#### Binomial Model for Forward and Futures Options

- Futures price behaves *like* a stock paying a continuous dividend yield of r.
  - The futures price at time 0 is (p. 437)

$$F = Se^{rT}.$$

- From Lemma 10 (p. 275), the expected value of S at time  $\Delta t$  in a risk-neutral economy is

$$Se^{r\Delta t}$$

– So the expected futures price at time  $\Delta t$  is

$$Se^{r\Delta t}e^{r(T-\Delta t)} = Se^{rT} = F.$$

# Binomial Model for Forward and Futures Options (continued)

- The above observation continues to hold if S pays a dividend yield!<sup>a</sup>
  - By Eq. (39) on p. 445, the futures price at time 0 is

 $F = Se^{(r-q)T}.$ 

- From Lemma 10 (p. 275), the expected value of S at time  $\Delta t$  in a risk-neutral economy is

$$Se^{(r-q)\Delta t}$$

– So the expected futures price at time  $\Delta t$  is

$$Se^{(r-q)\Delta t}e^{(r-q)(T-\Delta t)} = Se^{(r-q)T} = F.$$

<sup>a</sup>Contributed by Mr. Liu, Yi-Wei (R02723084) on April 16, 2014.

# Binomial Model for Forward and Futures Options (concluded)

• Now, under the BOPM, the risk-neutral probability for the futures price is

$$p_{\rm f} \equiv (1-d)/(u-d)$$

by Eq. (30) on p. 302.

- The futures price moves from F to Fu with probability  $p_{\rm f}$  and to Fd with probability  $1 - p_{\rm f}$ .

- Note that the *original* u and d are used!

• The binomial tree algorithm for forward options is identical except that Eq. (41) on p. 458 is the payoff.

#### Spot and Futures Prices under BOPM

• The futures price is related to the spot price via

$$F = Se^{rT}$$

if the underlying asset pays no dividends.

- Recall the futures price F moves to Fu with probability  $p_{\rm f}$  per period.
- So the stock price moves from  $S = Fe^{-rT}$  to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

with probability  $p_{\rm f}$  per period.

#### Spot and Futures Prices under BOPM (concluded)

• Similarly, the stock price moves from  $S = Fe^{-rT}$  to

$$Sde^{r\Delta t}$$

with probability  $1 - p_{\rm f}$  per period.

• Note that

$$S(ue^{r\Delta t})(de^{r\Delta t}) = Se^{2r\Delta t} \neq S.$$

- So the binomial model is not the CRR tree.
- This model may not be suitable for pricing barrier options (why?).

#### Negative Probabilities Revisited

- As  $0 < p_{\rm f} < 1$ , we have  $0 < 1 p_{\rm f} < 1$  as well.
- The problem of negative risk-neutral probabilities is now solved:
  - Suppose the stock pays a continuous dividend yield of q.
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - By Eq. (39) on p. 445, calculate S from F at each node via

$$S = Fe^{-(r-q)(T-t)}.$$

## Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).
- There are theories about why swaps exist.<sup>a</sup>

<sup>a</sup>Thanks to a lively discussion on April 16, 2014.

## Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
А	$D_{ m A}\%$	$Y_{ m A}\%$
В	$D_{ m B}\%$	$Y_{\rm B}\%$

• Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

## Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at  $Y_{\rm A}\%$  and B to borrow dollars at  $D_{\rm B}\%$ .
- But suppose A is *relatively* more competitive in the dollar market than the yen market, i.e.,

 $Y_{\rm B} - Y_{\rm A} < D_{\rm B} - D_{\rm A}.$ 

- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.

## Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is  $((D_{\rm B} D_{\rm A}) (Y_{\rm B} Y_{\rm A}))\%$ :
  - The total interest rate is originally  $(Y_{\rm A} + D_{\rm B})\%$ .
  - The new arrangement has a smaller total rate of  $(D_{\rm A} + Y_{\rm B})\%$ .
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

## Example

• A and B face the following borrowing rates:

	Dollars	Yen
А	9%	10%
В	12%	11%

- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

# Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).
- So a currency swap with a total saving of 3 1 = 2% is possible.
- A is relatively more competitive in the dollar market.
- B is relatively more competitive in the yen market.

# Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5% (lower than 10%).
  - B borrows dollars at 11.5% (lower than 12%).
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.



#### As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 477 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is  $SP_{\rm Y} P_{\rm D}$ .
  - $P_{\rm D}$  is the dollar bond's value in dollars.
  - $-P_{\rm Y}$  is the yen bond's value in yen.
  - -S is the \$/yen spot exchange rate.

# As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:
  - The term structures of interest rates in the currencies involved.
  - The spot exchange rate.
- It has zero value when

$$SP_{\rm Y} = P_{\rm D}.$$

#### Example

- Take a 3-year swap on p. 477 with principal amounts of US\$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/\$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09 \times 2} + 111 \times e^{-0.09 \times 3}\right)$$
$$-\left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08 \times 2} + 1.115 \times e^{-0.08 \times 3}\right) = 0.074.$$

#### As a Package of Forward Contracts

• From Eq. (38) on p. 445, the forward contract maturing *i* years from now has a dollar value of

$$f_i \equiv (SY_i) e^{-qi} - D_i e^{-ri}.$$
(43)

- $-Y_i$  is the yen inflow at year *i*.
- -S is the \$/yen spot exchange rate.
- -q is the yen interest rate.
- $-D_i$  is the dollar outflow at year *i*.
- -r is the dollar interest rate.

# As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

## Example

- Take the swap in the example on p. 480.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$ ,  $Y_3 = 111$ , S = 1/90,  $D_1 = D_2 = 0.115$ ,  $D_3 = 1.115$ , q = 0.09, and r = 0.08.
- Plug in these numbers to get  $f_1 + f_2 + f_3 = 0.074$ million dollars as before.

# Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind has confronted and has overcome in the last fifteen hundred years, the one which seems to me to have been the most amazing in character and the most stupendous in the scope of its consequences is the one relating to the problem of motion. — Herbert Butterfield (1900–1979)

#### Stochastic Processes

• A stochastic process

 $X = \{ X(t) \}$ 

is a time series of random variables.

- X(t) (or  $X_t$ ) is a random variable for each time t and is usually called the state of the process at time t.
- A realization of X is called a sample path.

## Stochastic Processes (concluded)

- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

• If the times form a continuum, X is called a continuous-time stochastic process.

### Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line,  $0, \pm 1, \pm 2, \ldots$
- In each time step, it can make one move to the right with probability p or one move to the left with probability 1 - p.

- This random walk is symmetric when p = 1/2.

• Connection with the BOPM: The particle's position denotes the number of up moves minus that of down moves up to that time.



#### Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- $\xi_n$  are independent and identically distributed with zero mean.
- Drift  $\mu$  is the expected change per period.
- Note that this process is continuous in space.

#### $\mathsf{Martingales}^{\mathrm{a}}$

• { $X(t), t \ge 0$ } is a martingale if  $E[|X(t)|] < \infty$  for  $t \ge 0$  and

$$E[X(t) | X(u), 0 \le u \le s] = X(s), \quad s \le t.$$
(44)

• In the discrete-time setting, a martingale means

$$E[X_{n+1} | X_1, X_2, \dots, X_n] = X_n.$$
 (45)

- $X_n$  can be interpreted as a gambler's fortune after the *n*th gamble.
- Identity (45) then says the expected fortune after the (n+1)th gamble equals the fortune after the nth gamble regardless of what may have occurred before.

<sup>a</sup>The origin of the name is somewhat obscure.

## Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (45) on p. 491 to yield

$$E[X_n] = E[X_1] \tag{46}$$

for all n.

• Similarly,

E[X(t)] = E[X(0)]

in the continuous-time case.

#### Still a Martingale?

• Suppose we replace Eq. (45) on p. 491 with

$$E[X_{n+1} \mid X_n] = X_n.$$

- It also says past history cannot affect the future.
- But is it equivalent to the original definition (45) on p. 491?<sup>a</sup>

<sup>a</sup>Contributed by Mr. Hsieh, Chicheng (M9007304) on April 13, 2005.

## Still a Martingale? (continued)

- Well, no.<sup>a</sup>
- Consider this random walk with drift:

$$X_{i} = \begin{cases} X_{i-1} + \xi_{i}, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

• Above,  $\xi_n$  are random variables with zero mean.

<sup>a</sup>Contributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

### Still a Martingale? (concluded)

• It is not hard to see that

$$E[X_i | X_{i-1}] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

- It is a martingale by the "new" definition.

• But

$$E[X_i \mid \dots, X_{i-2}, X_{i-1}] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even}, \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- It is not a martingale by the original definition.

#### Example

• Consider the stochastic process

$$\{Z_n \equiv \sum_{i=1}^n X_i, n \ge 1\},\$$

where  $X_i$  are independent random variables with zero mean.

• This process is a martingale because

$$E[Z_{n+1} | Z_1, Z_2, \dots, Z_n]$$
  
=  $E[Z_n + X_{n+1} | Z_1, Z_2, \dots, Z_n]$   
=  $E[Z_n | Z_1, Z_2, \dots, Z_n] + E[X_{n+1} | Z_1, Z_2, \dots, Z_n]$   
=  $Z_n + E[X_{n+1}] = Z_n.$ 

#### Probability Measure

- A probability measure assigns probabilities to states of the world.
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- A martingale is also defined with respect to an information set.
  - In the characterizations (44)–(45) on p. 491, the information set contains the current and past values of X by default.
  - But it need not be so.

#### Probability Measure (continued)

• A stochastic process  $\{X(t), t \ge 0\}$  is a martingale with respect to information sets  $\{I_t\}$  if, for all  $t \ge 0$ ,  $E[|X(t)|] < \infty$  and

$$E[X(u) \mid I_t] = X(t)$$

for all u > t.

• The discrete-time version: For all n > 0,

$$E[X_{n+1} \mid I_n] = X_n,$$

given the information sets  $\{I_n\}$ .

## Probability Measure (concluded)

• The above implies

 $E[X_{n+m} \mid I_n] = X_n$ 

for any m > 0 by Eq. (19) on p. 152.

- A typical  $I_n$  is the price information up to time n.
- Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

### Example

• Consider the stochastic process  $\{Z_n - n\mu, n \ge 1\}$ .

$$-Z_n \equiv \sum_{i=1}^n X_i.$$

 $-X_1, X_2, \ldots$  are independent random variables with mean  $\mu$ .

• Now,

$$E[Z_{n+1} - (n+1)\mu | X_1, X_2, \dots, X_n]$$
  
=  $E[Z_{n+1} | X_1, X_2, \dots, X_n] - (n+1)\mu$   
=  $E[Z_n + X_{n+1} | X_1, X_2, \dots, X_n] - (n+1)\mu$   
=  $Z_n + \mu - (n+1)\mu$   
=  $Z_n - n\mu$ .

## Example (concluded)

• Define

$$I_n \equiv \{X_1, X_2, \ldots, X_n\}.$$

• Then

$$\{Z_n - n\mu, n \ge 1\}$$

is a martingale with respect to  $\{I_n\}$ .

#### Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [pC_u + (1-p)C_d]/R.$$

- -p is the risk-neutral probability.
- \$1 grows to \$*R* in a period.

- Let C(i) denote the value of the option at time *i*.
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \dots, n \right\}.$$

• Then,

$$E\left[\left.\frac{C(i+1)}{R^{i+1}}\right| C(i) = C\right] = \frac{pC_u + (1-p)C_d}{R^{i+1}} = \frac{C}{R^i}.$$

• It is easy to show that

$$E\left[\left.\frac{C(k)}{R^k}\right|C(i)=C\right] = \frac{C}{R^i}, \quad i \le k.$$
(47)

- This formulation assumes:<sup>a</sup>
  - The model is Markovian: The distribution of the future is determined by the present (time i) and not the past.
  - 2. The payoff depends only on the terminal price of the underlying asset (Asian options do not qualify).

<sup>&</sup>lt;sup>a</sup>Contributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (B90902081) on May 3, 2006.

• In general, the discount process is a martingale in that<sup>a</sup>

$$E_i^{\pi} \left[ \frac{C(k)}{R^k} \right] = \frac{C(i)}{R^i}, \quad i \le k.$$
(48)

- $-E_i^{\pi}$  is taken under the risk-neutral probability conditional on the price information up to time *i*.
- This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

<sup>a</sup>In this general formulation, Asian options do qualify.

- Equation (48) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi} \left[ \frac{C(k)}{M(k)} \right], \quad i \le k.$$
(49)

- -M(j) is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
- It is called the bank account process.
- It says the discount process is a martingale under  $\pi$ .

- If interest rates are stochastic, then M(j) is a random variable.
  - M(0) = 1.
  - M(j) is known at time j 1.
- Identity (49) on p. 506 is the general formulation of risk-neutral valuation.

## Martingale Pricing (concluded)

**Theorem 17** A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale.<sup>a</sup>

<sup>a</sup>This probability measure is called the risk-neutral probability measure.

## Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is

$$p_{\rm f}Fu + (1-p_{\rm f})Fd = F\left(\frac{1-d}{u-d}u + \frac{u-1}{u-d}d\right) = F$$

(p. 464).

• Can be generalized to

$$F_i = E_i^{\pi} [F_k], \quad i \le k,$$

where  $F_i$  is the futures price at time *i*.

• This equation holds under stochastic interest rates, too.

## Martingale Pricing and Numeraire $^{\rm a}$

- The martingale pricing formula (49) on p. 506 uses the money market account as numeraire.<sup>b</sup>
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is positive at all times.

<sup>a</sup>John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value." <sup>b</sup>Leon Walras (1834–1910).

## Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi} \left[ \frac{C(k)}{S(k)} \right], \quad i \le k.$$

- S(j) denotes the price of S at time j.

• So the discount process remains a martingale.<sup>a</sup>

<sup>a</sup>This result is related to Girsanov's theorem.

## Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to  $S_1$  or  $S_2$ .
- In a period, asset two's price can go from P to  $P_1$  or  $P_2$ .
- Both assets must move up or down at the same time.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}$$

to rule out arbitrage opportunities.

## Example (continued)

- For any derivative security, let  $C_1$  be its price at time one if asset one's price moves to  $S_1$ .
- Let  $C_2$  be its price at time one if asset one's price moves to  $S_2$ .
- Replicate the derivative by solving

$$\alpha S_1 + \beta P_1 = C_1,$$
  
$$\alpha S_2 + \beta P_2 = C_2,$$

using  $\alpha$  units of asset one and  $\beta$  units of asset two.

## Example (continued)

• This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2}$$
 and  $\beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}$ .

• The derivative costs

$$C = \alpha S + \beta P$$
  
=  $\frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S_1}{P_2 S_1 - P_1 S_2} C_2.$ 

## Example (concluded)

• It is easy to verify that

$$\frac{C}{P} = p \, \frac{C_1}{P_1} + (1-p) \, \frac{C_2}{P_2}.$$

- Above,

$$p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- The derivative's price using asset two as numeraire (i.e., C/P) is a martingale under the risk-neutral probability p.
- The expected returns of the two assets are irrelevant.

#### Brownian Motion $^{\rm a}$

- Brownian motion is a stochastic process  $\{X(t), t \ge 0\}$ with the following properties.
  - **1.** X(0) = 0, unless stated otherwise.
  - **2.** for any  $0 \le t_0 < t_1 < \cdots < t_n$ , the random variables

 $X(t_k) - X(t_{k-1})$ 

for  $1 \le k \le n$  are independent.<sup>b</sup>

**3.** for  $0 \le s < t$ , X(t) - X(s) is normally distributed with mean  $\mu(t-s)$  and variance  $\sigma^2(t-s)$ , where  $\mu$ and  $\sigma \ne 0$  are real numbers.

<sup>a</sup>Robert Brown (1773–1858).

<sup>b</sup>So X(t) - X(s) is independent of X(r) for  $r \le s < t$ .

## Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.<sup>a</sup>
- This process will be called a  $(\mu, \sigma)$  Brownian motion with drift  $\mu$  and variance  $\sigma^2$ .
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The (0,1) Brownian motion is called the Wiener process.

<sup>a</sup>Norbert Wiener (1894–1964).

#### Example

• If  $\{X(t), t \ge 0\}$  is the Wiener process, then

$$X(t) - X(s) \sim N(0, t - s).$$

• A  $(\mu, \sigma)$  Brownian motion  $Y = \{Y(t), t \ge 0\}$  can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{50}$$

• Note that

$$Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s).$$