Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Define

$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - X_i are independent with

$$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,].$$

Brownian Motion as Limit of Random Walk (continued)

- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \, (X_1 + X_2 + \cdots + X_n) \, .$$

- Recall

$$\begin{aligned} E[\, X_i \,] &= 2p - 1, \\ \text{Var}[\, X_i \,] &= 1 - (2p - 1)^2. \end{aligned}$$

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$

$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} &\text{Var}[\, Y(t + \Delta t) - Y(t) \,] \\ =&\text{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \text{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (28) on p. 271 and $\Delta x = \ln u$.

Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process

$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$

is called geometric Brownian motion.

- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

$$E\left[e^{sX(t)} \right] = E\left[Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$

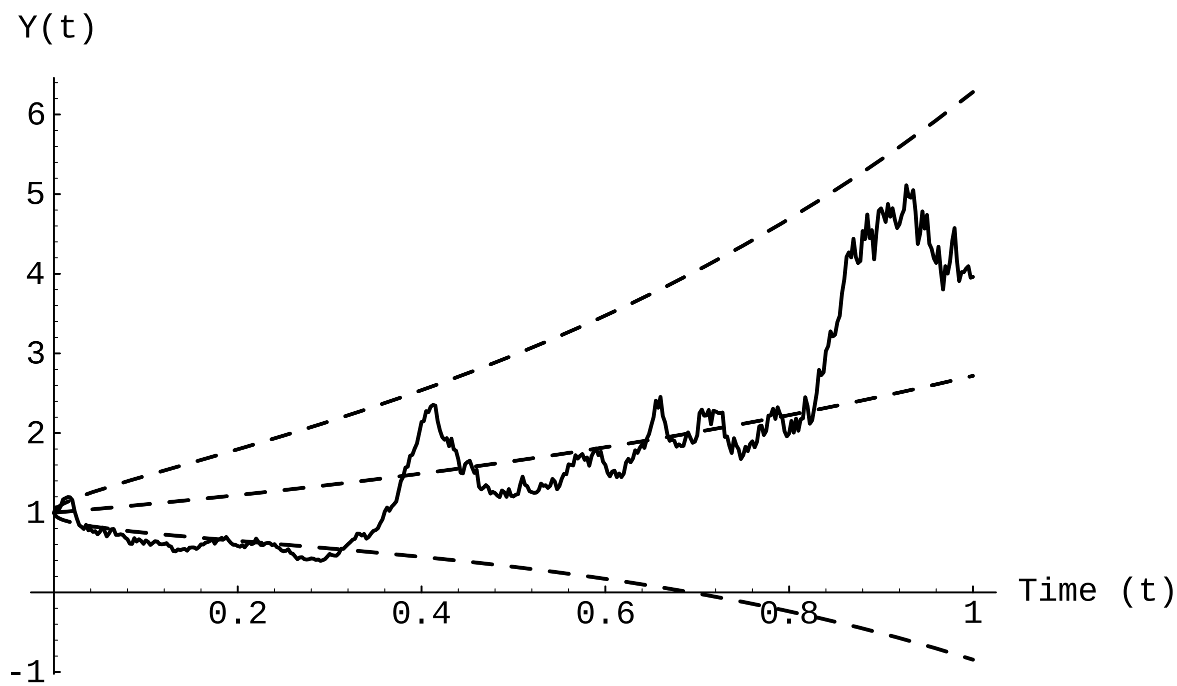
from Eq. (20) on p 154.

Geometric Brownian Motion (concluded)

- In particular,

$$E[\,Y(t)\,] = e^{\mu t + (\sigma^2 t/2)},$$

$$\begin{aligned}\mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right).\end{aligned}$$


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X\, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X\, dW$.

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.

Ito Integral

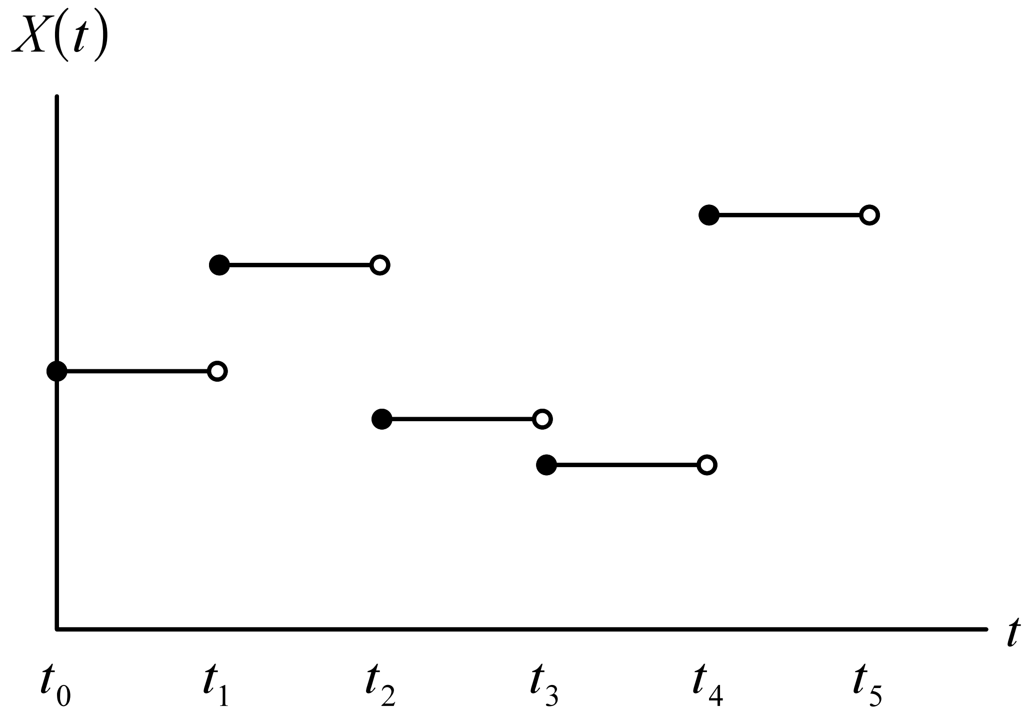
- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist

$$0 = t_0 < t_1 < \cdots$$

such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [\, t_{k-1}, t_k), \; k = 1, 2, \ldots$$

for any realization (see figure on next page).


X(t)
t
t0
t1
t2
t3
t4
t5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{51}$$

  where $t_n = t$.

  – The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as

$$\delta_n \equiv \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X \, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
  - A corollary is the mean value formula

$$E\left[\int_a^b X \, dW\right] = 0.$$

Theorem 18 *The Ito integral* $\int X \, dW$ *is a martingale.*

Discrete Approximation

- Recall Eq. (51) on p. 532.
- The following simple stochastic process $\{\widehat{X}(t)\}$ can be used in place of X to approximate $\int_0^t X\,dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [\,t_{k-1}, t_k),\, k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{\,\widehat{X}(t), W(t), 0 \leq t \leq s\,\},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

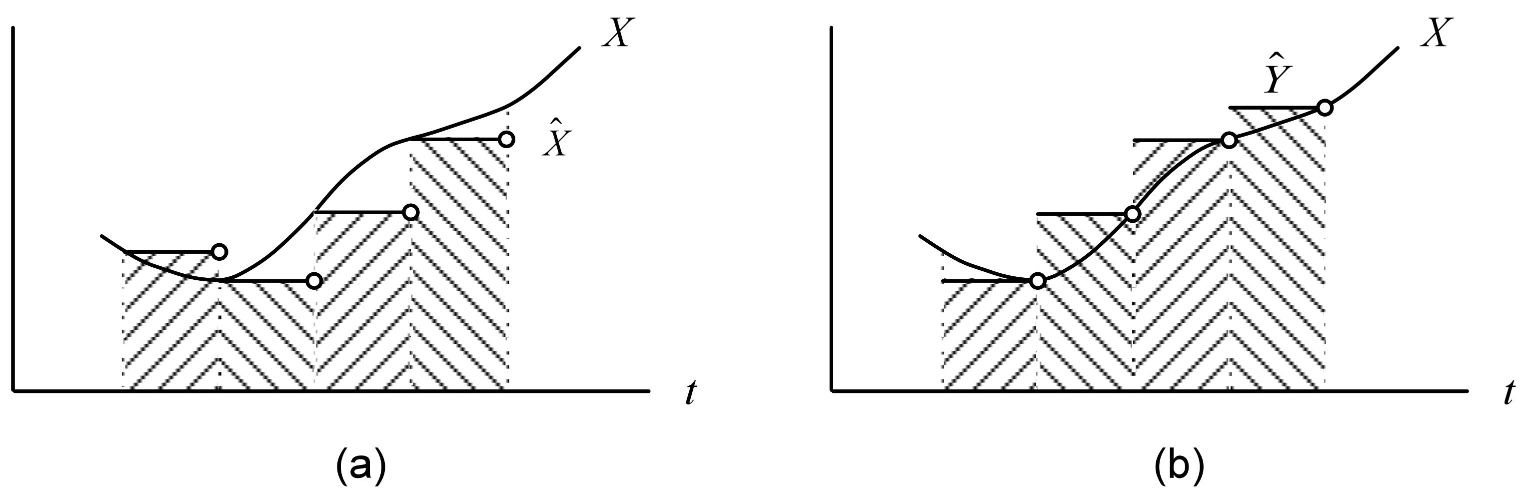
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \quad \text{for} \quad s \in [\,t_{k-1}, t_k),\ k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.


X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  – X_0 is a scalar starting point.

  – $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

  – $a(X_t, t)$: the drift.

  – $b(X_t, t)$: the diffusion.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{52}$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.

- X is a martingale if $a_t = 0$ (Theorem 18 on p. 534).

[a]Paul Langevin (1872–1946) in 1904.

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form of Eq. (52) is

$$dX_t = a_t \, dt + b_t \sqrt{dt} \; \xi, \tag{53}$$

where $\xi \sim N(0, 1)$.

Euler Approximation

- The following approximation follows from Eq. (53),

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n), \tag{54}$$

  where $t_n \equiv n\Delta t$.

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$, not $W(t_n) - W(t_{n-1})$.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (54) on p. 541 can be replaced by

$$\begin{aligned} &\widehat{X}(t_{n+1}) \\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n). \end{aligned}$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This is a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 19 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{55}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.

- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$.

- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 20 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

where $f_i \equiv \partial f / \partial X_i$ *and* $f_{ik} \equiv \partial^2 f / \partial X_i \partial X_k$*.*

Ito's Lemma (continued)

- The multiplication table for Theorem 20 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t\, dt + b_t\, dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$
\begin{aligned}
df &= \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\, \frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 \\
&= \frac{\partial f}{\partial X_t}\, (a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
&\quad + \frac{1}{2}\, \frac{\partial^2 f}{\partial X_t^2}\, (a_t\, dt + b_t\, dW_t)^2 \\
&= \left(\frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\, \frac{\partial^2 f}{\partial X_t^2}\, b_t^2 \right) dt \\
&\quad + \frac{\partial f}{\partial X_t}\, b_t\, dW_t.
\end{aligned}
$$

Ito's Lemma (continued)

Theorem 21 (Alternative Ito's Lemma) *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + b_i\, dW_i$*. Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider geometric Brownian motion $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
  - Hence $dX = \mu\, dt + \sigma\, dW$ by Eq. (50) on p. 518.
- Note that

$$\begin{aligned} \frac{\partial Y}{\partial X} &= Y, \\ \frac{\partial^2 Y}{\partial X^2} &= Y. \end{aligned}$$

Geometric Brownian Motion (concluded)

- Ito's formula (55) on p. 547 implies

$$\begin{aligned} dY &= Y\,dX + (1/2)\,Y\,(dX)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\,(\mu\,dt + \sigma\,dW)^2 \\ &= Y\,(\mu\,dt + \sigma\,dW) + (1/2)\,Y\sigma^2\,dt. \end{aligned}$$

- Hence

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\,dW. \tag{56}$$

- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ (not μ).

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned} dY/Y &= a\,dt + b\,dW_Y, \\ dZ/Z &= f\,dt + g\,dW_Z. \end{aligned}$$

- Consider the Ito process $U \equiv YZ$.
- Apply Ito's lemma (Theorem 21 on p. 553):

$$\begin{aligned} dU &= Z\,dY + Y\,dZ + dY\,dZ \\ &= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\ &\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\ &= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z. \end{aligned}$$

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.
- Note that

$$\begin{aligned} Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\ Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\ U &= \exp\left[\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\right]. \end{aligned}$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 557.
- Let $U \equiv Y/Z$.
- We now show that[a]

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{57}$$

- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a]Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 21 on p. 553) can be employed to show that

$$
\begin{aligned}
& dU \\
= \;& (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \;& (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(f Z\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgY Z \rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \;& U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \;& U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}
$$

Forward Price

- Suppose S follows

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

- Consider $F(S,t) \equiv Se^{y(T-t)}$.
- Observe that

$$\begin{aligned}
\frac{\partial F}{\partial S} &= e^{y(T-t)}, \\
\frac{\partial^2 F}{\partial S^2} &= 0, \\
\frac{\partial F}{\partial t} &= -ySe^{y(T-t)}.
\end{aligned}$$

Forward Prices (concluded)

- Then

$$\begin{aligned} dF &= e^{y(T-t)}\, dS - ySe^{y(T-t)}\, dt \\ &= Se^{y(T-t)}\, (\mu\, dt + \sigma\, dW) - ySe^{y(T-t)}\, dt \\ &= F(\mu - y)\, dt + F\sigma\, dW. \end{aligned}$$

- Thus F follows

$$\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.$$

- This result has applications in forward and futures contracts.[a]

[a] It is also consistent with p. 509.

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

  where $\kappa, \sigma \geq 0$.
- It is known that

$$\begin{aligned}
E[\,X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\,x_0\,], \\
\mathrm{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\,x_0\,], \\
\mathrm{Cov}[\,X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[\,1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\,x_0\,],
\end{aligned}$$

  for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- A generalized version:

$$dX = \kappa(\mu - X)\,dt + \sigma\,dW,$$

where $\kappa, \sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\,X(t)\,] &= \mu + (x_0 - \mu)\,e^{-\kappa(t-t_0)}, \qquad (58) \\ \text{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right], \end{aligned}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Square-Root Process

- Suppose X is an Ornstein-Uhlenbeck process.
- Ito's lemma says $V \equiv X^2$ has the differential,

$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right) dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

$$dX = \kappa(\mu - X)\, dt + \sigma\sqrt{X}\, dW,$$

where $\kappa, \sigma \geq 0$ and $X(0)$ is a nonnegative constant.

- Like the Ornstein-Uhlenbeck process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rates.[a]
- The Ornstein-Uhlenbeck process, in contrast, allows negative interest rates.
- The two processes are related (see p. 568).

[a] Cox, Ingersoll, and Ross (1985).

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \equiv (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\,\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\,\left(1 - e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a] William Feller (1906–1970) in 1951.

Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$
- The continuously compounded rate of return $X \equiv \ln S$ follows
$$dX = (\mu - \sigma^2/2)\, dt + \sigma\, dW$$
by Ito's lemma.[a]

---

[a]See also Eq. (56) on p. 556. Also consistent with Lemma 10 (p. 275).

Local-Volatility Models

- The more general deterministic volatility model posits

$$\frac{dS}{S} = (r_t - q_t)\, dt + \sigma(S, t)\, dW,$$

where $\sigma(S, t)$ is called the local volatility function.[a]

- A (weak) solution exists if $S\sigma(S, t)$ is continuous and grows at most linearly in S and t.[b]

---

[a] Derman and Kani (1994); Dupire (1994).

[b] Skorokhod (1961).

Local-Volatility Models (continued)

- Theoretically,[a]

$$\sigma(X,T)^2 = 2\,\frac{\frac{\partial C}{\partial T} + (r_T - q_T)X\,\frac{\partial C}{\partial X} + q_T C}{X^2\,\frac{\partial^2 C}{\partial X^2}}. \qquad (59)$$

- C is the call price at time $t = 0$ (today) with strike price X and time to maturity T.
- $\sigma(X,T)$ is the local volatility that will prevail at *future time* T and *stock price* $S_T = X$.

[a]Dupire (1994); Andersen and Brotherton-Ratcliffe (1998).

Local-Volatility Models (continued)

- In practice, $\sigma(S, T)^2$ may have spikes, vary wildly, or even be negative.
- The term $\partial^2 C/\partial X^2$ in the denominator often results in numerical instability.
- Denote the implied volatility surface by $\Sigma(X, T)$.
- Denote the local volatility surface by $\sigma(S, T)$.

Local-Volatility Models (continued)

- The relation between $\Sigma(X, T)$ and $\sigma(X, T)$ is[a]

$$
\begin{aligned}
\sigma(X,T)^2 &= \frac{\Sigma^2 + 2\Sigma\tau\left[\frac{\partial\Sigma}{\partial T} + (r_T - q_T)X\frac{\partial\Sigma}{\partial X}\right]}{\left(1 - \frac{Xy}{\Sigma}\frac{\partial\Sigma}{\partial X}\right)^2 + X\Sigma\tau\left[\frac{\partial\Sigma}{\partial X} - \frac{X\Sigma\tau}{4}\left(\frac{\partial\Sigma}{\partial X}\right)^2 + X\frac{\partial^2\Sigma}{\partial X^2}\right]}, \\
\tau &\equiv T - t, \\
y &\equiv \ln(X/S_t) + \int_t^T (q_s - r_s)\,ds.
\end{aligned}
$$

- Although this version may be more stable than Eq. (59) on p. 574, it is expected to suffer from similar problems.

[a] Andreasen (1996); Andersen and Brotherton-Ratcliffe (1998); Gatheral (2003); Wilmott (2006); Kamp (2009).

Local-Volatility Models (continued)

- Small changes to the implied volatility surface may produce big changes to the local volatility surface.
- In reality, option prices only exist for a finite set of maturities and strike prices.
- Hence interpolation and extrapolation may be needed to construct the volatility surface.
- But some implied volatility surfaces generate option prices that allow arbitrage profits.

Local-Volatility Models (continued)

- For example, consider the following implied volatility surface:[a]

$$\Sigma(X, T)^2 = a_{\text{ATM}}(T) + b(X - S_0)^2, \quad b > 0.$$

- It generates higher prices for out-of-the-money options than in-the-money options for T large enough.[b]

---

[a] ATM means at-the-money.

[b] Rebonato (2004).

Local-Volatility Models (continued)

- Let $x \equiv \ln(X/S_0) - rT$.
- For X large enough,[a]

$$\Sigma(X,T)^2 < 2\,\frac{|\,x\,|}{T}.$$

- For X small enough,[b]

$$\Sigma(X,T)^2 < \beta\,\frac{|\,x\,|}{T} \quad \text{for any } \beta > 2.$$

---

[a]Lee (2004).
[b]Lee (2004).

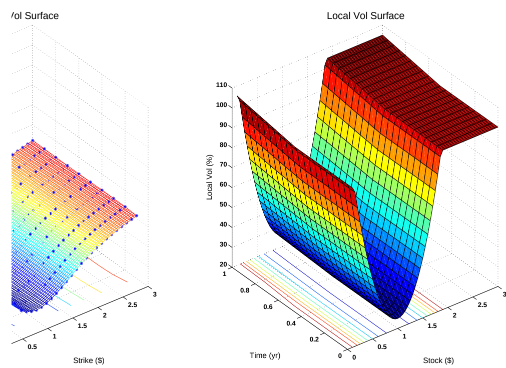
Local-Volatility Models (concluded)

- There exist conditions for a set of option prices to be arbitrage-free.[a]
- For some vanilla equity options, the Black-Scholes model "seems" better than the local volatility model.[b]

---

[a] Davis and Hobson (2007).
[b] Dumas, Fleming, and Whaley (1998).

Implied and Local Volatility Surfaces[a]



[a]Contributed by Mr. Lok, U Hou (D99922028) on April 5, 2014.

Implied Trees

- The trees for the local volatility model are called implied trees.[a]
- Their construction requires option prices at all strike prices and maturities.
  - That is, a volatility surface.
- The local volatility model does *not* require that the implied tree combine.

[a] Derman and Kani (1994); Dupire (1994); Rubinstein (1994).

Implied Trees (concluded)

- How to construct an implied tree with efficiency, valid probabilities and stability has been open for a long time.[a]
  - Reasons may include: noise and nonsynchrony in data, arbitrage opportunities in the smoothed and interpolated/extrapolated implied volatility surface, wrong model, etc.
- Numerically, inversion is an ill-posed problem.
- It is partially solved recently.[b]

---

[a]Derman and Kani (1994); Derman, Kani, and Chriss (1996); Coleman, Kim, Li, and Verma (2000); Ayache, Henrotte, Nassar, and Wang (2004); Kamp (2009).

[b]February 12, 2013.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation (PDE).
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.
- PDEs allow many numerical methods to be applicable.

Assumptions[a]

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

[a]Derman and Taleb (2005) summarizes criticisms on these assumptions and the replication argument.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 549),

$$dC = \left(\mu S \, \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \, \frac{\partial C}{\partial S} \, dW.$$

  – The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

[a]Mathematically speaking, it is not quite right (Bergman, 1982).

Black-Scholes Differential Equation (concluded)

- So
$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}\right) dt = r\left(C - S\,\frac{\partial C}{\partial S}\right) dt.$$

- Equate the terms to finally obtain
$$\frac{\partial C}{\partial t} + rS\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC.$$

- When there is a dividend yield q,
$$\frac{\partial C}{\partial t} + (r-q)\,S\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC. \tag{60}$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{61}$$

- Identity (61) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

Black-Scholes Differential Equation: An Alternative

- Perform the change of variable $V \equiv \ln S$.
- The option value becomes $U(V, t) \equiv C(e^V, t)$.
- Furthermore,

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{\partial U}{\partial t}, \\
\frac{\partial C}{\partial S} &= \frac{1}{S}\,\frac{\partial U}{\partial V}, \\
\frac{\partial^2 C}{\partial^2 S} &= \frac{1}{S^2}\,\frac{\partial^2 U}{\partial V^2} - \frac{1}{S^2}\,\frac{\partial U}{\partial V}. \qquad (62)
\end{aligned}
$$

Black-Scholes Differential Equation: An Alternative (concluded)

- The Black-Scholes differential equation (60) becomes

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 U}{\partial V^2} + \left(r - q - \frac{\sigma^2}{2} \right) \frac{\partial U}{\partial V} - rU + \frac{\partial U}{\partial t} = 0$$

  subject to $U(V, T)$ being the payoff such as $\max(X - e^V, 0)$.

- Equation (62) is an alternative way to calculate the gamma numerically.

[Black] got the equation [in 1969] but then was unable to solve it. Had he been a better physicist he would have recognized it as a form of the familiar heat exchange equation, and applied the known solution. Had he been a better mathematician, he could have solved the equation from first principles. Certainly Merton would have known exactly what to do with the equation had he ever seen it.

— Perry Mehrling (2005)

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T, S, A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T, S, A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a]Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 395ff.
- But one-dimensional PDEs are available for Asian options.[a]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a] Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

