The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:

$$
\begin{tikzcd}
 & Pd \\
P \arrow[ur, "1-q"] \arrow[dr, "q"'] & \\
 & Pu
\end{tikzcd}
$$

The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{112}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{113}$$

The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate r.
- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- As in the continuous-time case, it can be shown that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \qquad (114)$$

  which is independent of bond maturity and q.

  – Recall the BOPM.

- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

$$\begin{tikzcd} & 8\% \\ 4\% \arrow[ur] \arrow[dr] & \\ & 2\% \end{tikzcd}$$

Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 925.

Numerical Examples (continued)

$$
\begin{tikzcd}[column sep=small]
 & 92.593\ (= 100/1.08) & & & 100 \\
90.703 \arrow[ur] \arrow[dr] & & & 96.154 \arrow[ur] \arrow[dr] & \\
 & 98.039\ (= 100/1.02) & & & 100
\end{tikzcd}
$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$\begin{tikzcd} & 0.000 \\ C \arrow[ur] \arrow[dr] & \\ & 3.039 \end{tikzcd}$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:

$$
F \begin{array}{l} \nearrow \; 92 \; (= 100 - 8) \\ \\ \searrow \; 98 \; (= 100 - 2) \end{array}
$$

- As the futures price F is the expected future payoff (see text or p. 464),

$$
F = (1 - p) \times 92 + p \times 98 = 93.914.
$$

Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is[a]

$$90.703/96.154 = 94.331\%.$$

- The forward price exceeds the futures price.[b]

---

[a]See Eq. (100) on p. 898.
[b]Recall p. 410.

Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 935.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

- Identical results could have been obtained via arbitrage considerations.

$$
\begin{array}{ccccc}
 & & & & 105 \\
 & & & \nearrow & \\
 & & 5 & & \\
 & \nearrow & & \searrow & \\
 & & & & 105 \\
0 & & & & \\
 & & & & 105 \\
 & \searrow & & \nearrow & \\
 & & 5 & & \\
 & & & \searrow & \\
 & & & & 105
\end{array}
\qquad\qquad
\begin{array}{ccl}
 & & 102.222\ (= 5 + (105/1.08)) \\
 & \nearrow & \\
M & & \\
 & \searrow & \\
 & & 107.941\ (= 5 + (105/1.02))
\end{array}
$$

The left diagram depicts the cash flow; the right diagram illustrates the price process.

Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

$$\begin{array}{ccc} & & 102.222 \\ & \nearrow & \\ M & & \\ & \searrow & \\ & & 105 \end{array}$$

- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 938(a)).
- Their prices hence follow the processes on p. 938(b).
- The fair prices are

$$
\begin{aligned}
\text{PO} &= \frac{(1-p)\times 92.593 + p\times 100}{1.04} = 91.304,\\
\text{IO} &= \frac{(1-p)\times 9.630 + p\times 5}{1.04} = 7.839.
\end{aligned}
$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & \text{PO:}\ \ 0 \arrow[r] \arrow[dr] & 100 \\
0 \arrow[ur] \arrow[dr] & & 100 \\
 & 100 \arrow[r] \arrow[dr] & 0 \\
 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & \text{IO:}\ \ 5 \arrow[r] \arrow[dr] & 5 \\
0 \arrow[ur] \arrow[dr] & & 5 \\
 & 5 \arrow[r] \arrow[dr] & 0 \\
 & & 0
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & 92.593 \\
\text{PO} \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & 9.630 \\
\text{IO} \arrow[ur] \arrow[dr] & \\
 & 5
\end{tikzcd}
$$

(b)

The price 9.630 is derived from $5 + (5/1.08)$.

Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of

$$(10\% - \textsf{one-year rate})$$

  to make the overall coupon rate 5%.
- Their cash flows as *percentages of par* and values are shown on p. 940.

$$
\begin{tikzcd}[row sep=small]
 & \text{FLT:} & 108 \\
 & 4 \arrow[ur] \arrow[dr] & \\
 & & 108 \\
0 \arrow[uuur] \arrow[ddr] & & \\
 & & 0 \\
 & 104 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & \text{INV:} & 102 \\
 & 6 \arrow[ur] \arrow[dr] & \\
 & & 102 \\
0 \arrow[uuur] \arrow[ddr] & & \\
 & & 0 \\
 & 106 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}[row sep=small]
 & 104 \\
\text{FLT} \arrow[ur] \arrow[dr] & \\
 & 104
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & 100.444 \\
\text{INV} \arrow[ur] \arrow[dr] & \\
 & 106
\end{tikzcd}
$$

(b)

Numerical Examples: MBSs (concluded)

- On p. 940, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$
\begin{aligned}
\text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\
\text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142.
\end{aligned}
$$

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

  the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r \,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (58) on p. 523.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{115}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

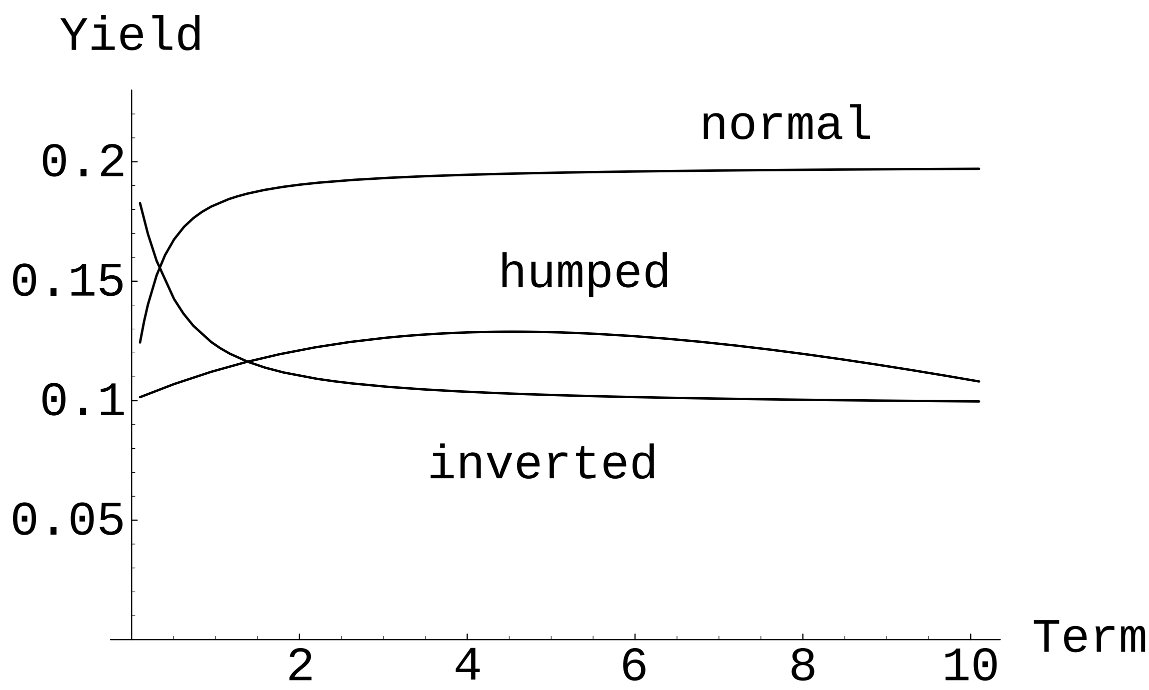
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve

$$(\partial r(t,T)/\partial r)\,\sigma = \sigma B(t,T)/(T-t).$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Indeed, higher β leads to greater attenuation of volatility with maturity.


Yield
normal
humped
inverted
0.2
0.15
0.1
0.05
2
4
6
8
10
Term

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t,s)\,N(x) - X P(t,T)\,N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

[a] Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine, as we will see next.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \tag{116}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2 \sigma^2 / 4$ (p. 958).

$$\begin{array}{ccccc} & & & & x+2\sqrt{\Delta t}\\ & & & \nearrow & \\ & & x+\sqrt{\Delta t} & & \\ & \nearrow & & \searrow & \\ x & & & & x \\ & \searrow & & \nearrow & \\ & & x-\sqrt{\Delta t} & & \\ & & & \searrow & \\ & & & & x-2\sqrt{\Delta t}\end{array} \qquad \begin{array}{ccccc} & & & & f(x+2\sqrt{\Delta t})\\ & & & \nearrow & \\ & & f(x+\sqrt{\Delta t}) & & \\ & \nearrow & & \searrow & \\ f(x) & & & & f(x) \\ & \searrow & & \nearrow & \\ & & f(x-\sqrt{\Delta t}) & & \\ & & & \searrow & \\ & & & & f(x-2\sqrt{\Delta t})\end{array}$$

Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{117}$$

  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

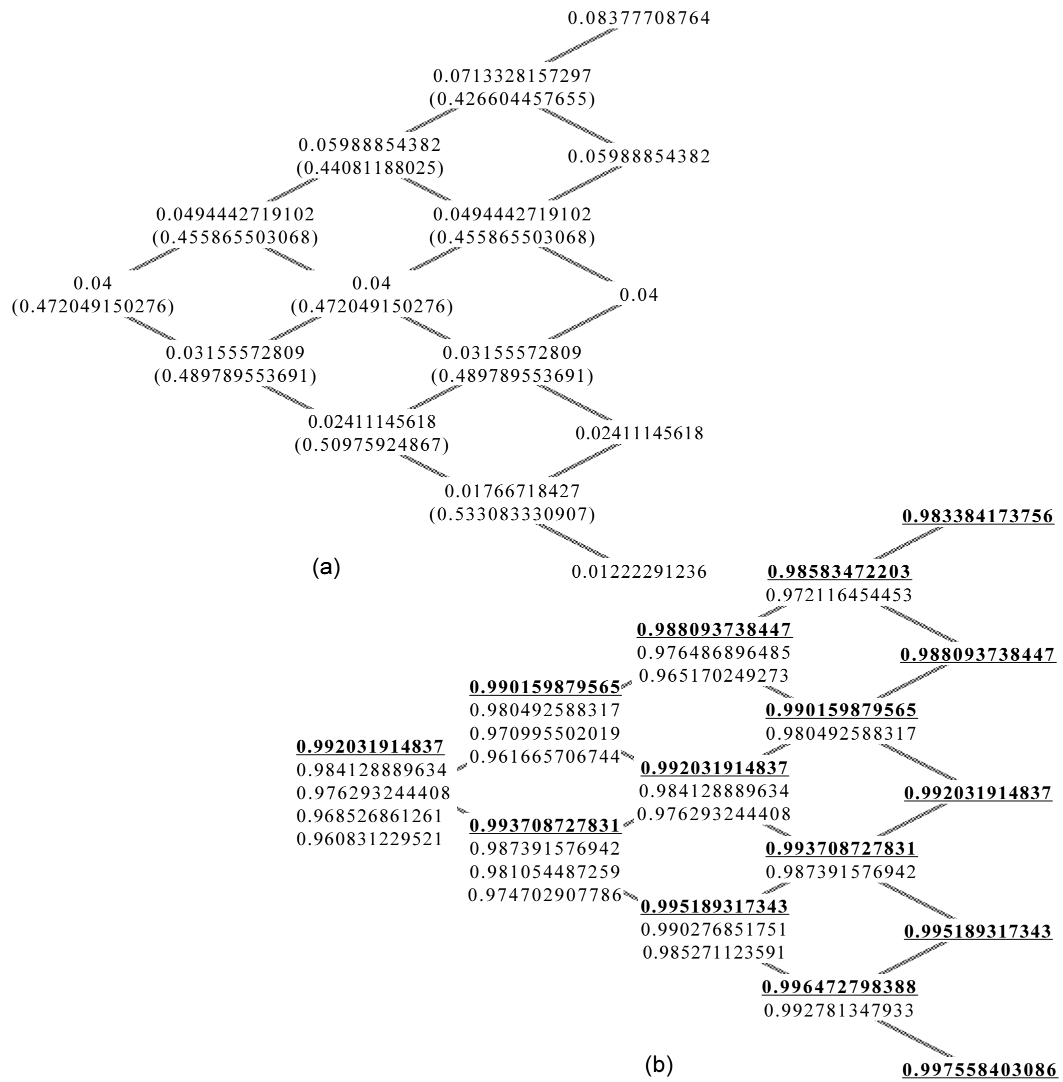
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.
- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 961(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.


0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
(0.44081188025)
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0.0494442719102
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(0.533083330907)
0.01222291236
(a)
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
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0.968526861261
0.960831229521
0.992031914837
0.984128889634
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0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW.$$

- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a]Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t+\Delta t)\sqrt{\Delta t} \\ \neq \;& -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t+\Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.

- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \equiv \int^y \sigma(z,t)^{-1}\, dz.$$

- Then x follows

$$dx = m(y,t)\, dt + dW$$

for some $m(y,t)$ (see text).

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The transformation that turns a 1-dim stochastic process into one with a constant diffusion term is unique.[a]

[a] Chiu (R98723059) (2012).

A General Method (concluded)

- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.

- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

Examples

- The transformation is
$$\int^r (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$
for the CIR model.
- The transformation is
$$\int^S (\sigma z)^{-1}\, dz = (1/\sigma) \ln S$$
for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.

[a]Jamshidian (1989).

Options on Coupon Bonds (continued)

- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for r.

Options on Coupon Bonds (continued)

- The solution is unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let
$$X_i \equiv P(r^*, T, t_i),$$
the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.
- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.

Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\begin{aligned} & \max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0\right) \\ = & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0). \end{aligned}$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

---

[a]Ho and Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a]Ho and Lee (1986).

$$
\begin{tikzcd}[row sep=small]
 & & & {} \\
 & & r_3 \arrow[ur] \arrow[dr] & \\
 & r_2 \arrow[ur] \arrow[dr] & & {} \\
r_1 \arrow[ur] \arrow[dr] & & r_3 + v_3 \arrow[ur] \arrow[dr] & \\
 & r_2 + v_2 \arrow[ur] \arrow[dr] & & {} \\
 & & r_3 + 2v_3 \arrow[ur] \arrow[dr] & \\
 & & & {}
\end{tikzcd}
$$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots & \text{if short rate moves down} \\ P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots & \text{if short rate moves up} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{118}$$

(see text).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$
\begin{aligned}
y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\
y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2+\cdots+v_n}{n-1}
\end{aligned}
$$

- The volatility of the yield to maturity for this bond is therefore

$$
\begin{aligned}
\kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\
&= \sqrt{p(1-p)}\, (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\
&= \sqrt{p(1-p)}\, \frac{v_2+\cdots+v_n}{n-1}.
\end{aligned}
$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{119}$$

- The variance of the short rate therefore equals $p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2$, where r_{u} and r_{d} are the two successor rates.[a]

[a] Contrast this with the lognormal model.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the r_i.
- It is easy to compute the v_is from the volatility structure, and vice versa.
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t{+}n) = (p P_{\mathrm{u}}(t{+}1, t{+}n) {+} (1{-}p)\, P_{\mathrm{d}}(t{+}1, t{+}n))\, P(t, t{+}1)$$

- Combine the above with Eq. (118) on p. 983 and assume $p = 1/2$ to obtain[a]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)}\, \frac{2 \times \exp[\, v_2 + \cdots + v_n\,]}{1 + \exp[\, v_2 + \cdots + v_n\,]}, \tag{120}$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)}\, \frac{2}{1 + \exp[\, v_2 + \cdots + v_n\,]}. \tag{120'}$$

---

[a]In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$\begin{aligned} P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\ P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}. \end{aligned}$$

- Short rate volatility σ equals $v/2$ by Eq. (119) on p. 985.
- Price derivatives by taking expectations under the risk-neutral probability.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is
$$r(t, t+n) \equiv \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$
- Its value is either $\ln \frac{P_{\mathrm{d}}(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_{\mathrm{u}}(t+1,t+n)}{P(t,t+n)}$.
- Thus the variance of return is
$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2\sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is $(n-1)(m-1)\,\sigma^2$ (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is
$$dr = \theta(t)\, dt + \sigma\, dW.$$
- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e., $dr = \theta(t)\, dt + \sigma(t)\, dW$.
- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 834ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

---

[a]Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

$$
\begin{tikzcd}
& & & r_4 \\
& & r_3 \arrow[ur] \arrow[dr] & \\
& r_2 \arrow[ur] \arrow[dr] & & r_4 v_4 \\
r_1 \arrow[ur] \arrow[dr] & & r_3 v_3 \arrow[ur] \arrow[dr] & \\
& r_2 v_2 \arrow[ur] \arrow[dr] & & r_4 v_4^2 \\
& & r_3 v_3^2 \arrow[ur] \arrow[dr] & \\
& & & r_4 v_4^3
\end{tikzcd}
$$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
  - A related model of Salomon Brothers takes v_i to be a given constant.[a]
- Lognormal models preclude negative short rates.

---

[a] Tuckman (2002).

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.
- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$
\begin{aligned}
y_{\mathrm{u}} &\equiv P_{\mathrm{u}}^{-1/(i-1)} - 1, \\
y_{\mathrm{d}} &\equiv P_{\mathrm{d}}^{-1/(i-1)} - 1.
\end{aligned}
$$

- The yield volatility is defined as

$$
\kappa_i \equiv \frac{\ln(y_{\mathrm{u}}/y_{\mathrm{d}})}{2}.
$$

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to P_{u} or P_{d} one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}}+P_{\mathrm{d}}}{2(1+r_1)}=\frac{1}{(1+y)^i}. \tag{121}$$

- Obviously, P_{u} and P_{d} are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively (p. 999).
- With κ^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{122}$$

The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving *forward* in time, how much \$1 at a node contributes to the price (review p. 860(a)).
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

[a] Chen (R84526007) and Lyuu (1997); Lyuu (1999).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$ for period i, respectively.
- One dollar at time i has a present value of

$$f(r, v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- The yield volatility is

$$g(r,v) \equiv \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right).$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 996).
- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 996).

The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices.
- Now solve

$$\begin{aligned} f(r,v) &= \frac{1}{(1+y)^i}, \\ g(r,v) &= \kappa_i, \end{aligned}$$

  for $r = r_i$ and $v = v_i$.
- This $O(n^2)$-time algorithm appears in the text.

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t)\, dW.$$

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant volatility will not attain mean reversion.