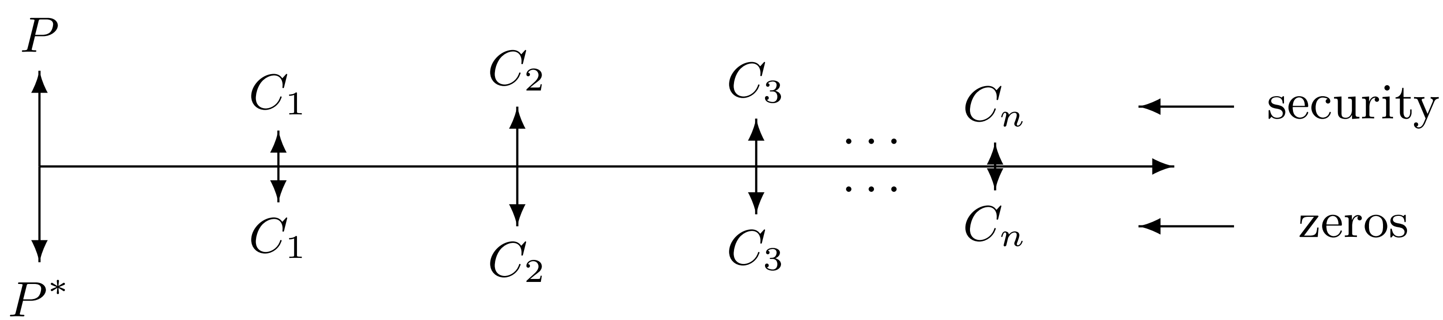
The PV Formula Justified

Theorem 1 *For a certain cash flow* $C_1, C_2, \ldots, C_n$,

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Suppose the price $P^* < P$.
- Short the zeros that match the security's n cash flows.
- The proceeds are P dollars.


P
C_1
C_2
C_3
C_n
security
$\cdots$
$\cdots$
zeros
C_1
C_2
C_3
C_n
P^*

The Proof (concluded)

- Then use P^* of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of $P - P^*$ dollars has been realized now.
- If $P^* > P$, just reverse the trades.

Two More Examples

- An American option cannot be worth less than the intrinsic value.[a]
  - Suppose the opposite is true.
  - So the American option is cheaper than its intrinsic value.
  - Now, buy the option, promptly exercise it, and close the stock position.
  - The cost of buying the option is less than the payoff: the intrinsic value.
  - So there is an immediate arbitrage profit.

[a] $\max(0, S_t - X)$ or $\max(0, X - S_t)$.

Two More Examples (concluded)

- A put or a call must have a nonnegative value.
  - Suppose otherwise and the option has a negative price.
  - Buy the option for a positive cash flow now.
  - It will end up with a nonnegative amount at expiration.
  - So an arbitrage profit is realized now.

Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.
- Let the current time be time zero.
- $\text{PV}(x)$ stands for the PV of x dollars at expiration.
- Hence $\text{PV}(x) = xd(\tau)$ where τ is the time to expiration.

Put-Call Parity[a]

$$C = P + S - \mathrm{PV}(X). \tag{22}$$

- Consider the portfolio of:
  - One short European call;
  - One long European put;
  - One share of stock;
  - A loan of PV(X).
- All options are assumed to carry the same strike price X and time to expiration, τ.
- The initial cash flow is therefore

$$C - P - S + \mathrm{PV}(X).$$

[a] Castelli (1877).

The Proof (continued)

- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.
- The loan is now X.
- The net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- The net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle implies that the initial investment to set up the portfolio must be nil as well (p. 195).

Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.
- $S = C - P + \mathrm{PV}(X)$: A stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$.
- $C - P = S - \mathrm{PV}(X)$: A long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

Intrinsic Value

Lemma 2 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- The put-call parity implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

- Recall $C \geq 0$ (p. 200).
- It follows that $C \geq \max(S - X, 0)$, the intrinsic value.
- An American call also cannot be worth less than its intrinsic value (p. 199).

Intrinsic Value (concluded)

A European put on a non-dividend-paying stock may be worth less than its intrinsic value.

Lemma 3 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

- Prove it with the put-call parity.
- Can explain the right figure on p. 172 why $P < X - S$ when S is small.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends.

Theorem 4 (Merton (1973)) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By an exercise in text, $C \geq \max(S - \text{PV}(X), 0)$.
- If the call is exercised, the value is $S - X$.
- But
$$\max(S - \text{PV}(X), 0) \geq S - X.$$

Remarks

- The above theorem does *not* mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock, however.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.[a]

Theorem 5 *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

---

[a]See Theorem 8.4.2 of the textbook.

A General Result[a]

Theorem 6 (Cox and Rubinstein (1985)) *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

Corollary 7 *Any sufficiently well-behaved payoff function can be approximated by a portfolio of calls and puts.*

---

[a]See Exercise 8.3.6 of the textbook.

Convexity of Option Prices[a]

Lemma 8 *For three otherwise identical calls or puts with strike prices* $X_1 < X_2 < X_3$,

$$\begin{aligned} C_{X_2} &\leq \omega C_{X_1} + (1-\omega)\, C_{X_3} \\ P_{X_2} &\leq \omega P_{X_1} + (1-\omega)\, P_{X_3} \end{aligned}$$

Here

$$\omega \equiv (X_3 - X_2)/(X_3 - X_1).$$

(*Equivalently,* $X_2 = \omega X_1 + (1-\omega)\, X_3$.)

[a] See Lemma 8.5.1 of the textbook.

The Intuition behind Lemma 8[a]

- Set up the following portfolio:

$$\omega C_{X_1} - C_{X_2} + (1-\omega)\, C_{X_3}.$$

- This is a butterfly spread (p. 183).
- It has a nonnegative value as, for any S at maturity,

$$\omega \max(S - X_1, 0) - \max(S - X_2, 0) + (1-\omega)\ \max(S - X_3, 0) \geq 0.$$

- Therefore,

$$\omega C_{X_1} - C_{X_2} + (1-\omega)\, C_{X_3} \geq 0.$$

[a]Contributed by Mr. Cheng, Jen-Chieh (B96703032) on March 17, 2010.

Option on a Portfolio vs. Portfolio of Options

- Consider a portfolio of non-dividend-paying assets with weights ω_i.
- Let C_i denote the price of a European call on asset i with strike price X_i.
- All options expire on the same date.

Option on a Portfolio vs. Portfolio of Options (concluded)

An option on a portfolio is cheaper than a portfolio of options.[a]

Theorem 9 *The call on the portfolio with a strike price*

$$X \equiv \sum_i \omega_i X_i$$

has a value at most

$$\sum_i \omega_i C_i.$$

The same holds for European puts.

[a] See Theorem 8.6.1 of the textbook.

Option Pricing Models

If the world of sense does not fit mathematics,
so much the worse for the world of sense.
— Bertrand Russell (1872–1970)

Black insisted that anything one could do
with a mouse could be done better
with macro redefinitions
of particular keys on the keyboard.
— Emanuel Derman,
My Life as a Quant (2004)

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[a]
  - Known as the Black-Scholes option pricing model.

[a]The results were obtained as early as June 1969.

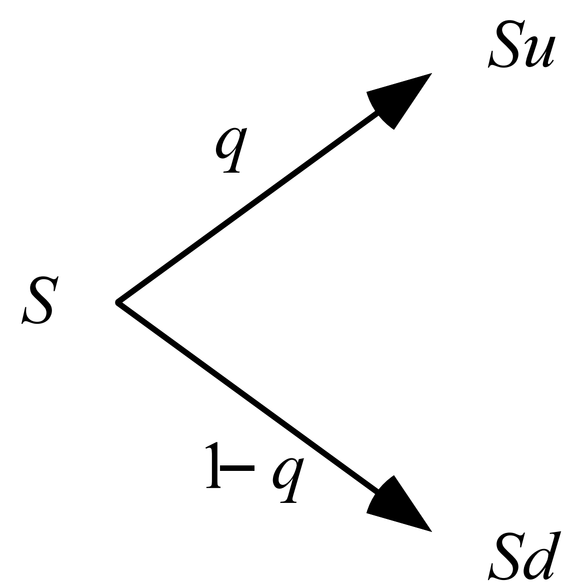
Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:

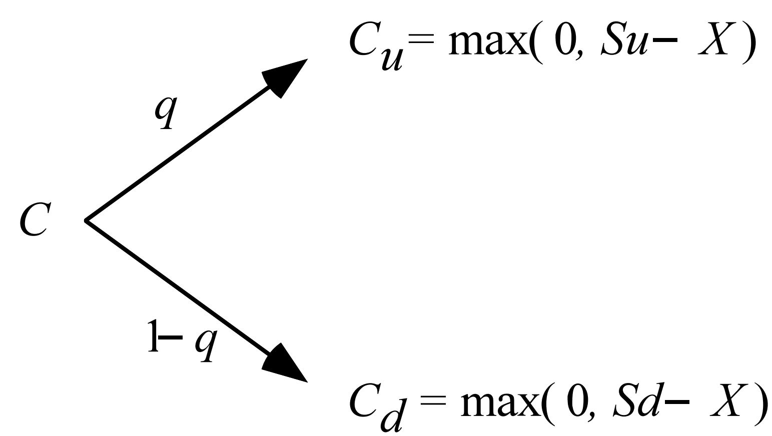
  S, u, d, X, $\hat{r}$, and the number of periods to expiration.


Su
q
S
1 − q
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time 1 if the stock price moves to Su.
- C_d is the call price at time 1 if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is

$$\begin{aligned} &hSu + RB \qquad \text{(up move)}, \\ &hSd + RB \qquad \text{(down move)}. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Choose h and B such that the portfolio *replicates* the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{23}$$

$$B = \frac{uC_d - dC_u}{(u - d)\, R}. \tag{24}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,[a]

$$C = hS + B.$$

- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

[a] Or the replicating portfolio, as it replicates the option.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 4 (p. 208).
- So

$$C = hS + B.$$

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q.[a]
- Hence it is independent of the expected gross return of the stock, $qSu + (1 - q)\, Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Then the set of possible stock prices is the same whatever q is.

---

[a] More precisely, not directly dependent on q. Thanks to a lively class discussion on March 16, 2011.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}.$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as

$$pSu + (1 - p)\,Sd = RS.$$

- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate *in a risk-neutral economy.*

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

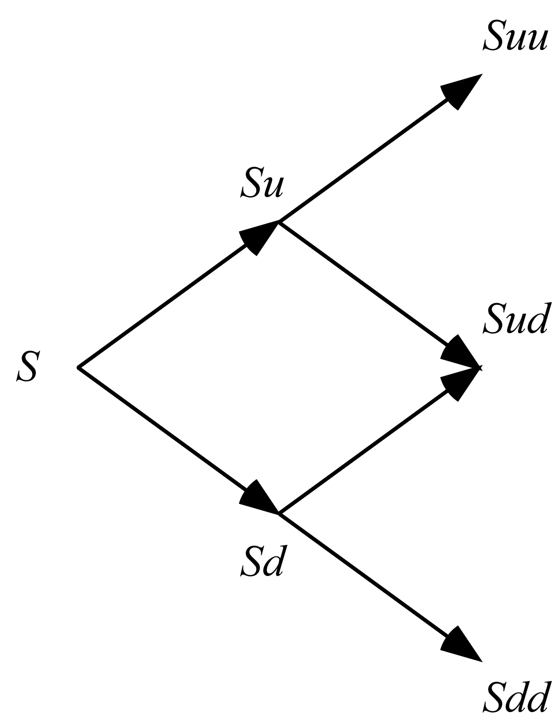
  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you flip a coin n times with p being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree *combines*.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.


Suu
Su
Sud
S
Sd
Sdd

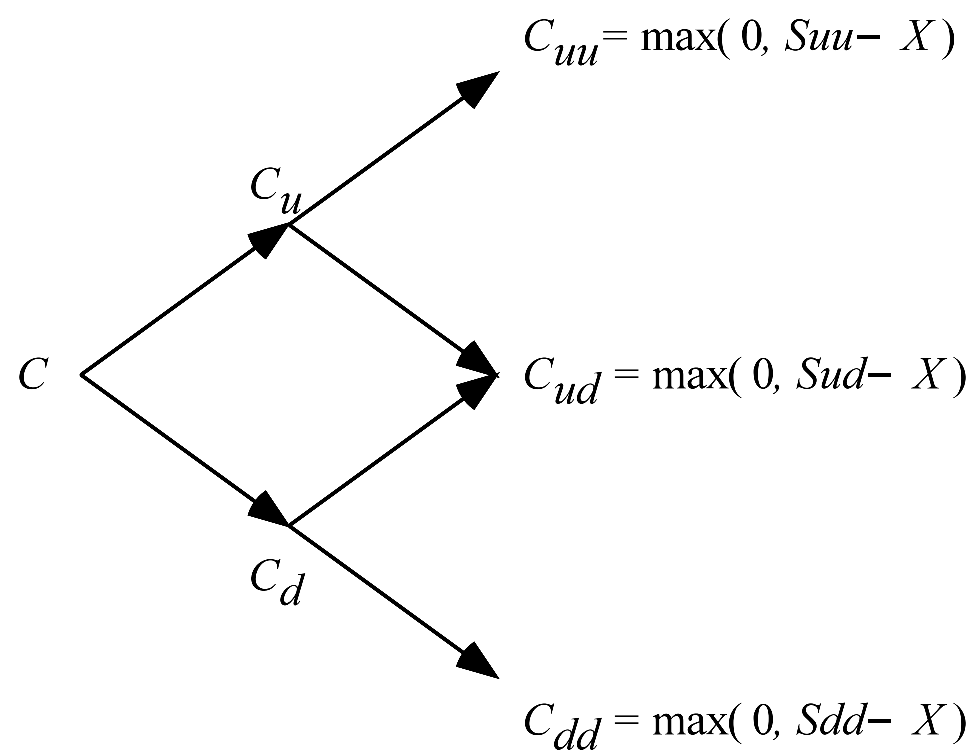
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$



$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \qquad (25) \\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (23) on p. 226.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute
$$\frac{pC_u + (1-p)\,C_d}{R}$$
as the option price.
- The values of delta h and B can be derived from Eqs. (23)–(24) on p. 226.

Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\, C_d}{R} \geq \frac{[\, pu + (1-p)\, d\,]\, S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  - The call again will not be exercised at present.[a]
- So

$$C = hS + B = \frac{pC_u + (1-p)\, C_d}{R}.$$

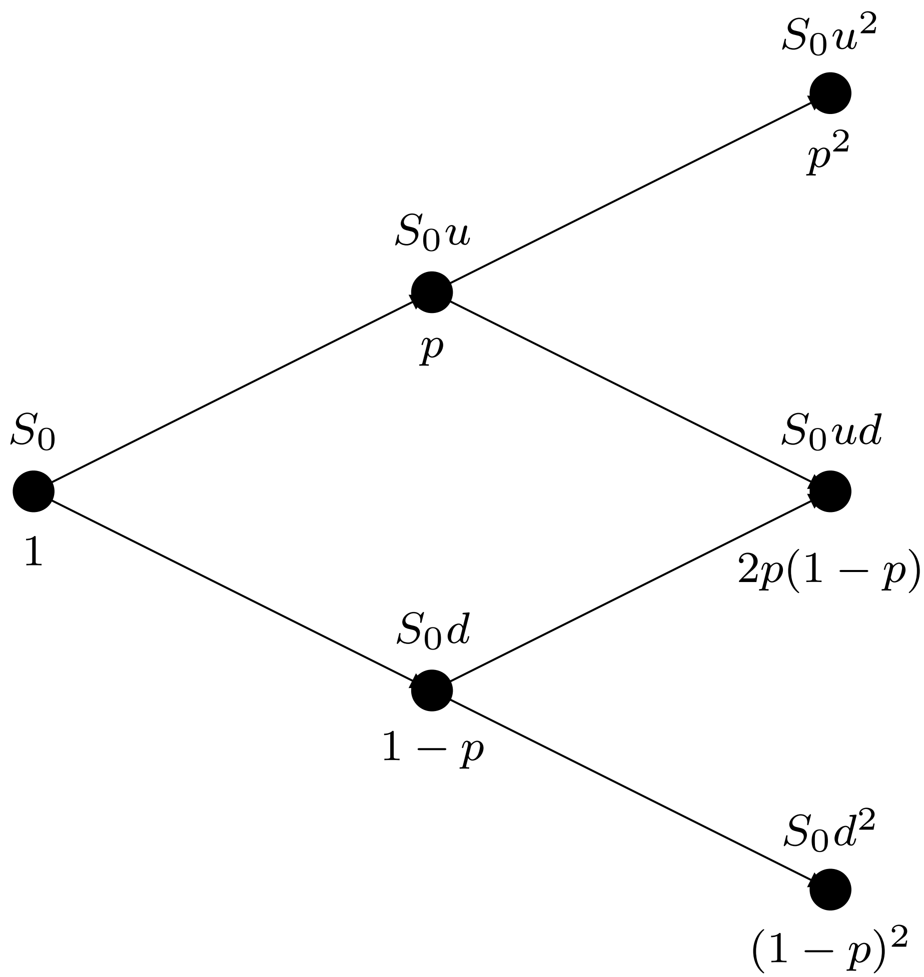
[a] Consistent with Theorem 4 (p. 208).

Backward Induction[a]

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (25) on p. 237.
- This recursive procedure is called backward induction.
- C equals

$$
\begin{aligned}
& [\,p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \; & [\,p^2 \max\big(0, Su^2 - X\big) + 2p(1-p) \max\left(0, Sud - X\right) \\
& +(1-p)^2 \max\big(0, Sd^2 - X\big)\,]/R^2.
\end{aligned}
$$

[a]Ernst Zermelo (1871–1953).


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (concluded)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, S u^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - S u^j d^{n-j}\right)}{R^n}.$$

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  - E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

---

[a]Except at the beginning, of course, when you have to put up the option value C or P before the replication starts.

Hakansson's Paradox[a]

- If options can be replicated, why are they needed at all?

[a] Hakansson (1979).

Can You Figure Out u, d without Knowing q?[a]

- Yes, you can, under BOPM.
- Let us observe the time series of past stock prices, e.g.,

$$\overbrace{S, Su,}^{u \text{ is available}} \quad Su^2, \underbrace{Su^3, Su^3 d}_{d \text{ is available}}, \ldots$$

- So with sufficiently long history, you will figure out u and d without knowing q.

[a]Contributed by Mr. Hsu, Jia-Shuo (D97945003) on March 11, 2009.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

  or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

- Hence,

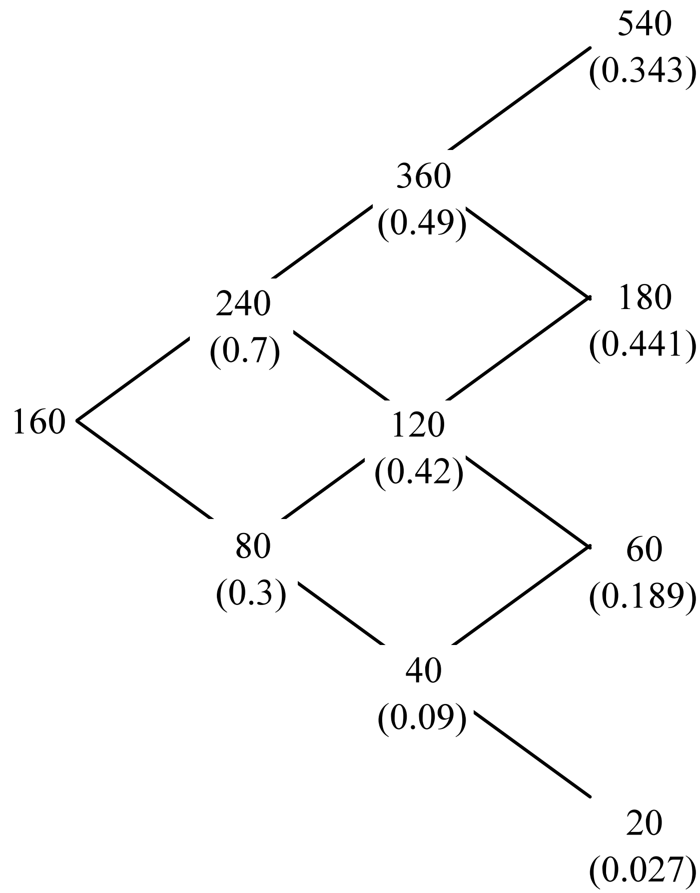
$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} \qquad (26) \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p).
\end{aligned}
$$

Numerical Examples

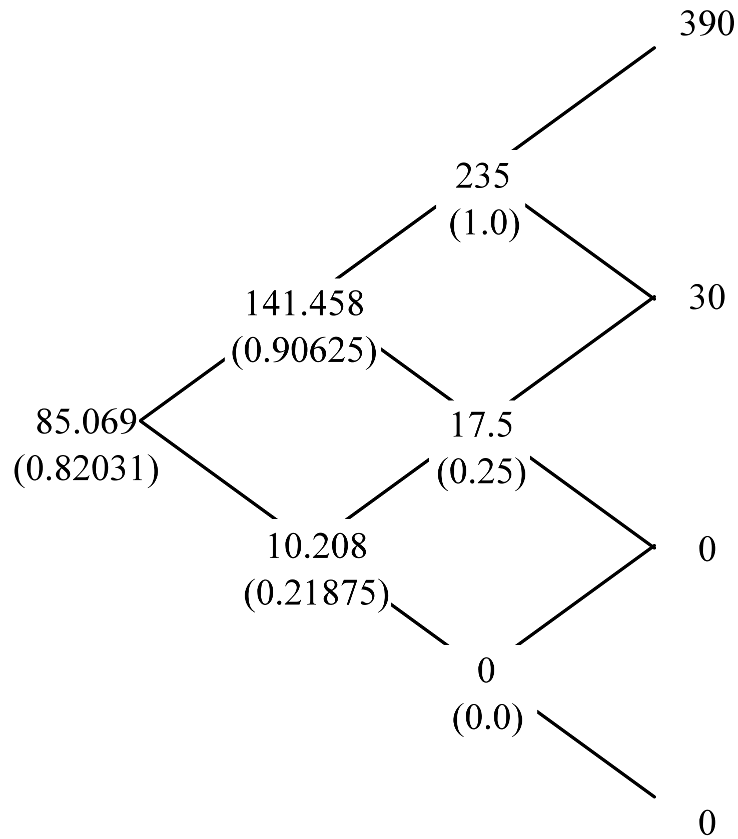
- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

Binomial process for the stock price
(probabilities in parentheses)



Binomial process for the call price
(hedge ratios in parentheses)



Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

  is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy

$$0.90625 - 0.82031 = 0.08594$$

  more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.

- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value!

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

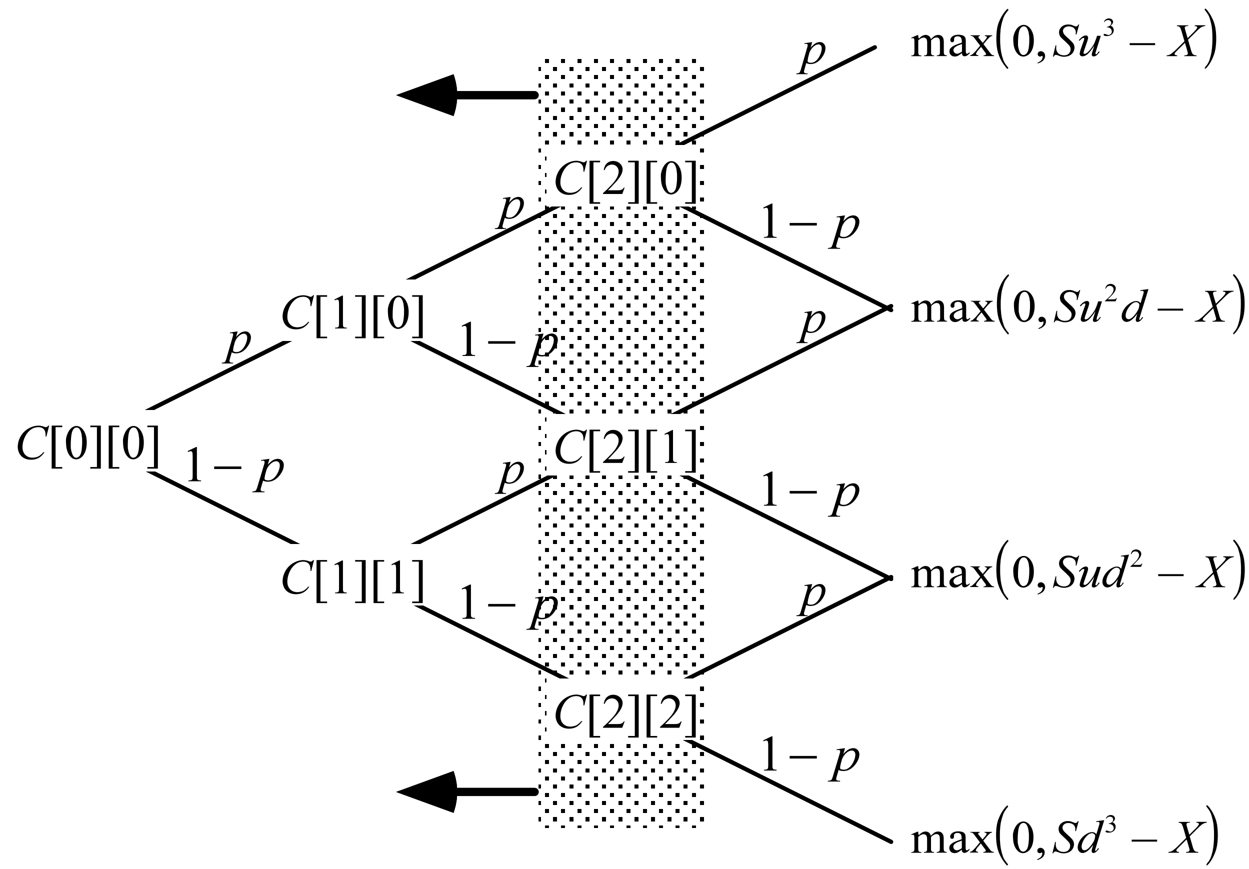
- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued (the mirror image of replication).
  - Set up a portfolio to replicate the call with \$85.069 to counterbalance its values exactly.
- · · ·

[a]Thanks to a lively class discussion on March 16, 2011.

Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To price European puts, simply replace the payoff.

[a] But watch out for the proper updating of array entries.



C[0][0]
C[1][0]
C[1][1]
C[2][0]
C[2][1]
C[2][2]
p
1 − p
max(0, Su³ − X)
max(0, Su²d − X)
max(0, Sud² − X)
max(0, Sd³ − X)

Further Time Improvement for Calls