The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.

- Suppose the bond price $P$ can move with probability $q$ to $Pu$ and probability $1-q$ to $Pd$, where $u > d$:
The Binomial Model (continued)

- Over the period, the bond’s expected rate of return is
  \[ \hat{\mu} \equiv \frac{qP_u + (1 - q)P_d}{P} - 1 = qu + (1 - q)d - 1. \]  
  \[ \text{(112)} \]

- The variance of that return rate is
  \[ \hat{\sigma}^2 \equiv q(1 - q)(u - d)^2. \]  
  \[ \text{(113)} \]
The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value $1$.

- This is the money market account modeled by the short rate $r$.

- The market price of risk is defined as $\lambda \equiv (\hat{\mu} - r)/\hat{\sigma}$.

- As in the continuous-time case, it can be shown that $\lambda$ is independent of the maturity of the bond (see text).
The Binomial Model (concluded)

• Now change the probability from $q$ to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r) - d}{u-d}, \quad (114)$$

which is independent of bond maturity and $q$.

− Recall the BOPM.

• The bond’s expected rate of return becomes

$$\frac{pPu + (1-p)Pd}{P} - 1 = pu + (1-p)d - 1 = r.$$ 

• The local expectations theory hence holds under the new probability measure $p$. 
Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

4% ← 8% → 2%
Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

\[
\frac{100}{1.04} = 96.154, \\
\frac{100}{(1.05)^2} = 90.703.
\]
- They follow the binomial processes on p. 925.
Numerical Examples (continued)

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

• The pricing of derivatives can be simplified by assuming investors are risk-neutral.

• Suppose all securities have the same expected one-period rate of return, the riskless rate.

• Then

\[
(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,
\]

where \( p \) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,

\[ C \begin{cases} 0.000 \\ 3.039 \end{cases} \]

• To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.
Numerical Examples: Fixed-Income Options
(continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580 \).

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

\[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

$$F = (1 - p) \times 92 + p \times 98 = 93.914.$$

• As the futures price $F$ is the expected future payoff (see text or p. 464),
Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is\textsuperscript{a}

\[\frac{90.703}{96.154} = 94.331\% .\]

- The forward price exceeds the futures price.\textsuperscript{b}

\textsuperscript{a}See Eq. (100) on p. 898.
\textsuperscript{b}Recall p. 410.
Numerical Examples: Mortgage-Backed Securities

• Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.

• Its cash flow and price process are illustrated on p. 935.

• Its fair price is

\[ M = \frac{(1 - p) \times 102.222 + p \times 107.941}{1.04} = 100.045. \]

• Identical results could have been obtained via arbitrage considerations.
The left diagram depicts the cash flow; the right diagram illustrates the price process.
Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the “down” state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

\[
M = \begin{cases} 
102.222 & \text{if } M < 105 \\
105 & \text{if } M = 102.222 \\
\end{cases}
\]

- The security is worth

\[
M = \frac{(1 - p) \times 102.222 + p \times 105}{1.04} = 99.142.
\]
Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage’s principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 938(a)).
- Their prices hence follow the processes on p. 938(b).
- The fair prices are

\[
PO = \frac{(1 - p) \times 92.593 + p \times 100}{1.04} = 91.304,
\]

\[
IO = \frac{(1 - p) \times 9.630 + p \times 5}{1.04} = 7.839.
\]
The price 9.630 is derived from $5 + \left(\frac{5}{1.08}\right)$. 
Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of
  
  \[(10\% - \text{one-year rate})\]

  to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 940.
Numerical Examples: MBSs (concluded)

- On p. 940, the floater’s price in the up node, 104, is derived from $4 + (108/1.08)$.

- The inverse floater’s price 100.444 is derived from $6 + (102/1.08)$.

- The current prices are

  $$\text{FLT} = \frac{1}{2} \times \frac{104}{1.04} = 50,$$

  $$\text{INV} = \frac{1}{2} \times \frac{(1 - p) \times 100.444 + p \times 106}{1.04} = 49.142.$$
Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.

— Top Ten Lies Finance Professors Tell Their Students
Introduction

• This chapter surveys equilibrium models.

• Since the spot rates satisfy

\[
r(t, T) = -\frac{\ln P(t, T)}{T - t},
\]

the discount function \( P(t, T) \) suffices to establish the spot rate curve.

• All models to follow are short rate models.

• Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model

- The short rate follows
  \[ dr = \beta (\mu - r) \, dt + \sigma \, dW. \]
- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).
- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).
- Since the process is an Ornstein-Uhlenbeck process,
  \[ E [ r(T) \mid r(t) = r ] = \mu + (r - \mu) e^{-\beta (T-t)} \]
  from Eq. (58) on p. 523.

\(^{a}\text{Vasicek (1977).}\)
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[
P(t, T) = A(t, T) e^{-B(t, T) r(t)},
\]

(115)

where

\[
A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t, T) - T + t)(\beta^2 \mu - \sigma^2 / 2)}{\beta^2} - \frac{\sigma^2 B(t, T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0.
\end{cases}
\]

and

\[
B(t, T) = \begin{cases} 
\frac{1 - e^{-\beta (T - t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0.
\end{cases}
\]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve

$$
(\partial r(t,T)/\partial r) \sigma = \sigma B(t,T)/(T - t).
$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Indeed, higher $\beta$ leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\(^a\)

- Consider a European call with strike price \(X\) expiring at time \(T\) on a zero-coupon bond with par value $1 and maturing at time \(s > T\).

- Its price is given by

\[
P(t, s) N(x) - XP(t, T) N(x - \sigma_v).
\]

\(^a\)Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[ x \equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]

\[ \sigma_v \equiv v(t, T) B(T, s), \]

\[ v(t, T)^2 \equiv \begin{cases} \frac{\sigma^2[1-e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T - t), & \text{if } \beta = 0 \end{cases}. \]

- By the put-call parity, the price of a European put is

\[ XP(t, T) N(-x + \sigma_v) - P(t, s) N(-x). \]
Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.
- Let $\Delta t \equiv T/n$ and
  \[ p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r) \sqrt{\Delta t}}{2\sigma}. \]
- The following binomial model converges to the Vasicek model,\(^a\)
  \[ r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n. \]

\(^a\)Nelson and Ramaswamy (1990).
Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$
\text{Prob}[\xi(k) = 1] = \begin{cases} 
p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\
0 & \text{if } p(r(k)) < 0 \\
1 & \text{if } 1 < p(r(k)) 
\end{cases}
$$

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

- This is consistent with mean reversion.
The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.

The binomial tree combines.

The key feature of the model that makes it happen is its constant volatility, $\sigma$.

For a general process $Y$ with nonconstant volatility, the resulting binomial tree may not combine, as we will see next.
The Cox-Ingersoll-Ross Model

- It is the following square-root short rate model:

\[ dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \]  

(116)

- The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).
- The parameter \( \beta \) determines the speed of adjustment.
- The short rate can reach zero only if \( 2\beta \mu < \sigma^2 \).
- See text for the bond pricing formula.

\(^a\)Cox, Ingersoll, and Ross (1985).
Binomial CIR

• We want to approximate the short rate process in the time interval \([0, T]\).

• Divide it into \(n\) periods of duration \(\Delta t \equiv T/n\).

• Assume \(\mu, \beta \geq 0\).

• A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

• Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

• It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta \mu/(\sigma^2 x) - (\beta x/2) - 1/(2x). \]

• Since this new process has a constant volatility, its associated binomial tree combines.
Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (p. 958).
\[
\begin{align*}
&x + 2\sqrt{\Delta t} & f(x + 2\sqrt{\Delta t}) \\
&x + \sqrt{\Delta t} & f(x + \sqrt{\Delta t}) \\
&x & f(x) \\
&x - \sqrt{\Delta t} & f(x - \sqrt{\Delta t}) \\
&x - 2\sqrt{\Delta t} & f(x - 2\sqrt{\Delta t})
\end{align*}
\]
Binomial CIR (concluded)

• The probability of an up move at each node $r$ is

$$p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}.$$  \hfill (117)

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

• Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2\) (year) for the binomial approximation.

• See p. 961(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.

- Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$. 
Numerical Examples (concluded)

• Once the short rates are in place, computing the probabilities is easy.

• Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.

• This phenomenon agrees with mean reversion.

• Convergence is quite good (see text).
A General Method for Constructing Binomial Models\(^a\)

- We are given a continuous-time process,
  \[
  dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW.
  \]

- Make sure the binomial model’s drift and diffusion converge to the above process by setting the probability of an up move to
  \[
  \frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}.
  \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

\(^a\)Nelson and Ramaswamy (1990).
A General Method (continued)

- But the binomial tree may not combine as

\[
\sigma(y, t) \sqrt{\Delta t} - \sigma(y_u, t + \Delta t) \sqrt{\Delta t} \\
\neq -\sigma(y, t) \sqrt{\Delta t} + \sigma(y_d, t + \Delta t) \sqrt{\Delta t}
\]

in general.

- When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.
A General Method (continued)

• To achieve this, define the transformation

\[ x(y, t) \equiv \int_{y}^{y} \sigma(z, t)^{-1} \, dz. \]

• Then \( x \) follows

\[ dx = m(y, t) \, dt + dW \]

for some \( m(y, t) \) (see text).

• The key is that the diffusion term is now a constant, and the binomial tree for \( x \) combines.

• The transformation that turns a 1-dim stochastic process into one with a constant diffusion term is unique.\(^a\)

\(^a\)Chiu (R98723059) (2012).
A General Method (concluded)

• The probability of an up move remains

\[
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
\]

where \( y(x, t) \) is the inverse transformation of \( x(y, t) \) from \( x \) back to \( y \).

• Note that \( y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t) \) and \( y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t) \).
Examples

- The transformation is
  \[ \int_{r}^{r} (\sigma \sqrt{z})^{-1} \, dz = 2\sqrt{r}/\sigma \]
  for the CIR model.

- The transformation is
  \[ \int_{S}^{S} (\sigma z)^{-1} \, dz = (1/\sigma) \ln S \]
  for the Black-Scholes model.

- The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate levels only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

• One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.

• Derivatives whose values depend on the correlation structure will be mispriced.

• The calibrated models may not generate term structures as concave as the data suggest.

• The term structure empirically changes in slope and curvature as well as makes parallel moves.

• This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.
Options on Coupon Bonds\(^{a}\)

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.

\(^{a}\)Jamshidian (1989).
Options on Coupon Bonds (continued)

- The payoff for the option is
  \[ \max \left( \sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0 \right). \]

- At time \( T \), there is a unique value \( r^* \) for \( r(T) \) that renders the coupon bond’s price equal the strike price \( X \).

- This \( r^* \) can be obtained by solving
  \[ X = \sum_{i=1}^{n} c_i P(r, T, t_i) \]
  numerically for \( r \).
Options on Coupon Bonds (continued)

• The solution is unique for one-factor models whose bond price is a monotonically decreasing function of $r$.

• Let

$$X_i \equiv P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$.

• Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$. 
Options on Coupon Bonds (concluded)

- As \( X = \sum_i c_i X_i \), the option’s payoff equals
  \[
  \max \left( \sum_{i=1}^n c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0 \right)
  = \sum_{i=1}^n c_i \times \max(P(r(T), T, t_i) - X_i, 0).
  \]

- Thus the call is a package of \( n \) options on the underlying zero-coupon bond.

- Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?

— Arthur Eddington (1882–1944)
Motivations

• Recall the difficulties facing equilibrium models mentioned earlier.
  – They usually require the estimation of the market price of risk.
  – They cannot fit the market term structure.
  – But consistency with the market is often mandatory in practice.
No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

---

\(^a\)Ho and Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to Bloomberg on May 26, 2012.
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.

- Bond price and forward rate models are usually non-Markovian (path dependent).

- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).

- Markovian models are easier to handle computationally.
The Ho-Lee Model\(^a\)

- The short rates at any given time are evenly spaced.
- Let \( p \) denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

\(^a\)Ho and Lee (1986).
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t + 1), P(t, t + 2), \ldots$ at time $t$ identified with the root of the tree.

- Let the discount factors in the next period be
  
  $P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots$ if short rate moves down
  $P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots$ if short rate moves up

- By backward induction, it is not hard to see that for $n \geq 2$,
  
  $$P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{- (v_2 + \cdots + v_n)}$$

  (118)

  (see text).  

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The Ho-Lee Model (continued)

- It is also not hard to check that the $n$-period zero-coupon bond has yields

$$
y_d(n) \equiv -\frac{\ln P_d(t + 1, t + n)}{n - 1}
$$

$$
y_u(n) \equiv -\frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}
$$

- The volatility of the yield to maturity for this bond is therefore

$$
\kappa_n \equiv \sqrt{py_u(n)^2 + (1 - p)y_d(n)^2 - [py_u(n) + (1 - p)y_d(n)]^2}
$$

$$
= \sqrt{p(1 - p)} \left( y_u(n) - y_d(n) \right)
$$

$$
= \sqrt{p(1 - p)} \frac{v_2 + \cdots + v_n}{n - 1}.
$$
The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)} \, v_2. \quad (119)$$

- The variance of the short rate therefore equals $p(1-p)(r_u - r_d)^2$, where $r_u$ and $r_d$ are the two successor rates.$^a$

\[\text{\underline{a}} \text{Contrast this with the lognormal model.}\]
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the $r_i$.

- It is easy to compute the $v_i$s from the volatility structure, and vice versa.

- The $r_i$s can be computed by forward induction.

- The volatility structure is supplied by the market.
The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = (pP_u(t+1, t+n) + (1-p) P_d(t+1, t+n)) P(t, t+1) \]

• Combine the above with Eq. (118) on p. 983 and assume \( p = 1/2 \) to obtain\(^a\)

\[
\begin{align*}
P_d(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \\
P_u(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}.
\end{align*}
\]

\(^a\)In the limit, only the volatility matters.
The Ho-Lee Model: Bond Price Process (concluded)

• The bond price tree combines.

• Suppose all $v_i$ equal some constant $v$ and $\delta \equiv e^v > 0$.

• Then

$$P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}},$$

$$P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.$$

• Short rate volatility $\sigma$ equals $v/2$ by Eq. (119) on p. 985.

• Price derivatives by taking expectations under the risk-neutral probability.
The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an $n$-period zero-coupon bond is
  
  $$r(t, t + n) \equiv \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).$$

- Its value is either $\ln \frac{P_d(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_u(t+1,t+n)}{P(t,t+n)}$.

- Thus the variance of return is
  
  $$\text{Var}[r(t, t + n)] = p(1 - p)((n - 1) \nu)^2 = (n - 1)^2 \sigma^2.$$
The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t,t+n)$ and $r(t,t+m)$ is $(n-1)(m-1)\sigma^2$ (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.
The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

\[ dr = \theta(t) \, dt + \sigma \, dW. \]

- This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e.,

\[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

- This corresponds to the discrete-time model in which \( v_i \) are not all identical.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.

- Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.

- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.
Problems with No-Arbitrage Models in General (concluded)

• This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.

• Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.
The Black-Derman-Toy Model\textsuperscript{a}

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 834ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with $r_i$.

\textsuperscript{a}Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).
The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.
  - A related model of Salomon Brothers takes $v_i$ to be a given constant.$^a$

- Lognormal models preclude negative short rates.

$^a$Tuckman (2002).
The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.
- $P_u$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_d$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.
The BDT Model: Volatility Structure (concluded)

• Corresponding to these two prices are the following yields to maturity,

\[ y_u \equiv P_u^{i-1/(i-1)} - 1, \]
\[ y_d \equiv P_d^{i-1/(i-1)} - 1. \]

• The yield volatility is defined as

\[ \kappa_i \equiv \frac{\ln(y_u/y_d)}{2}. \]
The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

\[(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1})\].

- They define the binomial tree up to period \(i-1\).
- We now proceed to calculate \(r_i\) and \(v_i\) to extend the tree to period \(i\).
The BDT Model: Calibration (continued)

- Assume the price of the $i$-period zero can move to $P_u$ or $P_d$ one period from now.

- Let $y$ denote the current $i$-period spot rate, which is known.

- In a risk-neutral economy,

\[ \frac{P_u + P_d}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \]  

\[ (121) \]

- Obviously, $P_u$ and $P_d$ are functions of the unknown $r_i$ and $v_i$. 
The BDT Model: Calibration (continued)

- Viewed from now, the future \((i - 1)\)-period spot rate at time 1 is uncertain.

- Recall that \(y_u\) and \(y_d\) represent the spot rates at the up node and the down node, respectively (p. 999).

- With \(\kappa^2\) denoting their variance, we have

\[
\kappa_i = \frac{1}{2} \ln \left( \frac{P_u^{-1/(i-1)} - 1}{P_d^{-1/(i-1)} - 1} \right). \tag{122}
\]
The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.\(^a\)

- Recall that forward induction inductively figures out, by moving *forward* in time, how much $1 at a node contributes to the price (review p. 860(a)).

- This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

\(^a\)Chen (R84526007) and Lyuu (1997); Lyuu (1999).
The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period $i$ be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i - 1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$ for period $i$, respectively.
- One dollar at time $i$ has a present value of

$$f(r, v) \equiv \frac{P_1}{1 + r} + \frac{P_2}{1 + rv} + \frac{P_3}{1 + rv^2} + \cdots + \frac{P_i}{1 + rv^{i-1}}.$$
The BDT Model: Calibration (continued)

- The yield volatility is

\[ g(r, v) \equiv \frac{1}{2} \ln \left( \frac{\left( \frac{P_{u,1}}{1+rv} + \frac{P_{u,2}}{1+rv^2} + \cdots + \frac{P_{u,i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left( \frac{P_{d,1}}{1+r} + \frac{P_{d,2}}{1+rv} + \cdots + \frac{P_{d,i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right). \]

- Above, \( P_{u,1}, P_{u,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the up node (like \( r_v^2 v_2 \) on p. 996).

- And \( P_{d,1}, P_{d,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the down node (like \( r_v \) on p. 996).
The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices.
- Now solve

\[
\begin{align*}
    f(r, v) &= \frac{1}{(1 + y)^i}, \\
    g(r, v) &= \kappa_i,
\end{align*}
\]

for \( r = r_i \) and \( v = v_i \).
- This \( O(n^2) \)-time algorithm appears in the text.
The BDT Model: Continuous-Time Limit

• The continuous-time limit of the BDT model is
  \[ d \ln r = \left( \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dW. \]

• The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes \( \sigma'(t) < 0. \)

• In particular, constant volatility will not attain mean reversion.