Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

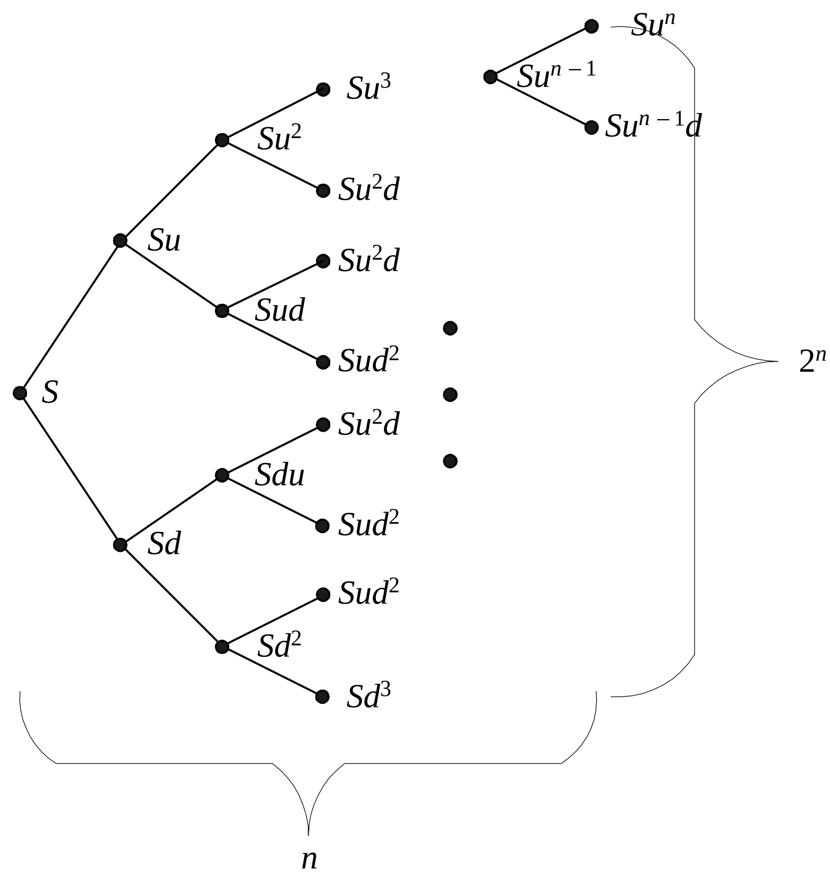
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
  $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (27) on p. 248 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Let
$$\widehat{\mu} \equiv \frac{1}{n}\, E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period.

- Let
$$\widehat{\sigma}^2 \equiv \frac{1}{n}\, \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of the that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$
\begin{aligned}
n\widehat{\mu} &= n[\, q \ln(u/d) + \ln d \,] \to \mu\tau, \\
n\widehat{\sigma}^2 &= nq(1-q) \ln^2(u/d) \to \sigma^2\tau.
\end{aligned}
$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 260).
  - Other choices are possible (see text).
  - Exact solutions for u, d, q are also feasible: 3 equations for 3 variables.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \tag{28}$$

- With Eqs. (28), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The choices (28) result in the CRR binomial model.[a]
- A more common choice for the probability is actually

$$q = \frac{R - d}{u - d}.$$

by Eq. (25) on p. 230.

- Their numerical properties are essentially identical.

[a] Cox, Ross, and Rubinstein (1979).

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (28) on p. 271.
  - If this happens, the risk-neutral probability may lie outside $[0, 1]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma \sqrt{\tau/n}} > e^{r\tau/n},$$

i.e., when $n > r^2 \tau / \sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions will be presented later.

[a] Many papers and programs forget to check this condition!

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

[a] The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.

Toward the Black-Scholes Formula (continued)

Lemma 10 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \equiv (e^{r\tau/n}-d)/(u-d).$$

- Let $n \to \infty$.[a]

[a] See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return[b] is thus the riskless rate r.

---

[a]By Lemma 10 (p. 275) and Eq. (21) on p. 161.

[b]In the sense of $(1/\tau) \ln E[\, S_\tau / S\,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S)\,]$ (geometric average rate of return).

Toward the Black-Scholes Formula (concluded)[a]

Theorem 11 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

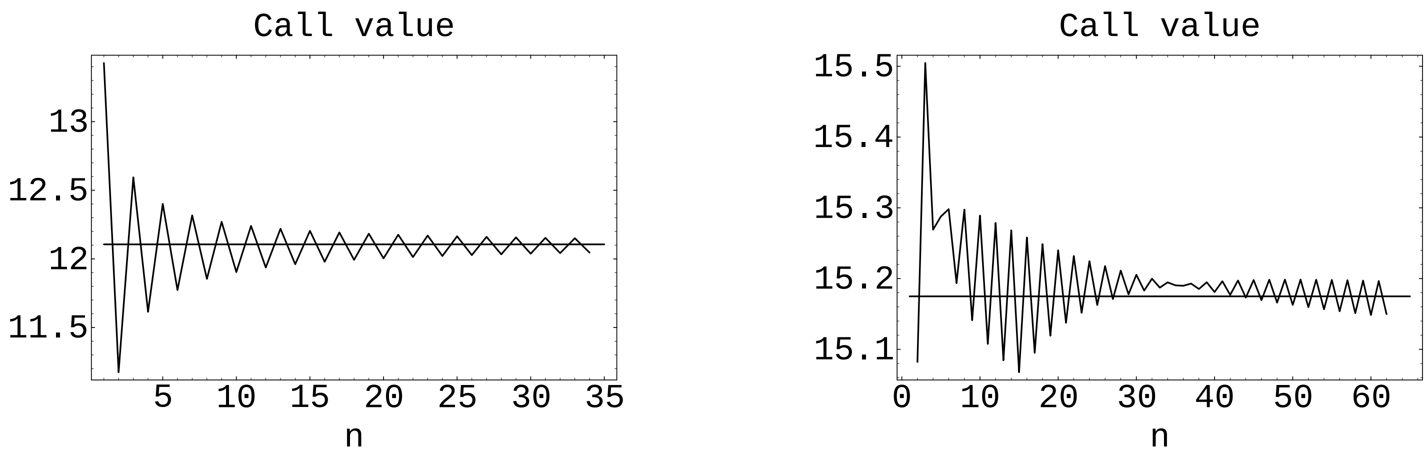
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$\begin{aligned} u &= e^{\sigma\sqrt{\tau/n}}, \\ d &= e^{-\sigma\sqrt{\tau/n}}, \\ \hat{r} &= r\tau/n. \end{aligned}$$



- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of u and d (see text).

[a] Chang and Palmer (2007).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

[a] Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  > the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

[a]Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.
- So?

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?[b]

---

[a] Fama (1965); French (1980); French and Roll (1986).

[b] Recall p. 146 about dating issues.

Trading Days and Calendar Days (concluded)

- Think of σ as measuring the volatility of stock price one year from now (regardless of what happens in between).
- Suppose a year has 260 trading days.
- So a heuristic is to replace σ in the Black-Scholes formula with[a]

$$\sigma \sqrt{\frac{365}{260} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

- How about binomial tree algorithms?

[a]French (1984).

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the continuation value.
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the *prevailing* stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\, u$, $(Su - D)\, d$, $(Sd - D)\, u$, $(Sd - D)\, d$.
  - The binomial tree no longer combines.

$$
\begin{array}{ccccc}
 & & & & (Su - D)\, u \\
 & & & \nearrow & \\
 & & Su - D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su - D)\, d \\
S & & & & \\
 & & & & (Sd - D)\, u \\
 & \searrow & & \nearrow & \\
 & & Sd - D & & \\
 & & & \searrow & \\
 & & & & (Sd - D)\, d
\end{array}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.[a]
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, σ is the volatility of the process followed by the *risky* component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

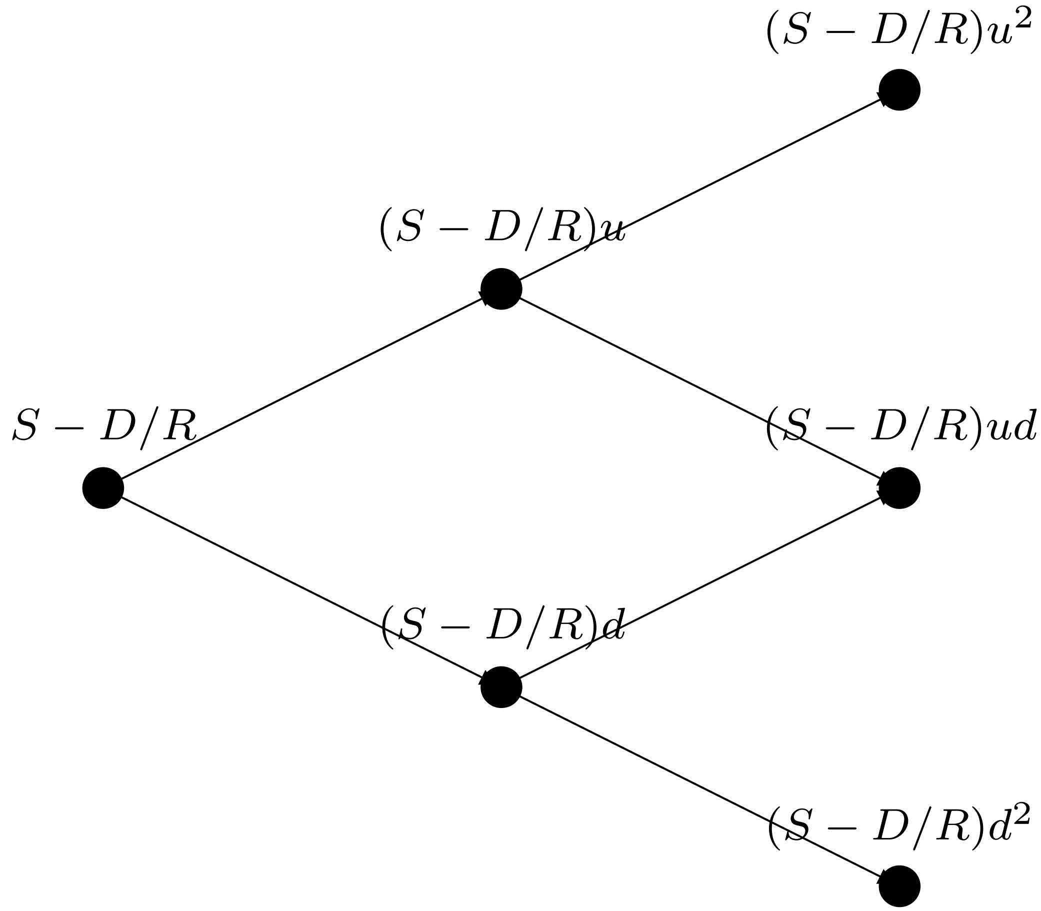
---

[a]Roll (1977).

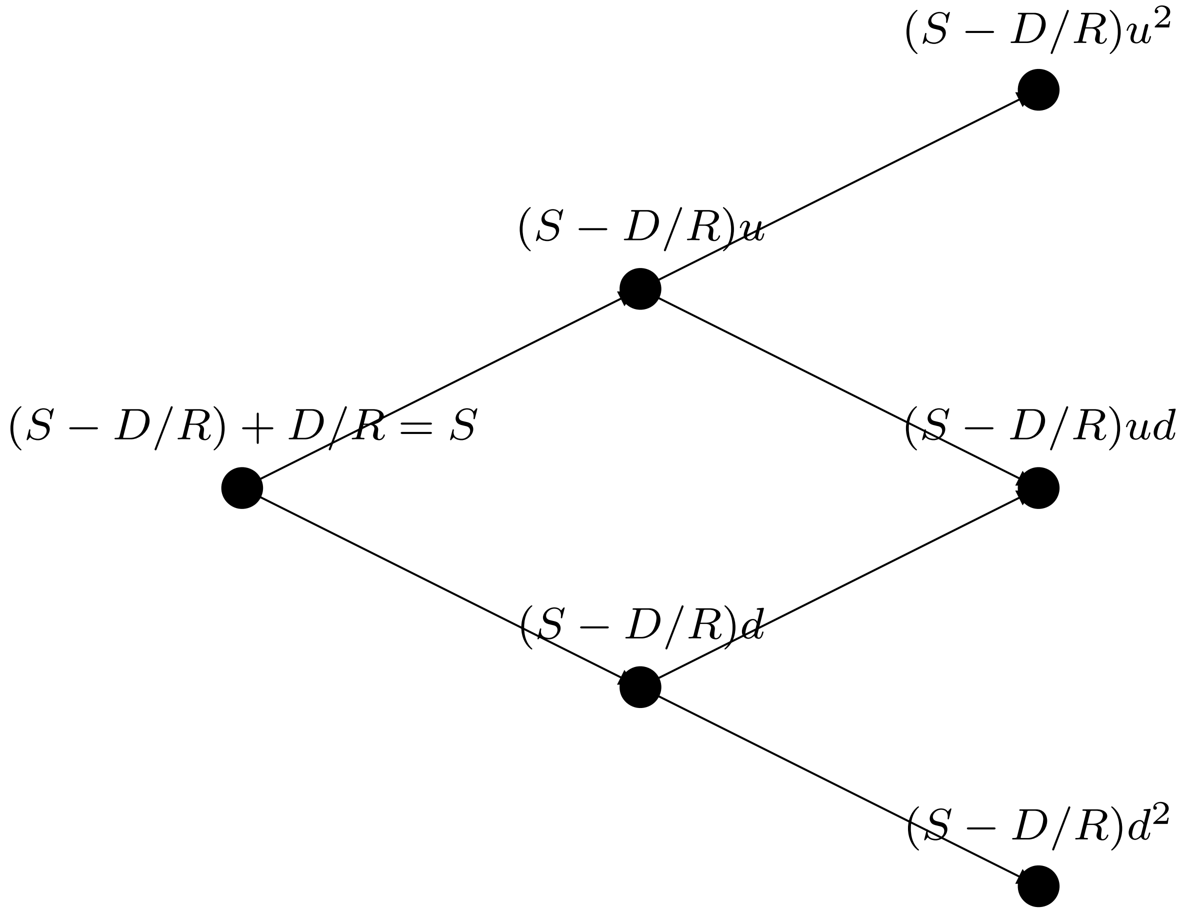
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

The Ad-Hoc Approximation vs. P. 291 (Step 1)



The Ad-Hoc Approximation vs. P. 291 (Step 2)



The Ad-Hoc Approximation vs. P. 291[a]

- The trees are different.
- The stock prices at maturity are also different.
  - $(Su - D)\, u, (Su - D)\, d, (Sd - D)\, u, (Sd - D)\, d$ (p. 291).
  - $(S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2$ (ad hoc).
- Note that, as $d < R < u$,

$$\begin{aligned}(Su - D)\, u &> (S - D/R)u^2,\\ (Sd - D)\, d &< (S - D/R)d^2,\end{aligned}$$

[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on March 18, 2009.

The Ad-Hoc Approximation vs. P. 291 (concluded)

- So the ad hoc approximation has a smaller dynamic range.
- This explains why in practice the volatility is usually increased when using the ad hoc approximation.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 686ff).
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.

---

[a] Dai (R86526008, D8852600) and Lyuu (2004).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.[a]

[a] In pricing European options, we care only about the distribution of S_τ.

Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$:[a]

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{29}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{29'}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (29) and (29′) remain valid as long as the dividend yield is predictable.

[a] Merton (1973).

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \equiv \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d![a]

---

[a] Contributed by Ms. Wang, Chuan-Ju (`F95922018`) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{30}$$

  where $\Delta t \equiv \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.

- Other than the change in Eq. (30), binomial tree algorithms stay the same *as if there were no dividends*.

Curtailing the Range of Tree Nodes[a]

- Those nodes can be skipped if they are extremely unlikely to be reached.
- The probability that the price will move more than a certain number of standard deviations from its initial value is negligible.
- Similarly, suppose the stock price at maturity is known.
- The probability that the price will move outside that number of standard deviations in working backward is also negligible.

[a] Andricopoulos, Widdicks, Duck, Newton (2004).

Curtailing the Range of Tree Nodes (continued)

- In summary, for certain stock prices, the strike price is so low or so high that it could not realistically be reached.
- For these prices the option value is basically deterministic.
- By working only within the said range of stock prices, we can save time without significant loss of accuracy.

Curtailing the Range of Tree Nodes (concluded)

- For time t, where $0 < t < T$, the maximum and minimum stock prices $S_{\max}(t)$ and $S_{\min}(t)$ are:

$$\begin{aligned} S_{\max}(t) &= \min\left(S_0 e^{rt+\eta\sigma\sqrt{t}}, X e^{-r(T-t)+\eta\sigma\sqrt{T-t}}\right), \\ S_{\min}(t) &= \max\left(S_0 e^{rt-\eta\sigma\sqrt{t}}, X e^{-r(T-t)-\eta\sigma\sqrt{T-t}}\right). \end{aligned}$$

Traversal Sequence

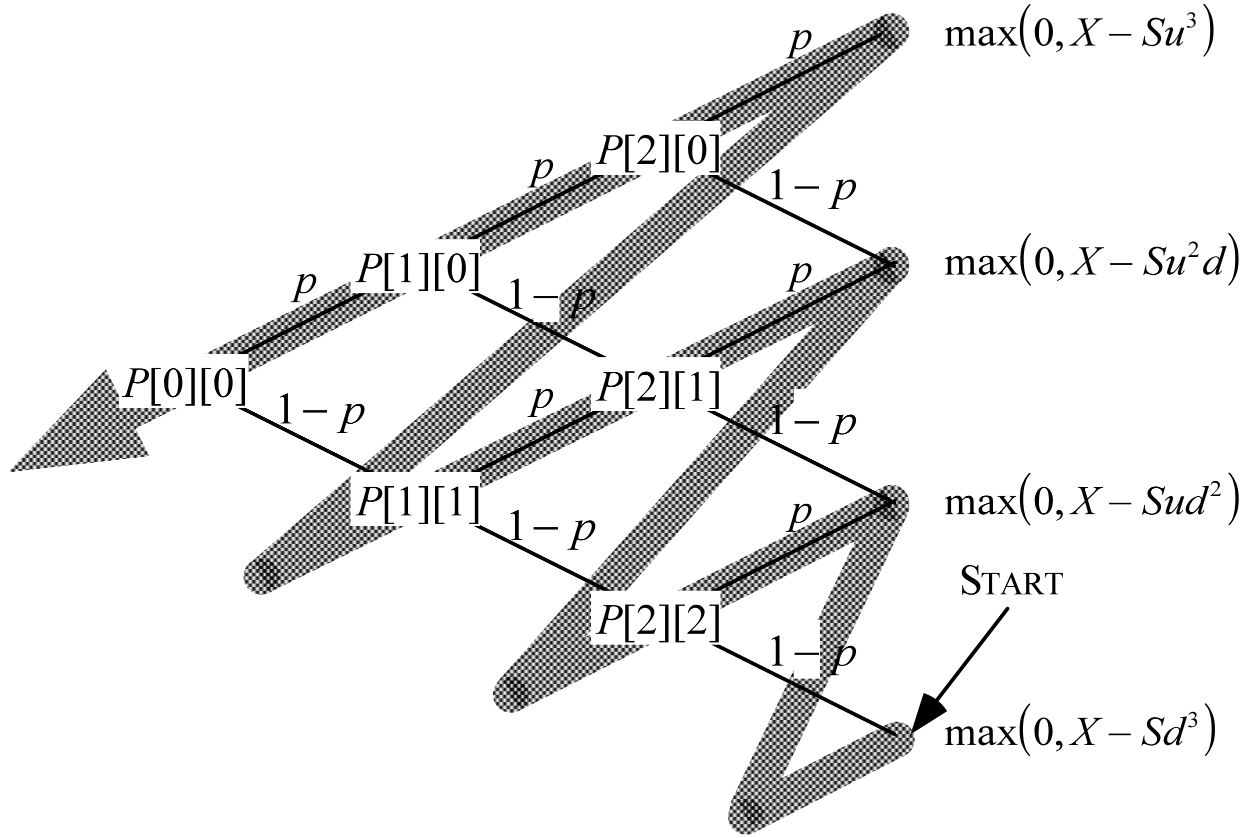
- Can the standard quadratic-time binomial tree algorithm for American options be improved?
  - By an order?
  - By a constant factor?
- In any case, it helps to skip nodes.
- Note the traversal sequence of backward induction on the tree.
  - It is by time (recall p. 259).

Diagonal Traversal of the Tree[a]

- Suppose we traverse the tree diagonally.
- Convince yourself that this procedure is well-defined.
- An early-exercise node is trivial to evaluate.
  - The difference of the strike price and the stock price.
- A non-early-exercise node must be evaluated by backward induction.

---

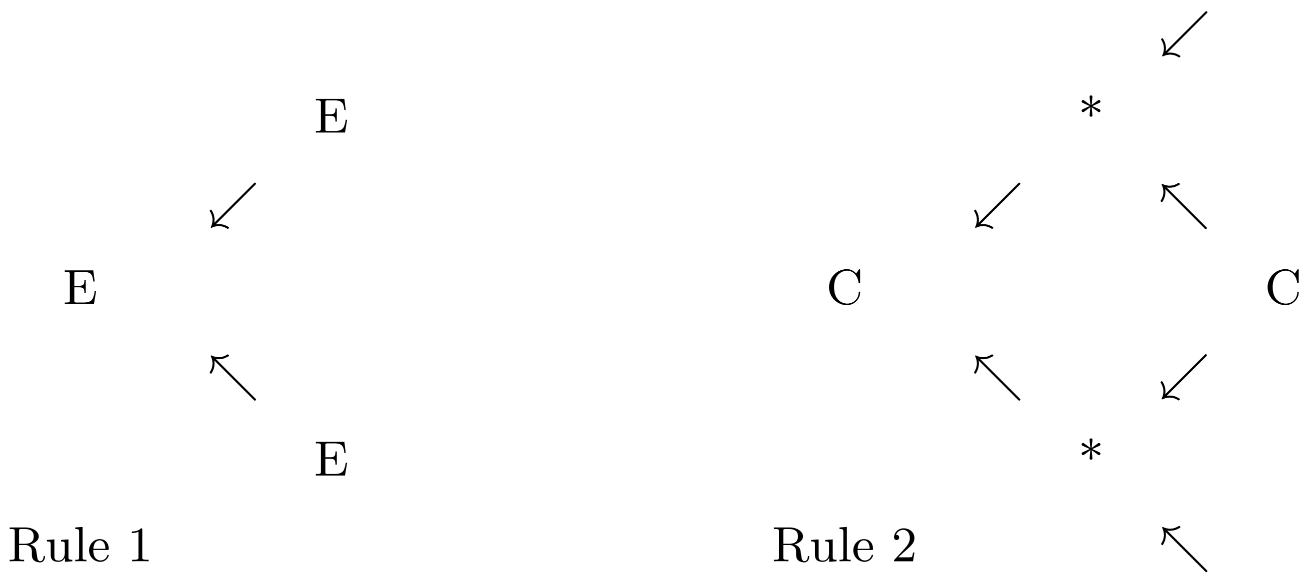
[a]Curran (1995).


p
$\max(0, X - Su^3)$
$P[2][0]$
p
$1-p$
$P[1][0]$
p
$1-p$
$\max(0, X - Su^2 d)$
p
$P[0][0]$
$P[2][1]$
$1-p$
p
$1-p$
$P[1][1]$
$1-p$
$\max(0, X - Sud^2)$
p
START
$P[2][2]$
$1-p$
$\max(0, X - Sd^3)$

Diagonal Traversal of the Tree (continued)

Two properties of the propagation of early exercise nodes (E) and non-early-exercise nodes (C) during backward induction are:

1. A node is an early-exercise node if both its successor nodes are exercised early.
   - A terminal node that is in-the-money is considered an early exercise node.
2. If a node is a non-early-exercise node, then all the earlier nodes at the same horizontal level are also non-early-exercise nodes.
   - Here we assume $ud = 1$.


E
E
E
Rule 1
*
C
C
*
Rule 2

Diagonal Traversal of the Tree (continued)

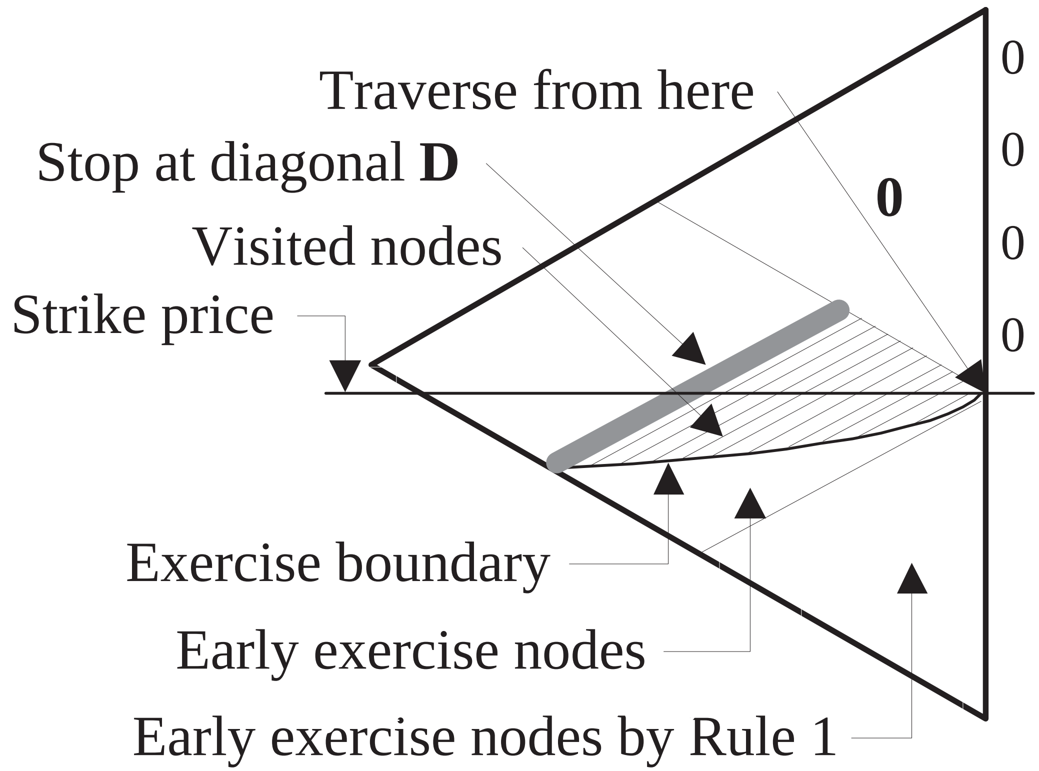
- Nothing is achieved if the whole tree needs to be explored.
- We need a stopping rule.
- The traversal stops when a diagonal D consisting *entirely* of non-early-exercise nodes is encountered.
  - By Rule 2, all early-exercise nodes have been found.

Diagonal Traversal of the Tree (continued)

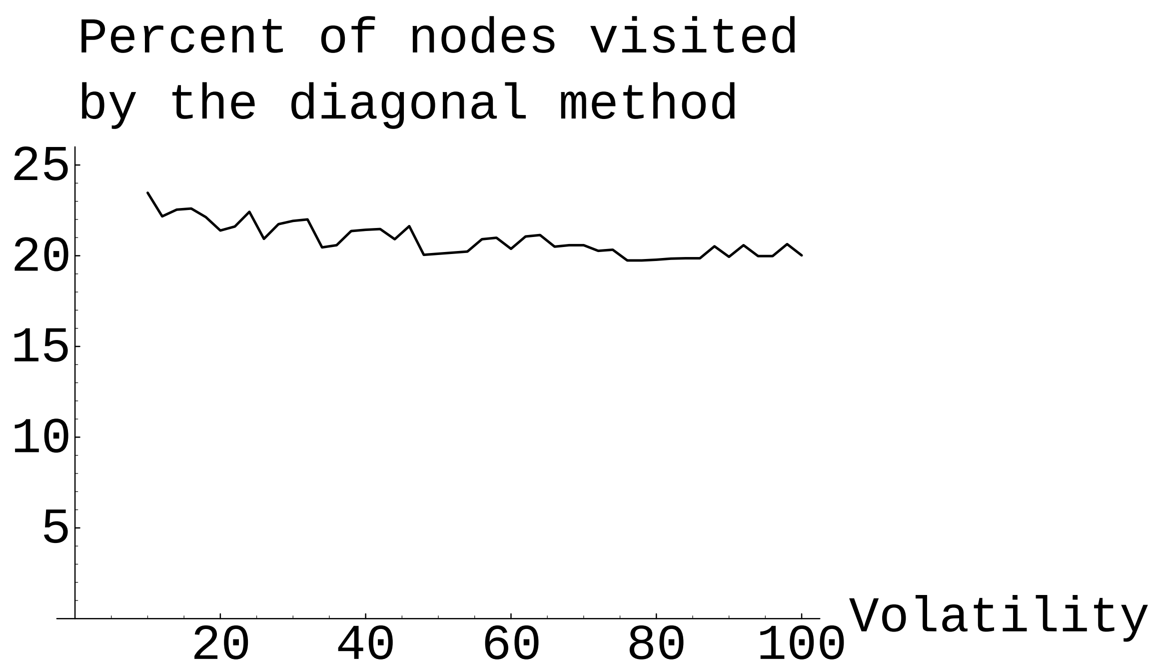
- When the algorithm finds an early exercise node N in traversing a diagonal, it can stop immediately and move on to the next diagonal.
  - By Rule 2, the node to the right of N must also be an early exercise node.
  - By Rule 1 and induction, the rest of the nodes on the current diagonal must all be early-exercise nodes.
  - They are hence computable on the fly when needed.

Diagonal Traversal of the Tree (continued)

- Also by Rule 1, the traversal can start from the zero-valued terminal node just above the strike price.
- The upper triangle above the strike price can be skipped since its nodes are all zero valued.



Traverse from here
Stop at diagonal **D**
Visited nodes
Strike price
0
0
0
0
0
Exercise boundary
Early exercise nodes
Early exercise nodes by Rule 1


Percent of nodes visited
by the diagonal method
25
20
15
10
5
20
40
60
80
100
Volatility

Diagonal Traversal of the Tree (continued)

- It remains to calculate the option value.
- It is the weighted sum of the discounted option values of the nodes on D.
  - How does the payoff influence the root?
  - We cannot go from the root to a node at which the option will be exercised without passing through D.
- The weight is the probability that the stock price hits the diagonal for the *first* time at that node.

Diagonal Traversal of the Tree (concluded)

- For a node on D which is the result of i up moves and j down moves, the said probability is

$$\binom{i+j-1}{i} p^i (1-p)^j.$$

  - A valid path must pass through the node which is the result of i up moves and $j-1$ down moves.

- Call the option value on this node P_i.

- The desired option value then equals

$$\sum_{i=0}^{a-1} \binom{i+j-1}{i} p^i (1-p)^j P_i e^{-(i+j)\, r\Delta t}.$$

The Analysis

- Since each node on D has been evaluated by that time, this part of the computation consumes $O(n)$ time.
- The space requirement is also linear in n since only the diagonal has to be allocated space.
- This idea can save computation time when D does not take long to find.

The Analysis (continued)

- Rule 2 is true with or without dividends.
- Suppose now that the stock pays a continuous dividend yield $q \leq r$ (or $r \leq q$ for calls by parity).
- Recall $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$.
- Rule 1 continues to hold since, for a current stock price of $Su^i d^j$:

The Analysis (concluded)

$$
\begin{aligned}
& \left(pP_u + (1-p)\,P_d\right) e^{-r\Delta t} \\
= \;& \left[\, p\left(X - Su^{i+1}d^{j}\right) + (1-p)\left(X - Su^{i}d^{j+1}\right)\right] e^{-r\Delta t} \\
= \;& Xe^{-r\Delta t} - Su^{i}d^{j}\left(pu + (1-p)\,d\right) e^{-r\Delta t} \\
= \;& Xe^{-r\Delta t} - Su^{i}d^{j}e^{-q\Delta t} \\
\leq \;& Xe^{-r\Delta t} - Su^{i}d^{j}e^{-r\Delta t} \\
\leq \;& X - Su^{i}d^{j}.
\end{aligned}
$$