Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,[a]

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

---

[a] Compare it with Eq. (16) on p. 129.

Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j-i}.$$

- The one-period forward rate:

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- Now,

$$\begin{aligned} f(T) &\equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) \\ &= S(T) + T\,\frac{\partial S}{\partial T}. \end{aligned}$$

- So $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$ (i.e., a normal spot rate curve).

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\,S(a,b)\,]. \tag{17}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

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Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (15) on p. 123.
  - So $E[\,S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

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A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1+S(2))^2 = (1+S(1))\, E[\,1+S(1,2)\,] \tag{18}$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[\,1+S(1,2)\,]} = \frac{1+S(1)}{(1+S(2))^2}.$$

A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for $(1+S(2))^{-2}$ dollars and sells it after one period.
  - The expected return is $E[\,(1+S(1,2))^{-1}\,]/(1+S(2))^{-2}$.
  - Strategy two buys a one-period bond with a return of $1+S(1)$.
- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2}=E\left[\frac{1}{1+S(1,2)}\right].$$

A "Bad" Expectations Theory (concluded)

- Combine this with Eq. (18) on p. 139 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right]=\frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use $g(x) \equiv (1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \quad \text{for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration Revisited

- To handle more general types of spot rate curve changes, define a vector $[\, c_1, c_2, \ldots, c_n \,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\, 1, 1, \ldots, 1 \,]$.
  - Twist: $[\, 1, 1, \ldots, 1, -1, \ldots, -1 \,]$, $[\, 0.8\%, 0.6\%, 0.4\%, 0.2\%, 0\%, -0.2\%, -0.4\% \ldots \,]$, etc.
  - $\ldots$

Duration Revisited (concluded)

- Let

$$P(y) \equiv \sum_i C_i/(1+S(i)+yc_i)^i$$

be the price associated with the cash flow $C_1, C_2, \ldots$.

- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0}.$$

- Modified duration equals the above when

$$[\,c_1, c_2, \ldots, c_n\,] = [\,1, 1, \ldots, 1\,],$$
$$S(1) = S(2) = \cdots = S(n).$$

Some Loose Ends on Dates

- Holidays.
- Weekends.
- Business days ($T + 2$, etc.).
- Shall we treat a year as 1 year whether it has 365 or 366 days?

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Benjamin Disraeli (1804–1881)

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as

$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\mathrm{Cov}[\,X, Y\,] \equiv E\left[\,(X - \mu_X)(Y - \mu_Y)\,\right],$$

  where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

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Correlation

- The standard deviation of X is the square root of the variance,

$$\sigma_X \equiv \sqrt{\text{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between X and Y is

$$\rho_{X,Y} \equiv \frac{\text{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

---

[a]Paul Wilmott (2009), "the correlations between financial quantities are notoriously unstable."

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\,X_i, X_j\,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\,X_i\,]$$

when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation $E[\, X \mid I \,]$ is the expected value of X conditional on I; it is a random variable.
- The law of iterated conditional expectations:

$$E[\, X \,] = E[\, E[\, X \mid I \,] \,].$$

- If I_2 contains at least as much information as I_1, then

$$E[\, X \mid I_1 \,] = E[\, E[\, X \mid I_2 \,] \mid I_1 \,]. \tag{19}$$

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The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is
$$\theta_X(t) \equiv E[\, e^{tX} \,].$$
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{20}$$

The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^n t_i X_i \sim N\left(\sum_{i=1}^n t_i\,\mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\,X_i, X_j\,]\right)$$

for every linear combination $\sum_{i=1}^n t_i X_i$.[a]

- X_i are said to have a multivariate normal distribution.

[a] Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0,1]$ so that

$$\text{Prob}[\,X \leq x\,] = x, \quad 0 < x \leq 1.$$

- Repeatedly draw two samples x_1 and x_2 from X until

$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where[a]

$$c \equiv \sqrt{-2(\ln \omega)/\omega}\,.$$

---

[a]As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\left(\sum_{i=1}^{12} \xi_i \right) - 6.$$

- But why use 12?
- Recall the mean and variance of ξ_i are $1/2$ and $1/12$, respectively.

[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

A Dirty Trick and a Right Attitude (concluded)

- So the general formula is

$$\frac{(\sum_{i=1}^{n} \xi_i) - (n/2)}{\sqrt{n/12}}.$$

- Choosing $n = 12$ yields a formula without the need of division and square-root operations.[a]
- Always blame your random number generator last.[b]
- Instead, check your programs first.

---

[a] Contributed by Mr. Chen, Shih-Hang (`R02723031`) on March 5, 2014.

[b] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated.
- Let X_1 and X_2 be independent standard normal variables.
- Set

$$\begin{aligned} U &\equiv aX_1, \\ V &\equiv \rho U + \sqrt{1-\rho^2}\, aX_2. \end{aligned}$$

Generation of Bivariate Normal Distributions (concluded)

- U and V are the desired random variables with

$$\begin{aligned} \text{Var}[\,U\,] &= \text{Var}[\,V\,] = a^2, \\ \text{Cov}[\,U,V\,] &= \rho a^2. \end{aligned}$$

- Note that the mapping between (X_1, X_2) and (U, V) is one-to-one.

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The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu+\sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \tag{21}$$

respectively.

  - They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.

Option Basics

The shift toward options as
the center of gravity of finance [. . .]
— Merton H. Miller (1923–2000)

Calls and Puts

- A call gives its holder the right to buy a number of the underlying asset by paying a strike price.
- A put gives its holder the right to sell a number of the underlying asset for the strike price.
- How to price options?[a]

[a]It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

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Payoff, Mathematically Speaking

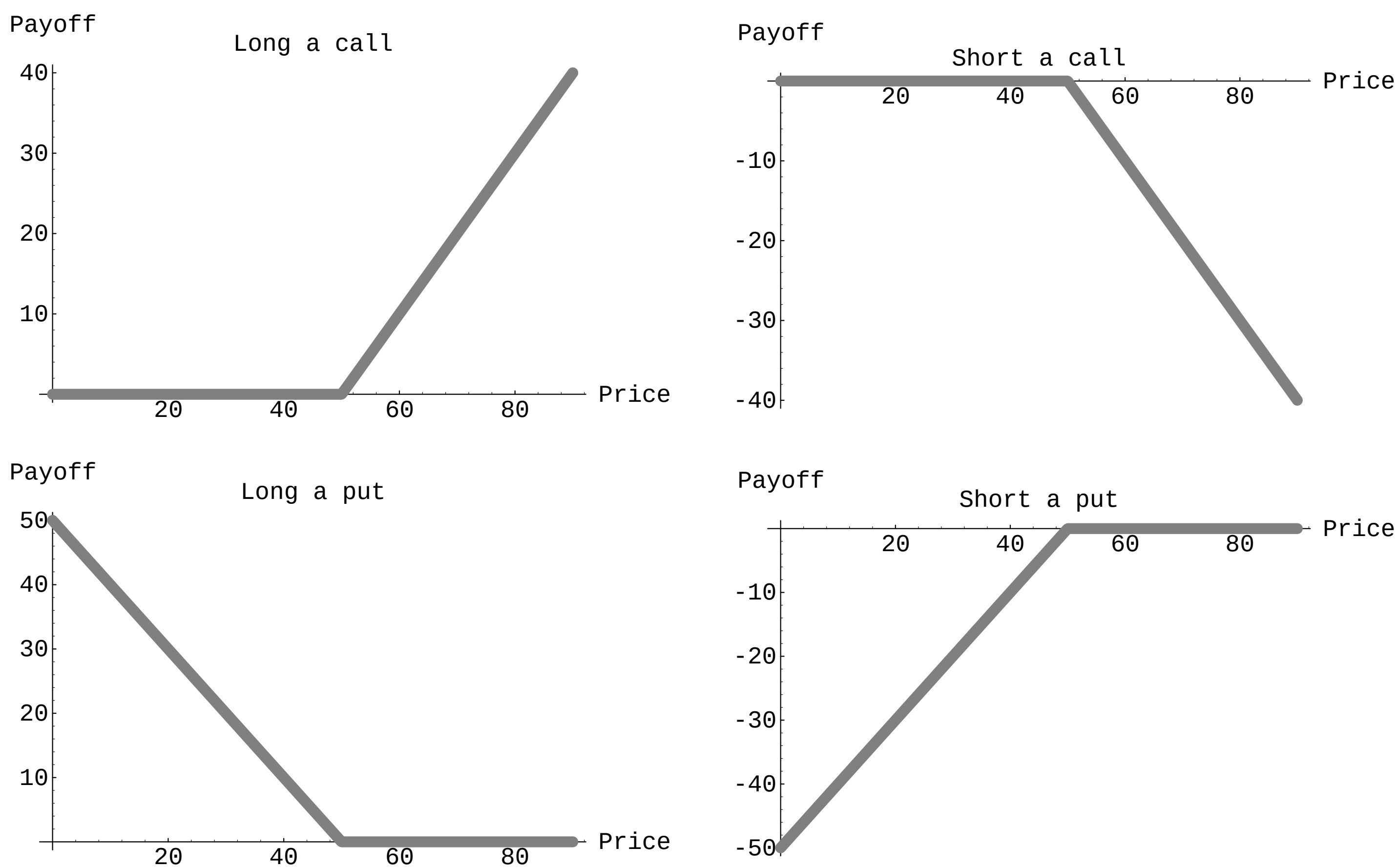
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.


Payoff
Long a call
40
30
20
10
20
40
60
80
Price
Payoff
Short a call
20
40
60
80
Price
-10
-20
-30
-40
Payoff
Long a put
50
40
30
20
10
20
40
60
80
Price
Payoff
Short a put
20
40
60
80
Price
-10
-20
-30
-40
-50

Payoff, Mathematically Speaking (continued)

- At any time t before the expiration date, we call

$$\max(0, S_t - X)$$

the intrinsic value of a call.

- At any time t before the expiration date, we call

$$\max(0, X - S_t)$$

the intrinsic value of a put.

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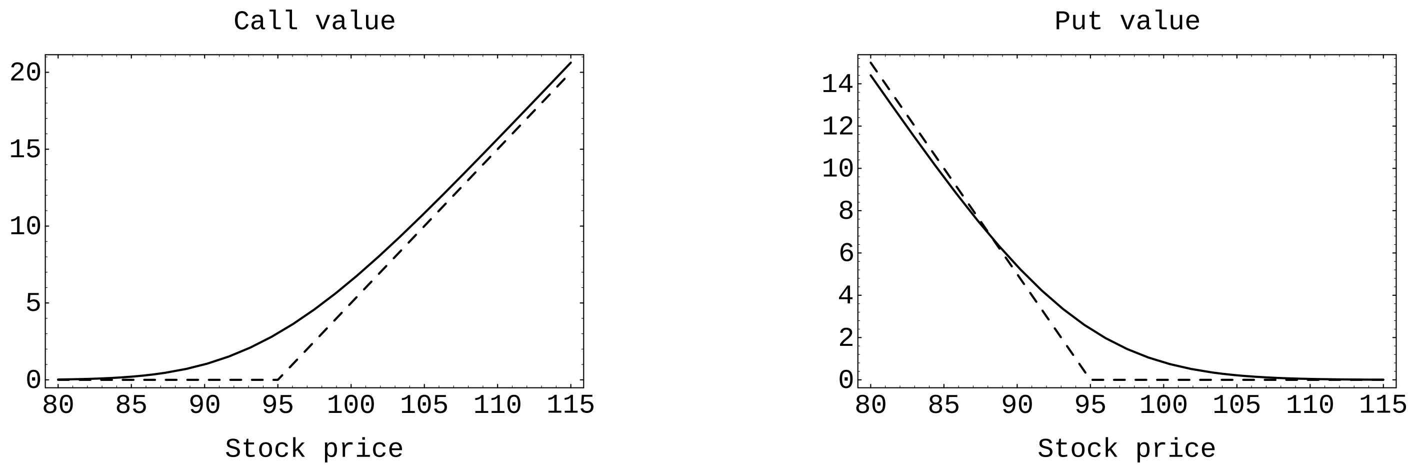
Payoff, Mathematically Speaking (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time before expiration is a major intellectual breakthrough.

[a] 11% of option holders let in-the-money options expire worthless.


Call value
20
15
10
5
0
80 85 90 95 100 105 110 115
Stock price
Put value
14
12
10
8
6
4
2
0
80 85 90 95 100 105 110 115
Stock price

Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n-for-m stock split, the strike price is only m/n times its previous value, and the number of shares covered by one contract becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- Options are assumed to be unprotected.

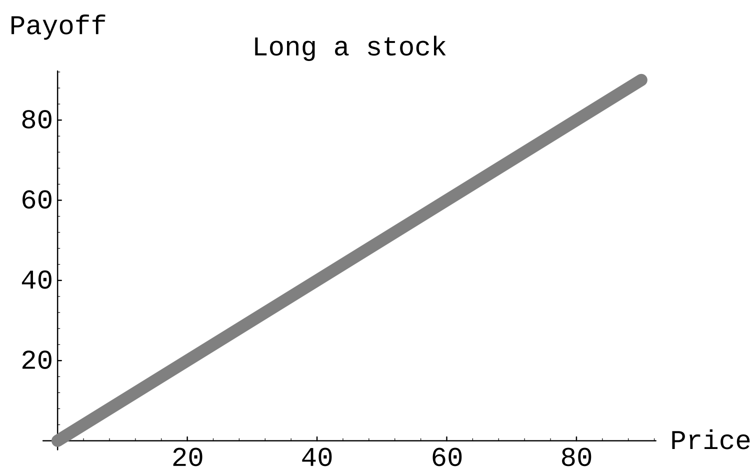
Example

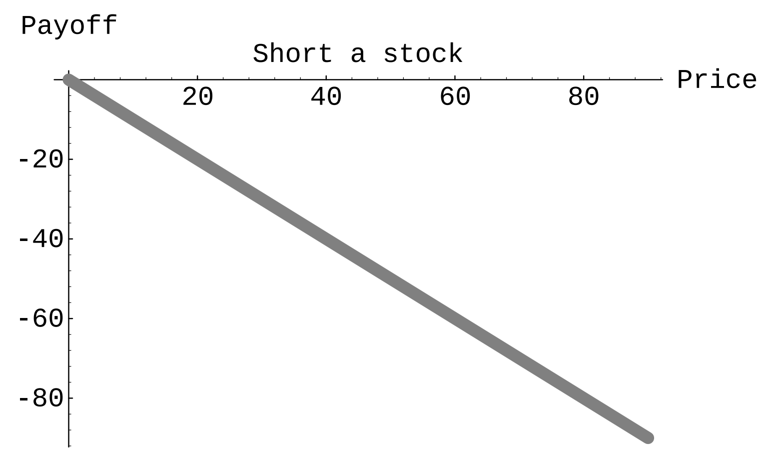
- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.
- Not all assets can be shorted.

Payoff of Stock



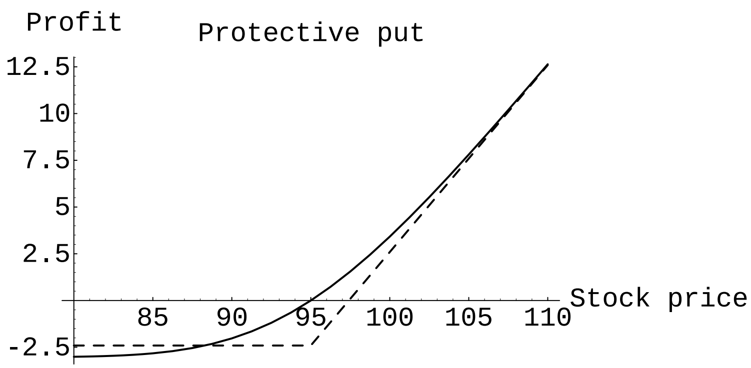


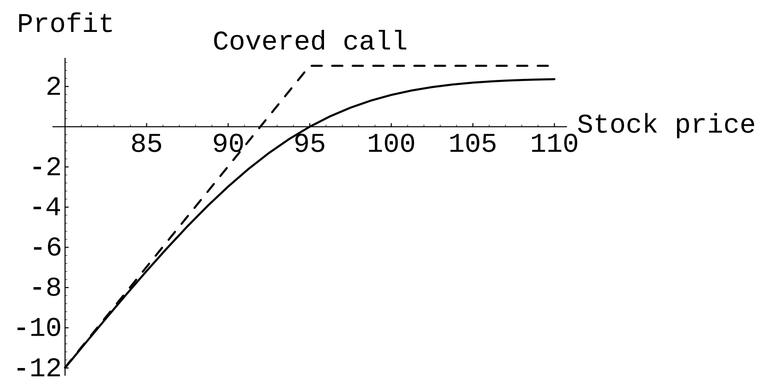
Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Protective put: A long position in stock with a long put.
- Covered call: A long position in stock with a short call.[a]
  - It is "covered" because the stock can be delivered to the buyer of the call if the call is exercised.
- Both strategies break even only if the stock price rises, so they are bullish.

---

[a] A short position has a payoff opposite in sign to that of a long position.





Solid lines are profits of the portfolio one month before maturity assuming the portfolio is set up when $S = 95$ then.

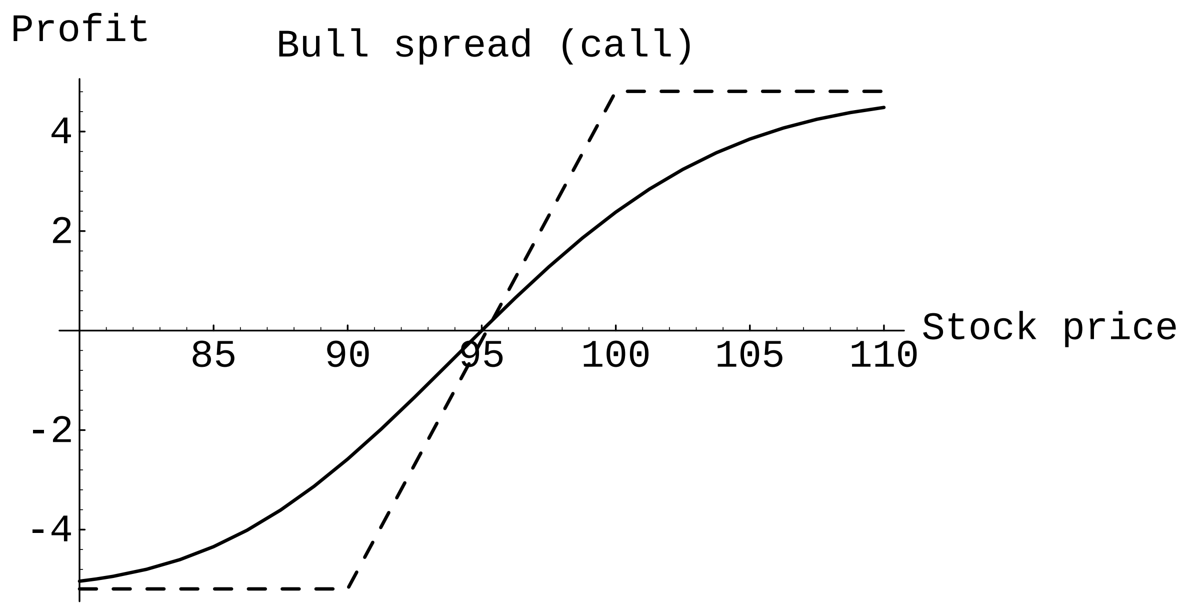
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with

$$X_L < X_M < X_H.$$

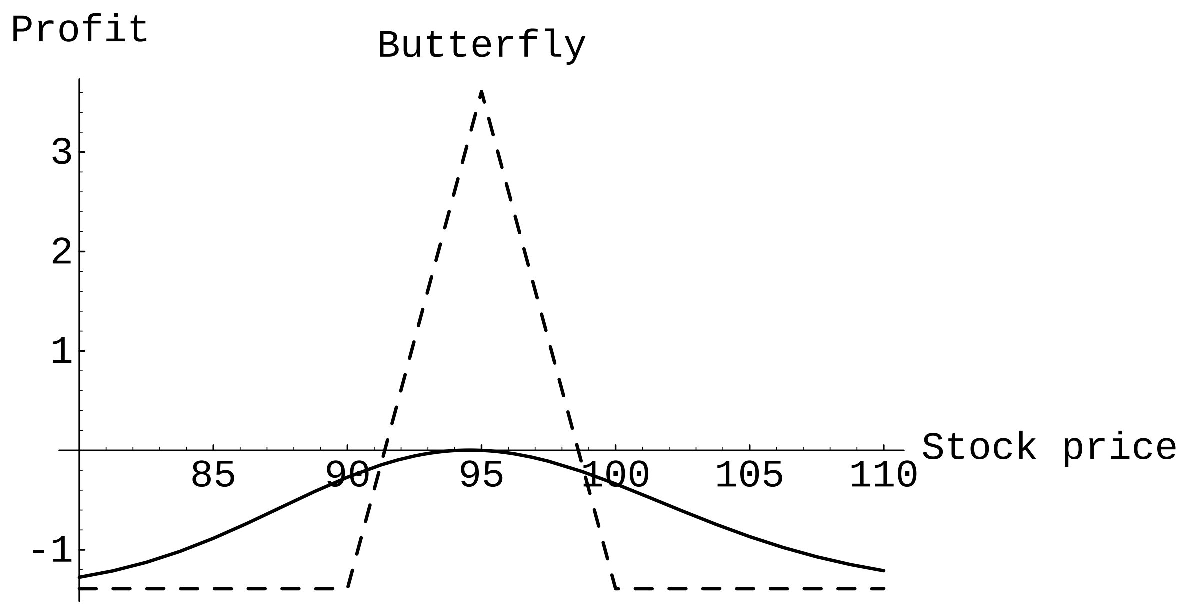
Covered Position: Spread (continued)

- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
  - The maximum loss is $C_L - C_H$.


Profit
Bull spread (call)
4
2
-2
-4
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short two X_M calls.



Profit
Butterfly
3
2
1
-1
85
90
95
100
105
110
Stock price

Covered Position: Spread (continued)

- A butterfly spread pays off a positive amount at expiration only if the asset price falls between X_L and X_H.
- A butterfly spread with a small $X_H - X_L$ approximates a state contingent claim,[a] which pays $1 only when a particular price results.[b]

[a] Alternatively, Arrow security.

[b] See Exercise 7.4.5 of the textbook.

Covered Position: Spread (concluded)

- The price of a state contingent claim is called a state price.
- The (undiscounted) state price equals

$$\frac{\partial^2 C}{\partial X^2}.$$

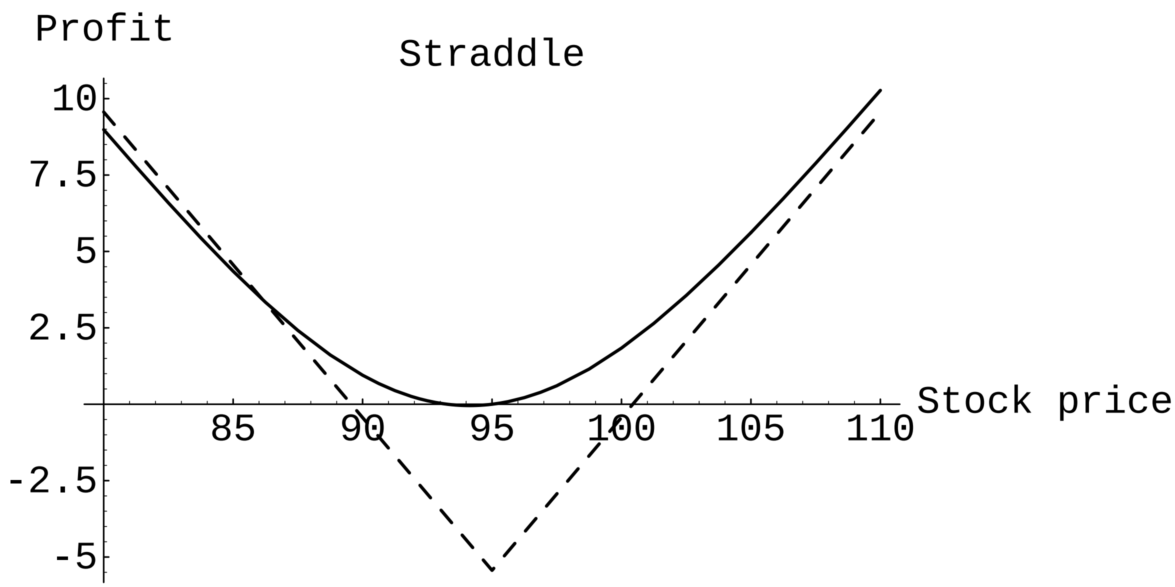
  - Recall that C is the call's price.[a]
- In fact, the PV of $\partial^2 C/\partial X^2$ is the probability density of the stock price $S_T = X$ at option's maturity.[b]

---

[a] One can also use the put (see Exercise 9.3.6 of the textbook).
[b] Breeden and Litzenberger (1978).

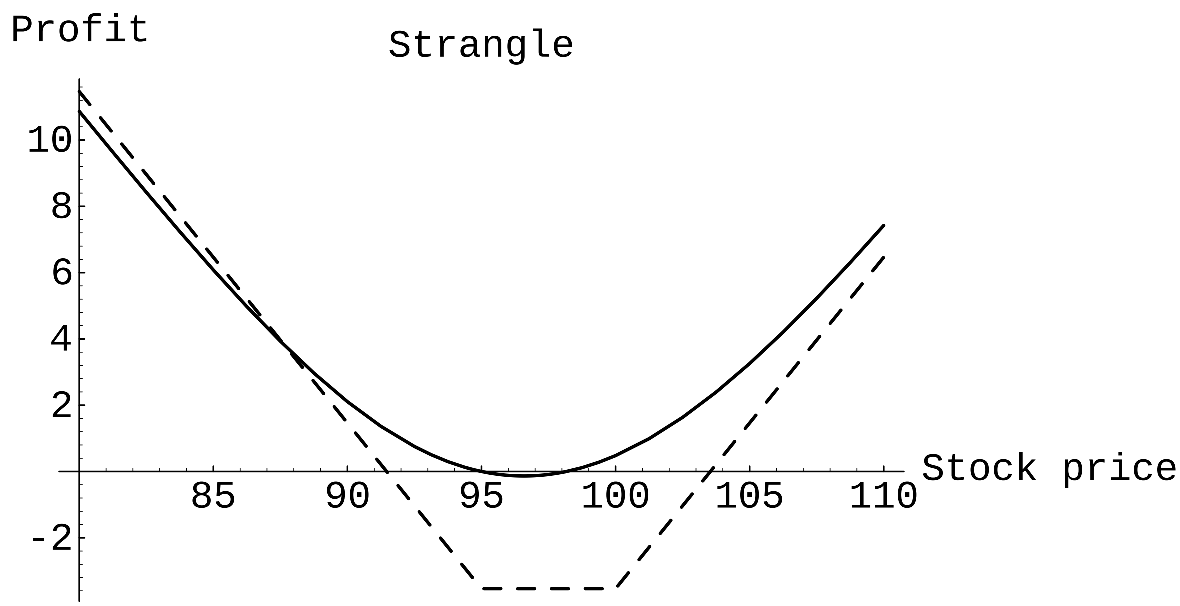
Covered Position: Combination

- A combination consists of options of different types on the same underlying asset.
  - These options must be either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
  - Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
  - Selling a straddle benefits from low volatility.


Profit
Straddle
10
7.5
5
2.5
-2.5
-5
Stock price
85
90
95
100
105
110

Covered Position: Combination (concluded)

- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.


Profit
Strangle
10
8
6
4
2
-2
85
90
95
100
105
110
Stock price

Arbitrage in Option Pricing

All general laws are attended with inconveniences,
when applied to particular cases.
— David Hume (1711–1776)

Arbitrage

- The no-arbitrage principle says there is no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).

Portfolio Dominance Principle

- Consider two portfolios A and B.
- A should be more valuable than B if A's payoff is at least as good as B's under all circumstances and better under some.

A Corollary

- A portfolio yielding a zero return in every possible scenario must have a zero PV.
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.