

Tree for Merton's Jump-Diffusion Model (continued)

- Finally,

$$\sum_{j=-m}^{m} q_j = 1, \tag{85}$$

  where $q_j \geq 0$.

- Solve the $2m + 1$ equations (84)–(85) to obtain the probabilities q_j.

Tree for Merton's Jump-Diffusion Model (continued)

- The diffusion component and the jump component are independent.

- So the probability of moving from M_t to $M_t + \sigma\sqrt{\Delta t} + jh$ is $p_u q_j$ and that to $M_t - \sigma\sqrt{\Delta t} + jh$ is $p_d q_j$.

- In the ensuing diffusion phase, the trinomial branches on p. 674 (the dashed lines) are used to link the jump nodes (like the gray nodes at time $\ell \Delta t$) with the diffusion nodes (like nodes C, D, E, F, and G).

Tree for Merton's Jump-Diffusion Model (concluded)

- The diffusion nodes like nodes a and b at time $\ell \Delta t$, however, continue to follow the structure of BOPM.

- The procedure is repeated until the maturity date.

- The size of the tree is $O(n^{2.5})$.

Multivariate Contingent Claims

- They depend on two or more underlying assets.

- The basket call on m assets has the terminal payoff

$$\max\left(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0\right),$$

  where α_i is the percentage of asset i.

- Basket options are essentially options on a portfolio of stocks or index options.

- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Multivariate Contingent Claims (concluded)[a]

Name	Payoff
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$
Better-off option	$\max(S_1(\tau), \ldots, S_k(\tau), 0)$
Worst-off option	$\min(S_1(\tau), \ldots, S_k(\tau), 0)$
Binary maximum option	$I\{\max(S_1(\tau), \ldots, S_k(\tau)) > X\}$
Maximum option	$\max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Minimum option	$\max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$
Basket average option	$\max((S_1(\tau), \ldots, S_k(\tau))/k - X, 0)$
Multi-strike option	$\max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0)$
Pyramid rainbow option	$\max(|S_1(\tau) - X_1| + \cdots + |S_k(\tau) - X_k| - X, 0)$
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0)$

[a]Lyuu and Teng (R91723054) (2011).

Correlated Trinomial Model[a]

- Two risky assets S_1 and S_2 follow

$$\frac{dS_i}{S_i} = r\,dt + \sigma_i\,dW_i$$

  in a risk-neutral economy, $i = 1, 2$.

- Let

$$
\begin{aligned}
M_i &\equiv e^{r\Delta t}, \\
V_i &\equiv M_i^2 (e^{\sigma_i^2 \Delta t} - 1).
\end{aligned}
$$

  - $S_i M_i$ is the mean of S_i at time Δt.
  - $S_i^2 V_i$ the variance of S_i at time Δt.

---

[a]Boyle, Evnine, and Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where ρ is the correlation between dW_1 and dW_2.

- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.

- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is (as usual, we impose $u_i d_i = 1$)

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$
\begin{aligned}
1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\
S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\
S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2.
\end{aligned}
$$

Correlated Trinomial Model (concluded)

- Let $R \equiv M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$.

- Match the variances and covariance:

$$
\begin{aligned}
S_1^2 V_1 &= (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5(S_1^2 - (S_1 M_1)^2) \\
&\quad + (p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2), \\
S_2^2 V_2 &= (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5(S_2^2 - (S_2 M_2)^2) \\
&\quad + (p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2), \\
S_1 S_2 R &= (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5)\, S_1 S_2.
\end{aligned}
$$

- The solutions are complex (see text).

Correlated Trinomial Model Simplified[a]

- Let $\mu_i' \equiv r - \sigma_i^2/2$ and $u_i \equiv e^{\lambda \sigma_i \sqrt{\Delta t}}$ for $i = 1, 2$.
- The following simpler scheme is good enough:

$$p_1 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right],$$

$$p_2 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right],$$

$$p_3 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right],$$

$$p_4 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right],$$

$$p_5 = 1 - \frac{1}{\lambda^2}.$$

---

[a]Madan, Milne, and Shefrin (1989).

Correlated Trinomial Model Simplified (continued)

- All of the probabilities lie between 0 and 1 if and only if

$$-1 + \lambda\sqrt{\Delta t}\left| \frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right| \leq \quad \rho \quad \leq 1 - \lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right|, \quad (86)$$

$$1 \leq \quad \lambda \quad\quad\quad\quad\quad\quad (87)$$

- We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as

$$-1 + O(\sqrt{\Delta t}) < \rho < 1 - O(\sqrt{\Delta t}),$$

such as the above one.[a]

---

[a]Kao (`R98922093`) (2011) and Kao (`R98922093`), Lyuu, and Wen (`D94922003`) (2014).

Correlated Trinomial Model Simplified (concluded)

- But this model cannot price 2-asset 2-barrier options accurately.[a]

- Few multivariate trees are both optimal and able to handle multiple barriers.[b]

- An alternative is to use orthogonalization.[c]

---

[a]See Chang (`B89704039`, `R93922034`), Hsu (`R7526001`, `D89922012`), and Lyuu (2006) and Kao (`R98922093`), Lyuu and Wen (`D94922003`) (2014) for solutions.

[b]See Kao (`R98922093`), Lyuu, and Wen (`D94922003`) (2014) for one.

[c]Hull and White (1990) and Dai (`R86526008`, `D8852600`), Lyuu, and Wang (`F95922018`) (2012).

Extrapolation

- It is a method to speed up numerical convergence.

- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{88}$$

- Assume c is an unknown constant independent of n.

  – Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and ignoring the smaller terms,

$$
\begin{aligned}
f(n_1) &= f + \frac{c}{n_1}, \\
f(n_2) &= f + \frac{c}{n_2}.
\end{aligned}
$$

- A better approximation to the desired f is

$$
f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \tag{89}
$$

- This estimate should converge faster than $1/n$.

- The Richardson extrapolation uses $n_2 = 2n_1$.

Improving BOPM with Extrapolation

- Consider standard European options.

- Denote the option value under BOPM using n time periods by $f(n)$.

- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (88) on p. 732.

- But the plots on p. 279 (redrawn on next page) demonstrate that convergence to the true option value oscillates with n.

- Extrapolation is inapplicable at this stage.
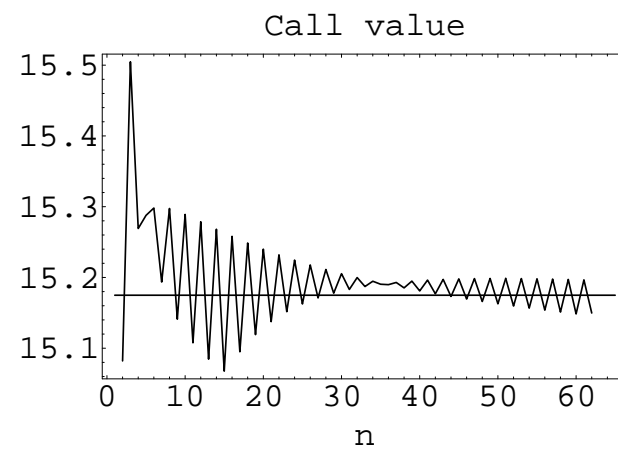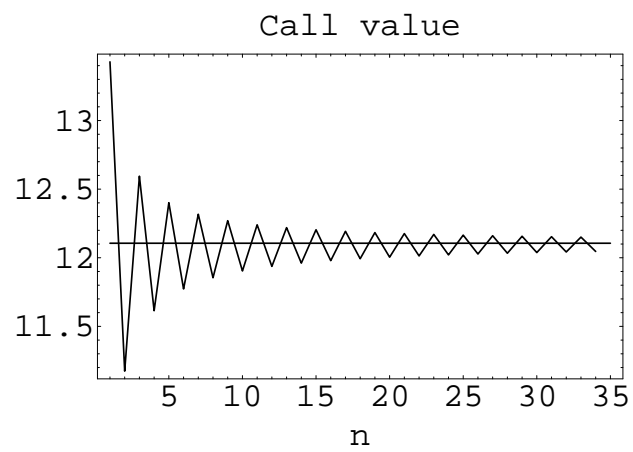
Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 735.

- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 737).[a]

- Apply extrapolation (89) on p. 733 with $n_2 = n_1 + 2$, where n_1 is odd.

- Result is shown in the right plot on p. 737.

- The convergence rate is amazing.

- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

[a]This can be proved; see Chang and Palmer (2007).

Call value
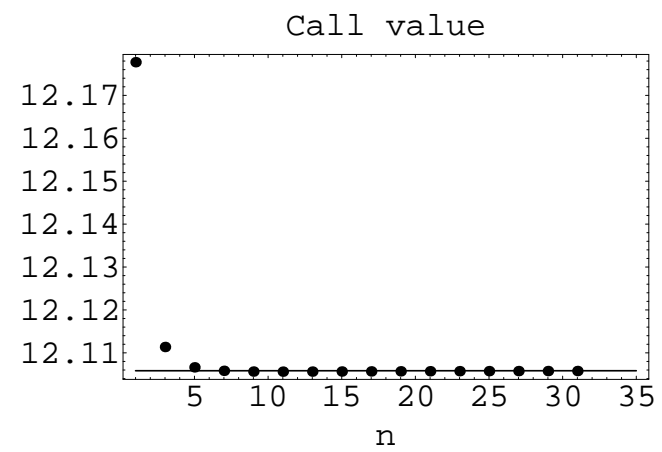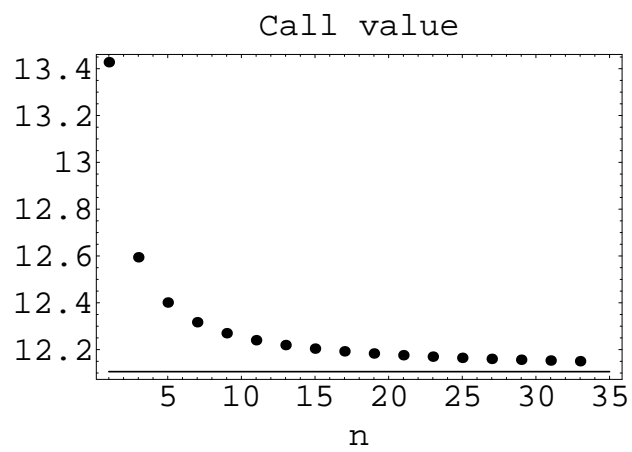
Numerical Methods

All science is dominated

by the idea of approximation.

— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.

- Then approximate the function value at each of these points (p. 741).

- Solve the equation numerically by introducing difference equations in place of derivatives.
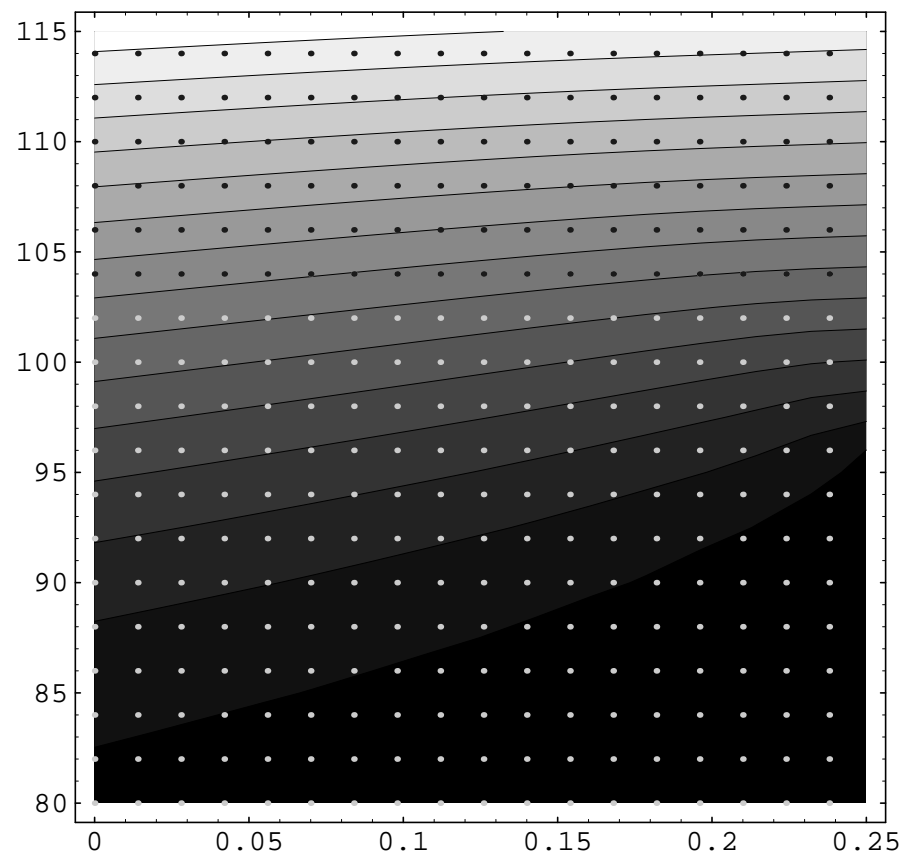
Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.

- Replace second derivatives with finite differences through central difference.

- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.

- The finite difference form is

$$-\rho(x_i, y_j) = \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2}$$
$$+ \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \dots$.

- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$-h^2 \rho(x_i, y_j) = \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j)$$
$$+ \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.

- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation
  $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$, $D > 0$.

- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.

- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x,t_{j+1}) - \theta(x,t_j)}{\Delta t} + \cdots, \tag{90}$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots. \tag{91}$$

Explicit Methods (continued)

- Next, assemble Eqs. (90) and (91) into a single equation at (x_i, t_j).

- But we need to decide how to evaluate x in the first equation and t in the second.

- Since central difference around x_i is used in Eq. (91), we might as well use x_i for x in Eq. (90).

- Two choices are possible for t in Eq. (91).

- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\,\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \qquad (92)$$
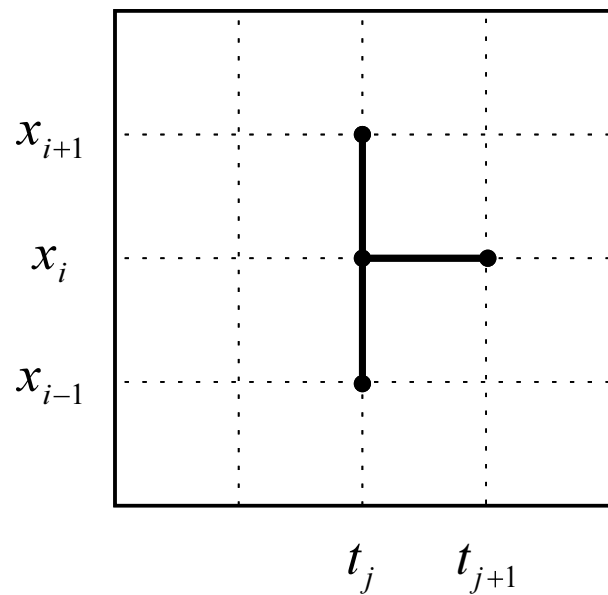
Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
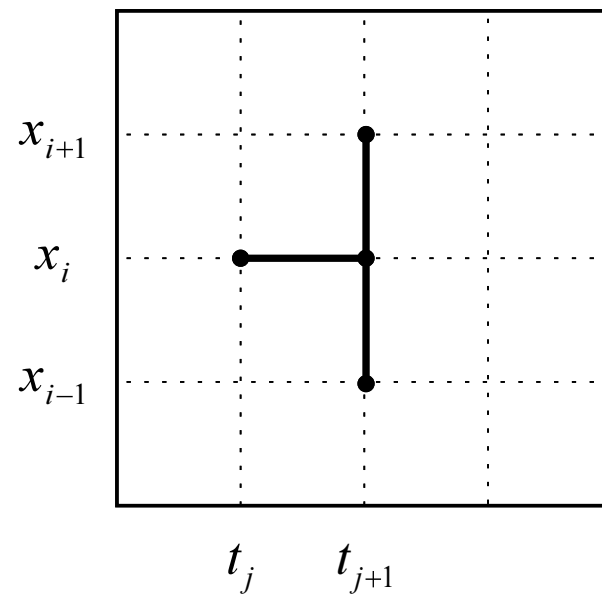
- Rearrange Eq. (92) on p. 745 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see exhibit (a) on next page).

Stencils

(a)   (b)

Explicit Methods (concluded)

- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

$$\theta_{i,1}, \quad i = 1, 2, \ldots.$$

- And then

$$\theta_{i,2}, \quad i = 1, 2, \ldots.$$

- And so on.

Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2/(2D).$$

  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

- The stability condition may lead to high running times and memory requirements.

- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time 8 times as much.

Explicit Method and Trinomial Tree

- Recall that

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.

- They can be interpreted as probabilities.

- So the finite-difference equation becomes identical to backward induction on trinomial trees!

- The freedom in choosing Δx corresponds to similar freedom in the construction of trinomial trees.