States and Their Transitions

- The tuple

$$(i, S, P)$$

captures the state[a] for the Asian option.

  - i: the time.
  - S: the prevailing stock price.
  - P: the running sum.[b]

[a] A "sufficient statistic," if you will.

[b] When the average is a moving average, a different technique is needed (Kao (R89723057) and Lyuu (2003)).

States and Their Transitions (concluded)

- For the binomial model, the state transition is:

$$
(i, S, P) \begin{array}{l} \nearrow \\ \\ \searrow \end{array} \begin{array}{ll} (i+1, Su, P+Su), & \text{for the up move} \\ \\ (i+1, Sd, P+Sd), & \text{for the down move} \end{array}
$$

- This leads to an exponential-time algorithm.

Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
- Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$

$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas:

$$C = Se^{-q_{\mathrm{a}}\tau} N(x) - Xe^{-r\tau} N(x - \sigma_{\mathrm{a}}\sqrt{\tau}), \tag{33}$$

$$P = Xe^{-r\tau} N(-x + \sigma_{\mathrm{a}}\sqrt{\tau}) - Se^{-q_{\mathrm{a}}\tau} N(-x), \tag{33'}$$

  - With the volatility set to $\sigma_{\mathrm{a}} \equiv \sigma/\sqrt{3}$.
  - With the dividend yield set to $q_{\mathrm{a}} \equiv (r + q + \sigma^2/6)/2$.
  - $x \equiv \frac{\ln(S/X) + \left(r - q_{\mathrm{a}} + \sigma_{\mathrm{a}}^2/2\right)\tau}{\sigma_{\mathrm{a}}\sqrt{\tau}}$.

An Approximate Formula for Asian Calls[a]

$$C = e^{-r\tau} \left[\frac{S}{\tau} \int_0^\tau e^{\mu t + \sigma^2 t/2} N\left(\frac{-\gamma + (\sigma t/\tau)(\tau - t/2)}{\sqrt{\tau/3}} \right) dt \right.$$
$$\left. - XN\left(\frac{-\gamma}{\sqrt{\tau/3}} \right) \right],$$

where

- $\mu \equiv r - \sigma^2/2$.
- γ is the unique value that satisfies

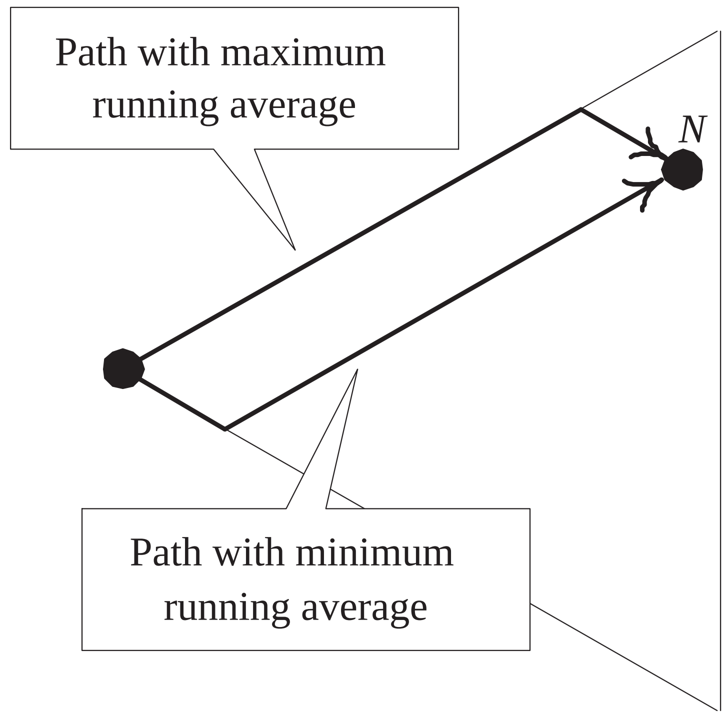
$$\frac{S}{\tau} \int_0^\tau e^{3\gamma\sigma t(\tau - t/2)/\tau^2 + \mu t + \sigma^2 [t - (3t^2/\tau^3)(\tau - t/2)^2]/2} \, dt = X.$$

[a]Rogers and Shi (1995); Thompson (1999); Chen (R92723061) (2005); Chen (R92723061) and Lyuu (2006).

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j, i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i}d + \cdots + u^{j-i}d^i}^{j})$$
$$= S_0 \frac{1 - u^{j-i+1}}{1 - u} + S_0 u^{j-i} d \frac{1 - d^i}{1 - d}.$$


Path with maximum running average
N
Path with minimum running average

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j+1$ and call it $A_{\max}(j,i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1+\overbrace{d+d^2+\cdots+d^i+d^iu+\cdots+d^iu^{j-i}}^{j})$$

$$= S_0\,\frac{1-d^{i+1}}{1-d}+S_0d^iu\,\frac{1-u^{j-i}}{1-u}.$$

- Divide this value by $j+1$ and call it $A_{\min}(j,i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.

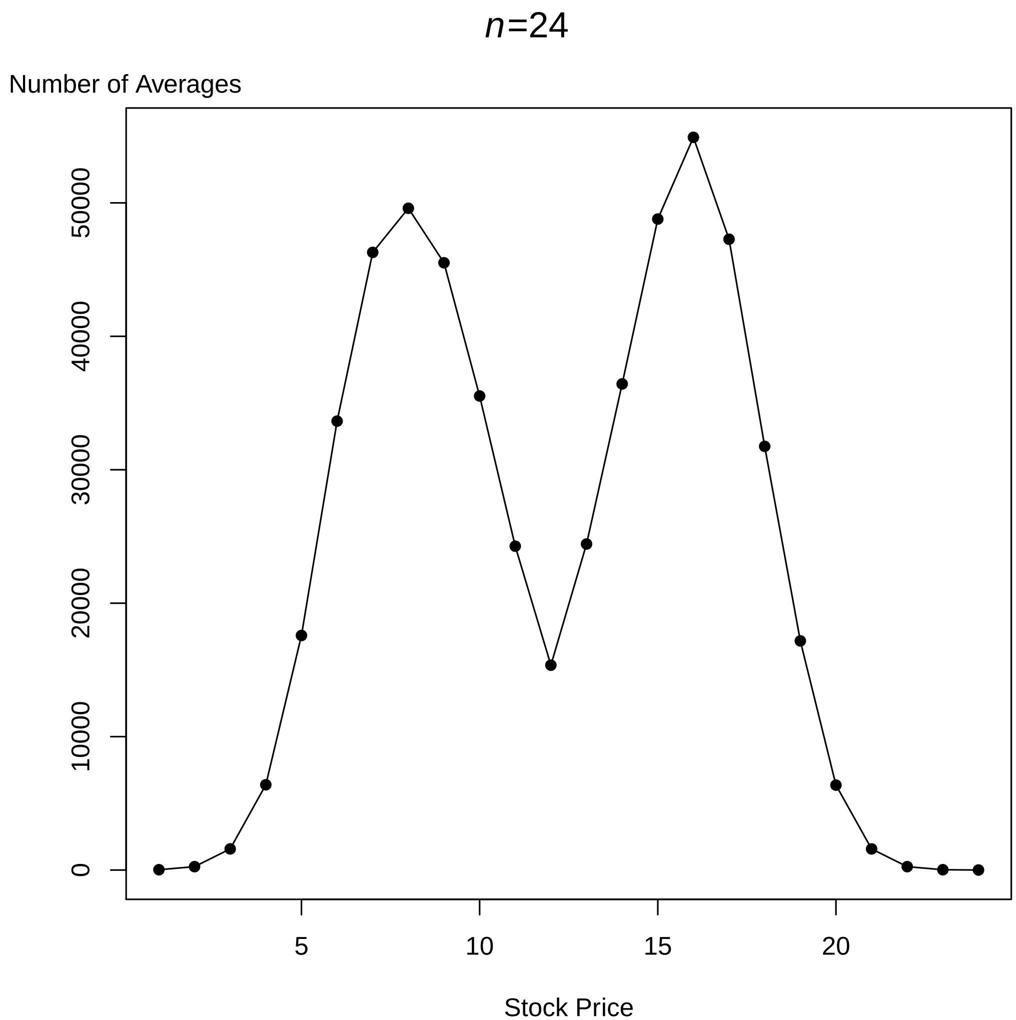
Approximation Algorithm for Asian Options (continued)

- The number of paths to $N(j, i)$ are far too many: $\binom{j}{i}$.
  - For example,

$$\binom{j}{j/2} \sim 2^j \sqrt{2/(\pi j)}\,.$$

- The number of distinct running averages for the nodes at any given time step n seems to be bimodal for n big enough.[a]
  - In the plot on the next page, $u = 5/4$ and $d = 4/5$.

[a]Contributed by Mr. Liu, Jun (R99944027) on April 15, 2014.

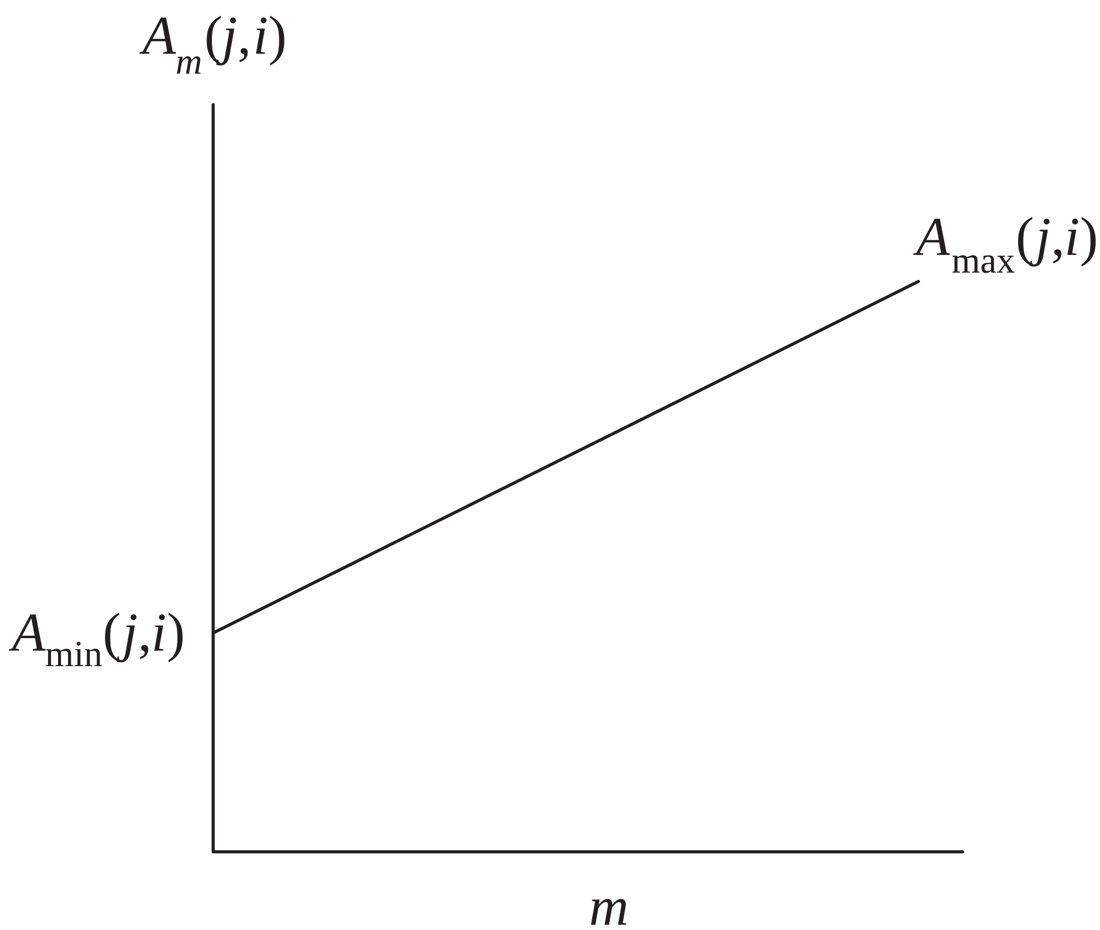

n=24
Number of Averages
0
10000
20000
30000
40000
50000
5
10
15
20
Stock Price

Approximation Algorithm for Asian Options (continued)

- But all averages must lie between $A_{\text{min}}(j, i)$ and $A_{\text{max}}(j, i)$.
- Pick $k + 1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j, i) \equiv \left(\frac{k - m}{k} \right) A_{\text{min}}(j, i) + \left(\frac{m}{k} \right) A_{\text{max}}(j, i)$$

  for $m = 0, 1, \ldots, k$.



$A_m(j,i)$
$A_{\max}(j,i)$
$A_{\min}(j,i)$
m

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" introduces errors, but it works reasonably well in practice.[a]
- A better alternative picks values whose logarithms are equally spaced.[b]
- Still other alternatives are possible.
- Generally, k must scale with at least n to show convergence.[c]

---

[a] Hull and White (1993).

[b] Called log-linear interpolation.

[c] Dai (`R86526008`, `D8852600`), Huang (`F83506075`), and Lyuu (2002).

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.
- Suppose the current node is $N(j, i)$ and the running average is a.
- Assume the next node is $N(j+1, i)$, after an up move.
- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the new running average

$$A_{\mathrm{u}} \equiv \frac{(j+1)\, a + S_0 u^{j+1-i} d^i}{j+2}.$$

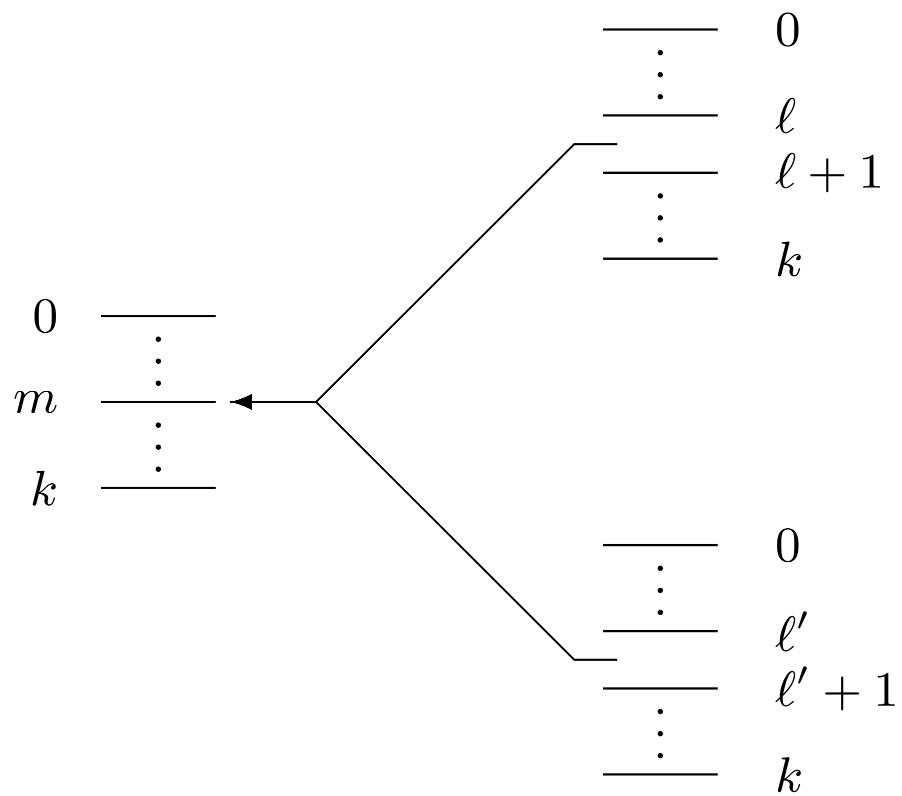
Approximation Algorithm for Asian Options (continued)

- But A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1, i)$!
- Find the 2 running averages that bracket it:

$$A_{\ell}(j+1, i) \leq A_{\mathrm{u}} < A_{\ell+1}(j+1, i).$$

- In "most" cases, the fastest way to nail ℓ is via

$$\ell = \left\lfloor \frac{A_{\mathrm{u}} - A_{\min}(j, i)}{[\, A_{\max}(j, i) - A_{\min}(j, i)\,]/k} \right\rfloor.$$


0
ℓ
ℓ + 1
k
0
m
k
0
ℓ′
ℓ′ + 1
k

Approximation Algorithm for Asian Options (continued)

- But watch out for the rare case where

$$A_{\mathrm{u}} = A_\ell(j+1, i)$$

for some ℓ.

- Also watch out for the case where

$$A_{\mathrm{u}} = A_{\max}(j, i).$$

- Finally, watch out for the degenerate case where $A_0(j+1, i) = \cdots = A_k(j+1, i)$.
  - It will happen along extreme paths!

Approximation Algorithm for Asian Options (continued)

- Express A_{u} as a linearly interpolated value of the two running averages,

$$A_{\mathrm{u}} = xA_{\ell}(j+1,i) + (1-x)\,A_{\ell+1}(j+1,i), \quad 0 < x \leq 1.$$

- Obtain the approximate option value given the running average A_{u} via

$$C_{\mathrm{u}} \equiv xC_{\ell}(j+1,i) + (1-x)\,C_{\ell+1}(j+1,i).$$

  - $C_{\ell}(t,s)$ denotes the option value at node $N(t,s)$ with running average $A_{\ell}(t,s)$.

- This interpolation introduces the second source of error.

Approximation Algorithm for Asian Options (continued)

- The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value C_{d}.
- Finally obtain the option value as
$$[\, pC_{\mathrm{u}} + (1-p)\, C_{\mathrm{d}} \,]\, e^{-r\Delta t}.$$
- The running time is $O(kn^2)$.
  - There are $O(n^2)$ nodes.
  - Each node has $O(k)$ buckets.

Approximation Algorithm for Asian Options (continued)

- For the calculations *from* time step $n-1$, no interpolation is needed.[a]
  - The option values are simply (for calls):

$$\begin{aligned} C_{\mathrm{u}} &= \max(A_{\mathrm{u}} - X, 0), \\ C_{\mathrm{d}} &= \max(A_{\mathrm{d}} - X, 0). \end{aligned}$$

  - That saves $O(nk)$ calculations.

[a]Contributed by Mr. Chen, Shih-Hang (R02723031) on April 9, 2014.

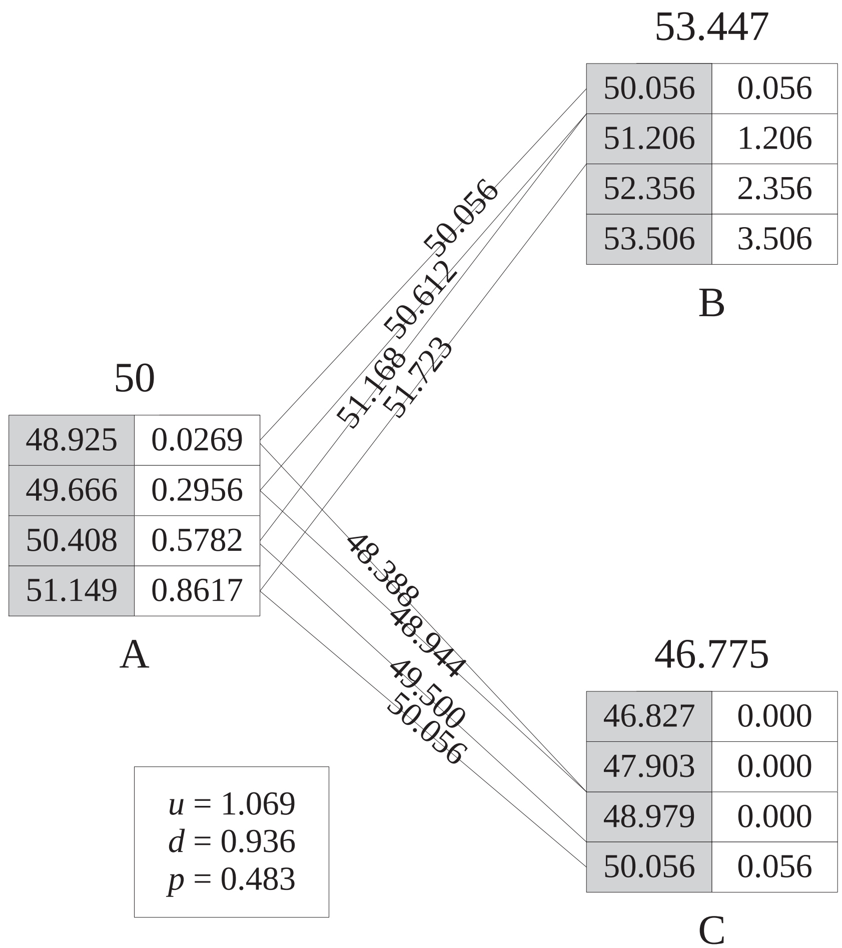
Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: no historical average to deal with.
- This problem can be easily addressed (see text).
- How about the Greeks?[a]

[a]Thanks to lively class discussions on March 31, 2004 and April 9, 2014.

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 412 is 48.925.
- The maximum running average at node A in the same figure is 51.149.


53.447
50.056 0.056
51.206 1.206
52.356 2.356
53.506 3.506
B
50
48.925 0.0269
49.666 0.2956
50.408 0.5782
51.149 0.8617
A
50.056
50.612
51.168
51.723
48.388
48.944
49.500
50.056
46.775
46.827 0.000
47.903 0.000
48.979 0.000
50.056 0.056
C
u = 1.069
d = 0.936
p = 0.483

A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option value corresponding to running average 50.056 at node B is 0.056.
- The option values corresponding to running average 51.206 at node B is 1.206.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be

$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$

- With 48.944 lying between 47.903 and 48.979 at node C, we solve

$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$

to obtain $x \approx 0.033$.

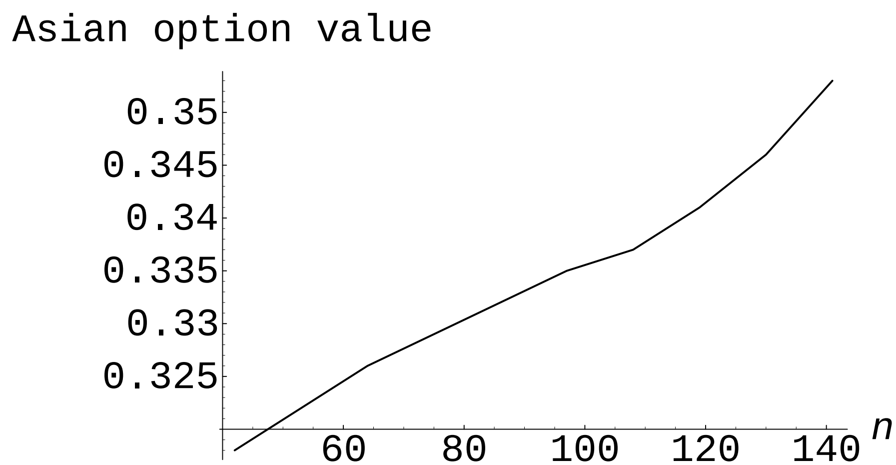
A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1 - p) \times 0.0 \approx 0.2956,$$

  where $p = 0.483$.
- The remaining three option values at node A can be computed similarly.

Convergence Behavior of the Approximation Algorithm[a]



[a]Dai (R86526008, D8852600) and Lyuu (2002).

Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value.[a]
- To guarantee convergence, k needs to grow with n at least.
- There is a convergent approximation algorithm that does away with interpolation with a running time of[b]

$$2^{O(\sqrt{n})}.$$

---

[a] Dai (R86526008, D8852600), Huang (F83506075), and Lyuu (2002).
[b] Dai (R86526008, D8852600) and Lyuu (2002, 2004).

Remarks on Asian Option Pricing (continued)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options.[a]
  - k can be varied for trade-off between time and accuracy.
  - If we pick $k = O(n^2)$, then the error is $O(1/n)$, and the running time is $O(n^4)$.

[a] Aingworth, Motwani (1962–2009), and Oldham (2000).

Remarks on Asian Option Pricing (continued)

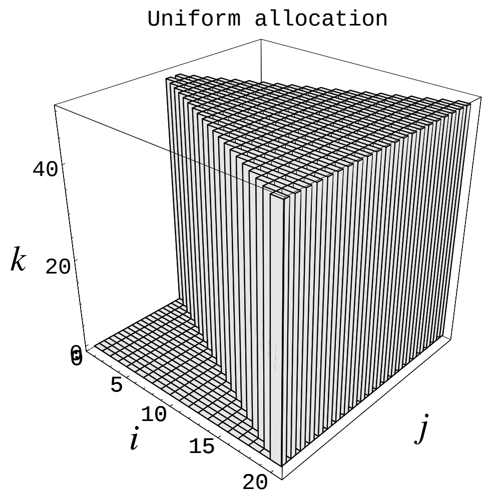
- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$.[a]
  - It varies the number of buckets per node.
  - If we pick $k = O(n)$, the error is $O(n^{-0.5})$.
  - If we pick $k = O(n^{1.5})$, then the error is $O(1/n)$, and the running time is $O(n^{3.5})$.
- Under "reasonable assumptions," an $O(n^2)$-time algorithm with an error bound of $O(1/n)$ exists.[b]

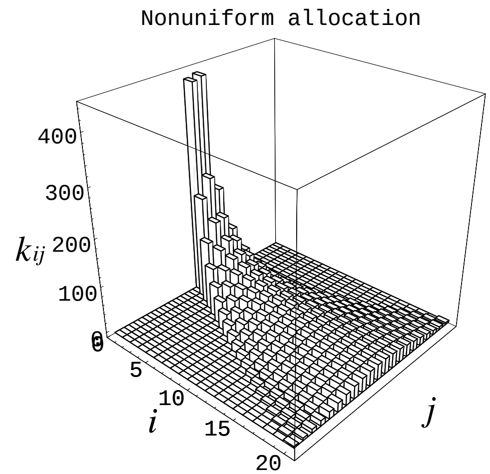
---

[a] Dai (`R86526008`, `D8852600`), Huang (`F83506075`), and Lyuu (2002).
[b] Hsu (`R7526001`, `D89922012`) and Lyuu (2004).

Remarks on Asian Option Pricing (concluded)

- The basic idea is a nonuniform allocation of running averages instead of a uniform k.
- It strikes a balance between error and complexity.





A Grand Comparison[a]

[a]Hsu (R7526001, D89922012) and Lyuu (2004); Zhang (2001,2003); Chen (R92723061) and Lyuu (2006).

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286

A Grand Comparison (concluded)

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.641598
100			8.4088330	8.4091957	8.4088744	8.408966	8.408519
110			3.5556100	3.5554997	3.5556415	3.556094	3.554687
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.952421
100			7.9456288	7.9459286	7.9458549	7.945918	7.944357
110			4.0717942	4.0702869	4.0720881	4.071945	4.070115
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.982782
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.694902
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.208724
110			5.7301225	5.7296982	5.7303567	5.730357	5.728161

Forwards, Futures, Futures Options, Swaps

Summon the nations to come to the trial.
Which of their gods can predict the future?
— Isaiah 43:9

The sure fun of the evening
outweighed the uncertain treasure[.]
— Mark Twain (1835–1910),
The Adventures of Tom Sawyer

Terms

- r will denote the riskless interest rate.
- The current time is t.
- The maturity date is T.
- The remaining time to maturity is $\tau \equiv T - t$ (all measured in years).
- The spot price S, the spot price at maturity is S_T.
- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f.
- A price with a subscript t usually refers to the price at time t.
- Continuous compounding will be assumed.

Forward Contracts

- Forward contracts are for the delivery of the underlying asset for a certain delivery price on a specific time.
  - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for $2.5 per bushel on September 27, 1995.[a]
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

[a] The farmer assumes a *short* position.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
  - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
  - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_{f} denotes the annual interest rate of the foreign currency.
- r_{ℓ} denotes the annual interest rate of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity[a]

$$\frac{F}{S} = e^{r_\ell - r_{\mathrm{f}}}. \tag{34}$$

- A holder of the local currency can do either of:
  - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.
  - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F, by selling forward foreign currency now.

[a]Keynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history.

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become $Fe^{r_{\mathrm{f}}}/S$ one year from now in the 2nd case.
- If $Fe^{r_{\mathrm{f}}}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $Fe^{r_{\mathrm{f}}}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

  - Contrast that with the payoff of a call, $\max(S_T - X, 0)$.
- Forward contracts do not involve any initial cash flow.
- The forward price is the delivery price which makes the forward contract zero valued.
  - That is,

$$f = 0 \quad \text{when} \quad X = F.$$

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
- So although the value of a forward contract, f, is 0 at the outset, it will fluctuate thereafter.
  - This value is enhanced when the spot price climbs.
  - It is depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 12 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{35}$$

- If $F > Se^{r\tau}$:
  - Borrow S dollars for τ years.
  - Buy the underlying asset.
  - Short the forward contract with delivery price F.

Proof (concluded)

- At maturity:
  - Deliver the asset for F.
  - Use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of
$$F - Se^{r\tau} > 0.$$
- If $F < Se^{r\tau}$, do the opposite.

Example

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on a 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}. \tag{36}$$

- Consider a portfolio consisting of:
  - One long forward contract;
  - Cash amount $Xe^{-r\tau}$;
  - One short position in the underlying asset.

Contract Value: The Underlying Pays No Income (concluded)

- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.[a]

---

[a]Recall p. 195.

Lemma 12 (p. 437) Revisited

- Set $f = 0$ in Eq. (36) on p. 440.
- Then $X = Se^{r\tau}$, the forward price.

Forward Price: Underlying Pays Predictable Income

Lemma 13 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I)\, e^{r\tau}. \tag{37}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, the asset is delivered for F, and $(S - I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - (S - I)\, e^{r\tau} > 0$.
- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}.$$

  - r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is[a]

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{38}$$

- One consequence of Eq. (38) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{39}$$

[a] See text for proof.

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the CBT.
  - One million U.S. dollars for the Eurodollar futures on the CME.
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell a food processor 100,000 bushels of corn at \$2.00 per bushel in November.
  - Suppose the price of corn rises to \$2.5 by November.

Daily Settlements (concluded)

- (continued)
  - The farmer has incentive to sell his harvest in the spot market at \$2.5.
  - With marking to market, the farmer has transferred \$0.5 per bushel from his futures account to that of the food processor by November.
  - When the farmer makes delivery, he is paid the spot price, \$2.5 per bushel.
  - The farmer has little incentive to default.
  - The net price remains \$2.00 per bushel, the original delivery price.

Delivery and Hedging

- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.[a]
- Changes in futures prices usually track those in spot prices.
- This makes hedging possible.

[a]But since early 2006, futures for corn, wheat and soybeans occasionally expired at a price much higher than that day's spot price.

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1})$$
$$= F_n - F_0 = S_T - F_0.$$

- A futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract.
- The actual payoff may vary because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Forward and Futures Prices[a]

- Surprisingly, futures price equals forward price if interest rates are nonstochastic!
  - See text for proof.
- This result "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

[a]Cox, Ingersoll, and Ross (1981).

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market.
- And the futures position of the two parties will be at the prevailing futures price.

Futures Options (concluded)

- It works as if the call holder received a futures contract plus cash equivalent to the prevailing futures price F_t minus the strike price X:

$$F_t - X.$$

  - This futures contract has zero value.

- It works as if the put holder sold a futures contract for

$$X - F_t$$

dollars.

Forward Options

- Similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.
- Exercising a forward option incurs no immediate cash flows.

Example

- Consider a call with strike \$100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.
- If an offsetting position is then taken in the forward market,[a] a \$10 profit *in December* will be assured.
- A call on the futures would realize the \$10 profit *in July*.

[a] The counterparty will pay you \$110 for the underlying asset.

Some Pricing Relations

- Let delivery take place at time T, the current time be 0, and the option on the futures or forward contract have expiration date t $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does not have the same value as a futures option.
- The payoffs of calls at time t are, respectively,

$$\text{futures option} = \max(F_t - X, 0), \tag{40}$$

$$\text{forward option} = \max(F_t - X, 0)\, e^{-r(T-t)}. \tag{41}$$

Some Pricing Relations (concluded)

- A European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the options.
  - Futures price equals spot price at maturity.
  - This conclusion is independent of the model for the spot price.

Put-Call Parity

The put-call parity is slightly different from the one in Eq. (22) on p. 202.

Theorem 14 *(1) For European options on futures contracts,*

$$C = P - (X - F)\, e^{-rt}.$$

(2) For European options on forward contracts,

$$C = P - (X - F)\, e^{-rT}.$$

- See text for proof.

Early Exercise

The early exercise feature is not valuable for and forward options.

Theorem 15 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

- See text for proof.

Early exercise may be optimal for American futures options even if the underlying asset generates no payouts.

Theorem 16 *American futures options may be exercised optimally before expiration.*

Black's Model[a]

- Formulas for European futures options:

$$
\begin{aligned}
C &= F e^{-rt} N(x) - X e^{-rt} N(x - \sigma\sqrt{t}), \qquad (42)\\
P &= X e^{-rt} N(-x + \sigma\sqrt{t}) - F e^{-rt} N(-x),
\end{aligned}
$$

  where $x \equiv \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.

- Formulas (42) are related to those for options on a stock paying a continuous dividend yield.

- They are exactly Eqs. (29) on p. 300 with q set to r and S replaced by F.

[a]Black (1976).

Black Model (concluded)

- This observation incidentally proves Theorem 16 (p. 461).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$ by Eqs. (40)–(41) on p. 458.

