Introduction to Term Structure Modeling

The fox often ran to the hole
by which they had come in,
to find out if his body was still thin enough
to slip through it.
— *Grimm's Fairy Tales*

And the worst thing you can have
is models and spreadsheets.
— Warren Buffet, May 3, 2008

Outline

- Use the binomial interest rate tree to model stochastic term structure.
  - Illustrates the basic ideas underlying future models.
  - Applications are generic in that pricing and hedging methodologies can be easily adapted to other models.
- Although the idea is similar to the earlier one used in option pricing, the current task is more complicated.
  - The evolution of an entire term structure, not just a single stock price, is to be modeled.
  - Interest rates of various maturities cannot evolve arbitrarily, or arbitrage profits may occur.

Issues

- A stochastic interest rate model performs two tasks.
  - Provides a stochastic process that defines future term structures without arbitrage profits.
  - "Consistent" with the observed term structures.

History

- Methodology founded by Merton (1970).
- Modern interest rate modeling is often traced to 1977 when Vasicek and Cox, Ingersoll, and Ross developed simultaneously their influential models.
- Early models have fitting problems because they may not price today's benchmark bonds correctly.
- An alternative approach pioneered by Ho and Lee (1986) makes fitting the market yield curve mandatory.
- Models based on such a paradigm are called (somewhat misleadingly) arbitrage-free or no-arbitrage models.

Binomial Interest Rate Tree

- Goal is to construct a no-arbitrage interest rate tree consistent with the yields and/or yield volatilities of zero-coupon bonds of all maturities.
  - This procedure is called calibration.[a]
- Pick a binomial tree model in which the logarithm of the future short rate obeys the binomial distribution.
  - Exactly like the CRR tree.
- The limiting distribution of the short rate at any future time is hence lognormal.

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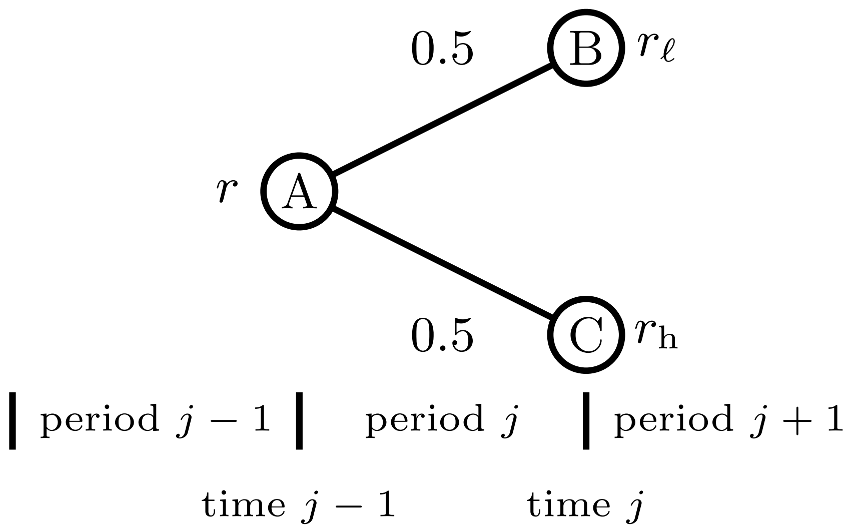
[a]Derman (2004), "complexity without calibration is pointless."

Binomial Interest Rate Tree (continued)

- A binomial tree of future short rates is constructed.
- Every short rate is followed by two short rates in the following period (p. 836).
- In the figure on p. 836, node A coincides with the start of period j during which the short rate r is in effect.



0.5
B
r_ℓ
r
A
0.5
C
$r_{\rm h}$
period $j-1$
period j
period $j+1$
time $j-1$
time j

Binomial Interest Rate Tree (continued)

- At the conclusion of period j, a new short rate goes into effect for period $j+1$.
- This may take one of two possible values:
  - r_ℓ: the "low" short-rate outcome at node B.
  - r_{h}: the "high" short-rate outcome at node C.
- Each branch has a 50% chance of occurring in a risk-neutral economy.

Binomial Interest Rate Tree (continued)

- We shall require that the paths combine as the binomial process unfolds.
- The short rate r can go to r_{h} and r_{ℓ} with equal risk-neutral probability $1/2$ in a period of length Δt.
- Hence the volatility of $\ln r$ after Δt time is

$$\sigma = \frac{1}{2} \frac{1}{\sqrt{\Delta t}} \ln \left(\frac{r_{\mathrm{h}}}{r_{\ell}} \right)$$

  (see Exercise 23.2.3 in text).
- Above, σ is annualized, whereas r_{ℓ} and r_{h} are period based.

Binomial Interest Rate Tree (continued)

- Note that
$$\frac{r_{\mathrm{h}}}{r_\ell} = e^{2\sigma\sqrt{\Delta t}}.$$
- Thus greater volatility, hence uncertainty, leads to larger r_{h}/r_ℓ and wider ranges of possible short rates.
- The ratio r_{h}/r_ℓ may depend on time if the volatility is a function of time.
- Note that r_{h}/r_ℓ has nothing to do with the current short rate r if σ is independent of r.

Binomial Interest Rate Tree (continued)

- In general there are j possible rates[a] in period j,

$$r_j,\, r_j v_j,\, r_j v_j^2,\, \ldots,\, r_j v_j^{j-1},$$

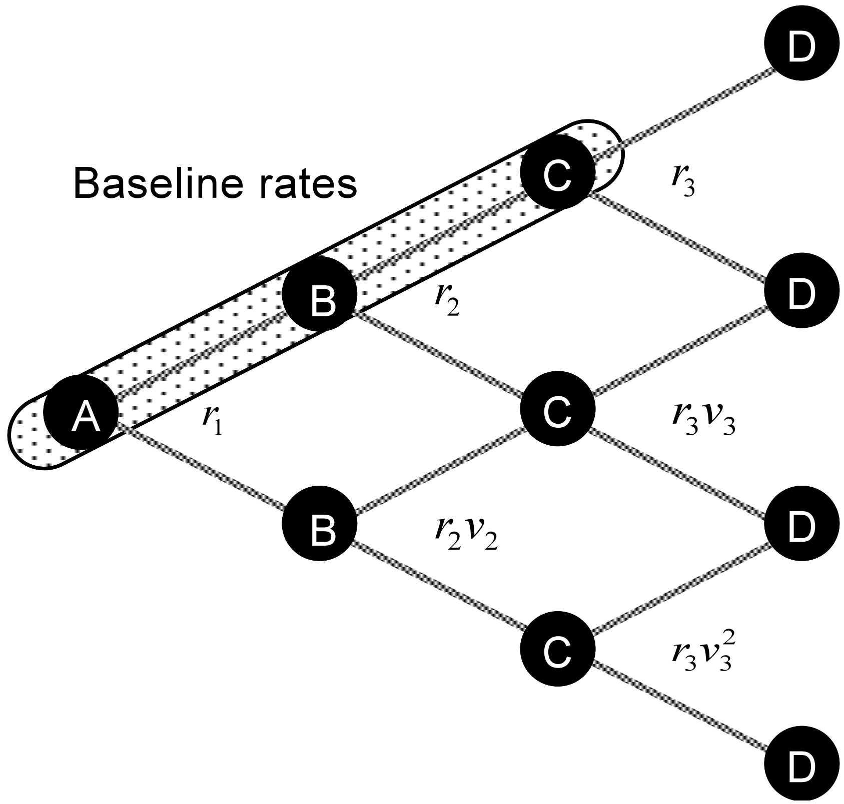
where

$$v_j \equiv e^{2\sigma_j \sqrt{\Delta t}} \tag{95}$$

is the multiplicative ratio for the rates in period j (see figure on next page).

- We shall call r_j the baseline rates.
- The subscript j in σ_j is meant to emphasize that the short rate volatility may be time dependent.

[a]Not $j + 1$.


Baseline rates
A
B
B
C
C
C
D
D
D
D
r_1
r_2
$r_2 v_2$
r_3
$r_3 v_3$
$r_3 v_3^2$

Binomial Interest Rate Tree (concluded)

- In the limit, the short rate follows the following process,

$$r(t) = \mu(t)\, e^{\sigma(t)\, W(t)}, \tag{96}$$

  in which the (percent) short rate volatility $\sigma(t)$ is a deterministic function of time.

- The expected value of $r(t)$ equals $\mu(t)\, e^{\sigma(t)^2 (t/2)}$.

- Hence a declining short rate volatility is usually imposed to preclude the short rate from assuming implausibly high values.

- Incidentally, this is how the binomial interest rate tree achieves mean reversion.

Memory Issues

- Path independency: The term structure at any node is independent of the path taken to reach it.
- So only the baseline rates r_i and the multiplicative ratios v_i need to be stored in computer memory.
- This takes up only $O(n)$ space.[a]
- Storing the whole tree would take up $O(n^2)$ space.
  - Daily interest rate movements for 30 years require roughly $(30 \times 365)^2/2 \approx 6 \times 10^7$ double-precision floating-point numbers (half a gigabyte!).

[a]Throughout, n denotes the depth of the tree.

Set Things in Motion

- The abstract process is now in place.
- We need the annualized rates of return of the riskless bonds that make up the benchmark yield curve and their volatilities.
- In the U.S., for example, the on-the-run yield curve obtained by the most recently issued Treasury securities may be used as the benchmark curve.

Set Things in Motion (concluded)

- The term structure of (yield) volatilities[a] can be estimated from:
  - Historical data (historical volatility).
  - Or interest rate option prices such as cap prices (implied volatility).
- The binomial tree should be consistent with both term structures.
- Here we focus on the term structure of interest rates.

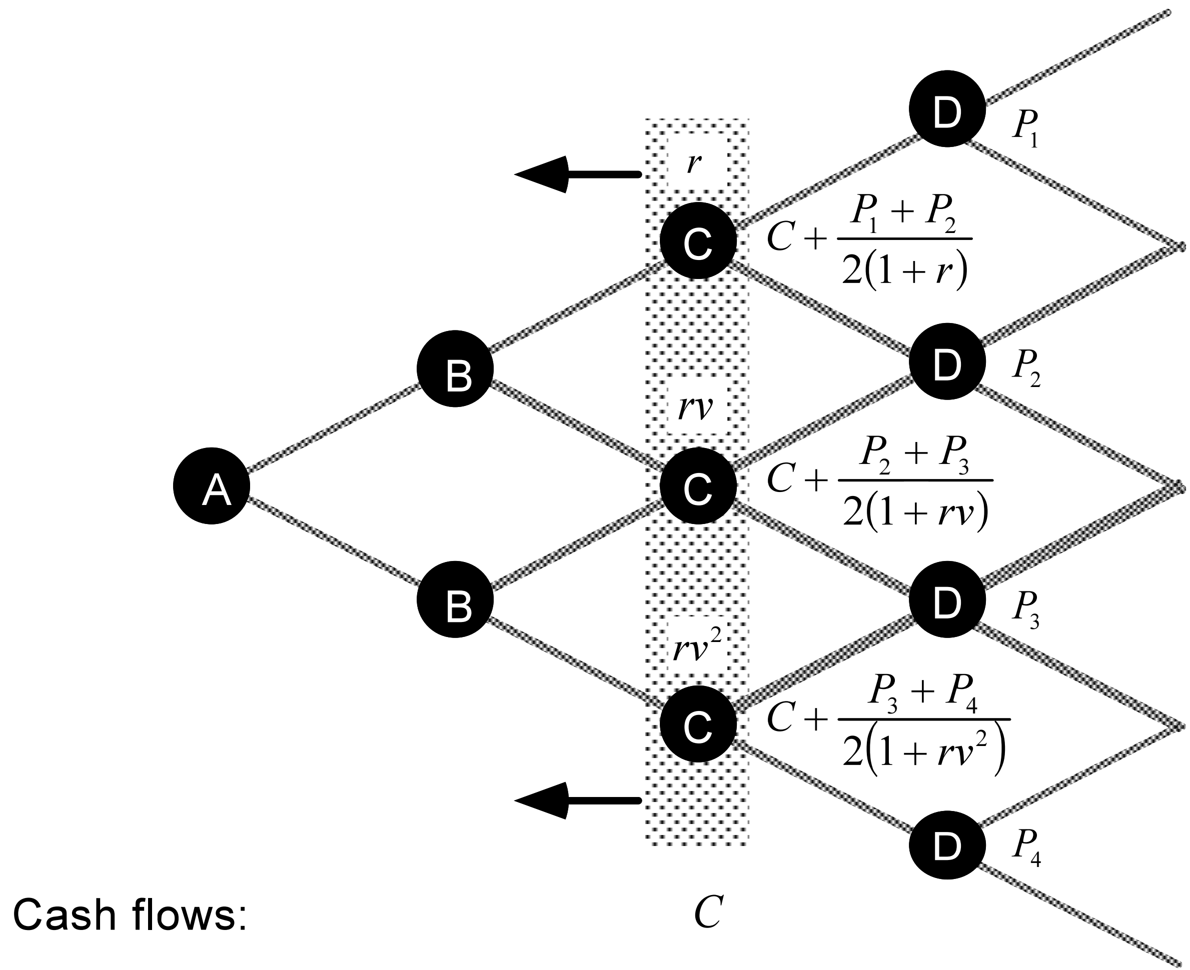
[a]Or simply the volatility (term) structure.

Model Term Structures

- The model price is computed by backward induction.
- Refer back to the figure on p. 836.
- Given that the values at nodes B and C are P_{B} and P_{C}, respectively, the value at node A is then

$$\frac{P_{\mathrm{B}} + P_{\mathrm{C}}}{2(1+r)} + \text{cash flow at node A.}$$

- We compute the values column by column without explicitly expanding the binomial interest rate tree (see next page).
- This takes $O(n^2)$ time and $O(n)$ space.



r
rv
rv^2
A
B
B
C
C
C
D
D
D
D
P_1
P_2
P_3
P_4
$C+\frac{P_1+P_2}{2(1+r)}$
$C+\frac{P_2+P_3}{2(1+rv)}$
$C+\frac{P_3+P_4}{2(1+rv^2)}$
Cash flows:
C

Term Structure Dynamics

- An n-period zero-coupon bond's price can be computed by assigning \$1 to every node at period n and then applying backward induction.
- Repeating this step for $n = 1, 2, \ldots$, one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics.
  - Taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.

Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
- Assume the short rate volatility is such that $v \equiv r_{\mathrm{h}}/r_{\ell} = 1.5$, independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let f_j denote the forward rate in period j.
- This forward rate can be derived from the market discount function via $f_j = (d(j)/d(j+1)) - 1$ (see Exercise 5.6.3 in text).

An Approximate Calibration Scheme (continued)

- Since the ith short rate $r_j v_j^{i-1}$, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left(\frac{2}{1+v_j} \right)^{j-1} f_j. \tag{97}$$

- The binomial interest rate tree is trivial to set up.

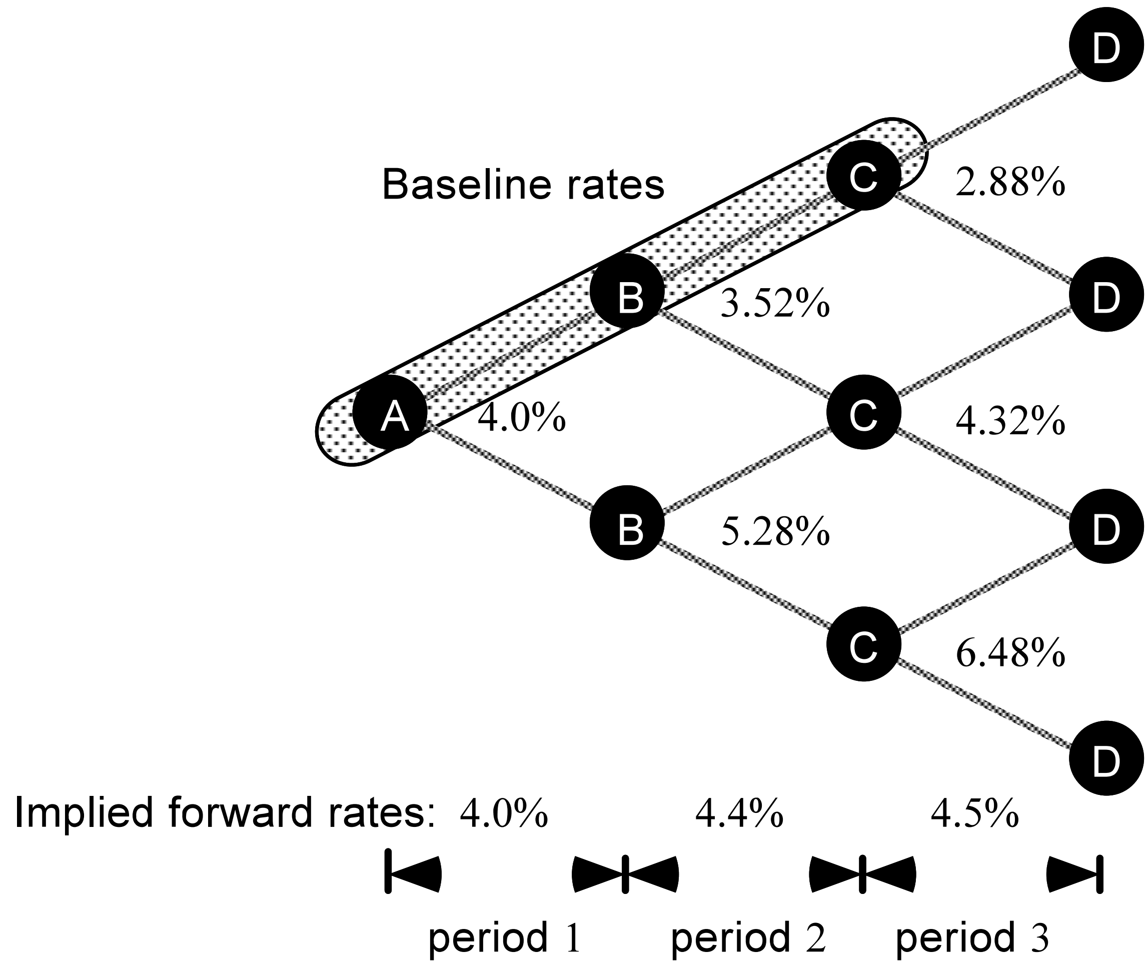
An Approximate Calibration Scheme (continued)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

$$\frac{1}{4} \times \frac{1}{1.04} \times \left(\frac{1}{1.0352} \times \left(\frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left(\frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

  or 0.88155, which exceeds discount factor 0.88135.

- The tree is thus *not* calibrated.


Baseline rates
A
B
B
C
C
C
D
D
D
D
4.0%
3.52%
5.28%
2.88%
4.32%
6.48%
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree overprices the bonds (see Exercise 23.2.4 in text).
- Suppose we replace the baseline rates r_j by $r_j v_j$.
- Then the resulting tree underprices the bonds.[a]
- The true baseline rates are thus bounded between r_j and $r_j v_j$.

[a] Lyuu and Wang (F95922018) (2009, 2011).

Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the m-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate r_m for period m.
- A root-finding method is applied to solve $f(r_m) = P$ for r_m given the zero's price P and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in $O(n^3)$ time.

Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.[a]
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price or the Arrow-Debreu price.
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 0 to time n.

[a]Jamshidian (1991).

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time j and there are $j+1$ nodes.
  - The unknown baseline rate for period j is $r \equiv r_j$.
  - The multiplicative ratio is $v \equiv v_j$.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at earlier time $j-1$, corresponding to rates $r, rv, \ldots, rv^{j-1}$ for period j.
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.
- We want to find r based on $P_1, P_2, \ldots, P_j$ and the price of the j-period zero-coupon bond.

Binomial Interest Rate Tree Calibration (continued)

- One dollar at time j has a known market value of $1/[\,1 + S(j)\,]^j$, where $S(j)$ is the j-period spot rate.
- Alternatively, this dollar has a present value of

$$g(r) \equiv \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}.$$

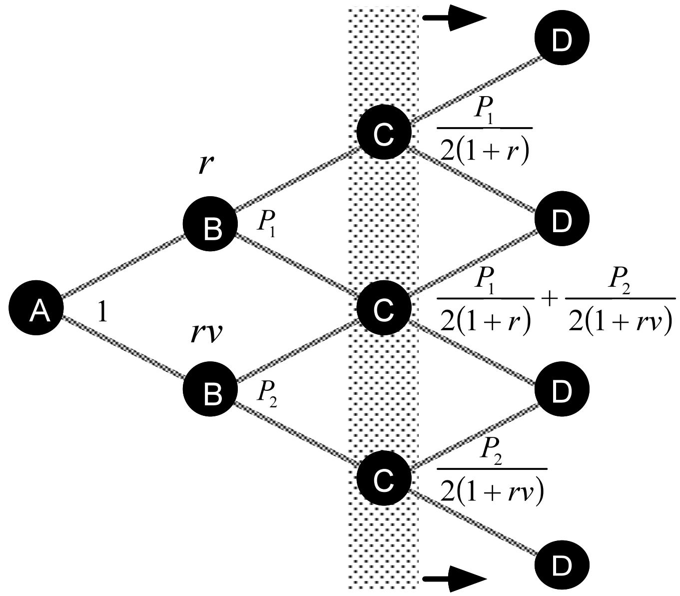
- So we solve

$$g(r) = \frac{1}{[\,1 + S(j)\,]^j} \tag{98}$$

for r.

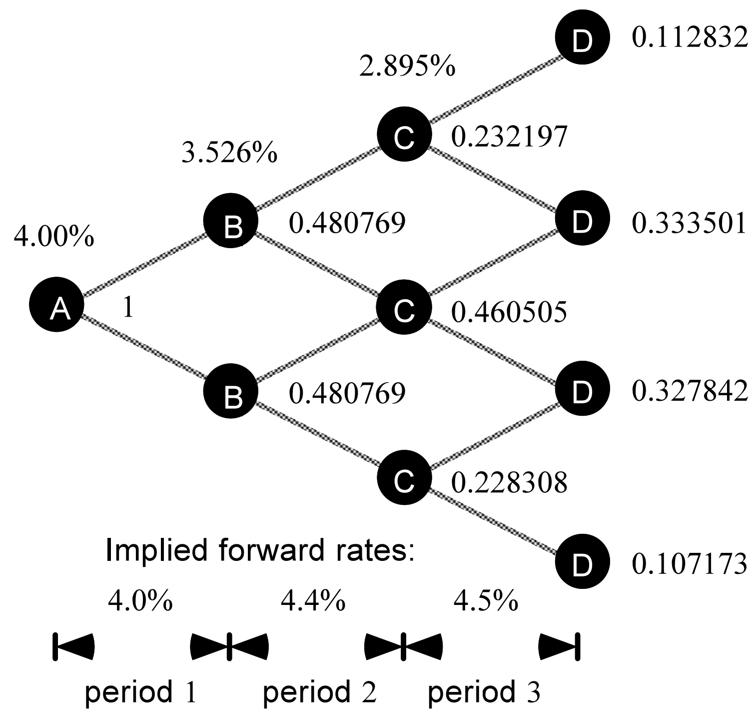
Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for r is guaranteed.
- The state prices at time j can now be calculated (see figure (a) next page).
- We call a tree with these state prices a binomial state price tree (see figure (b) next page).
- The calibrated tree is depicted on p. 861.



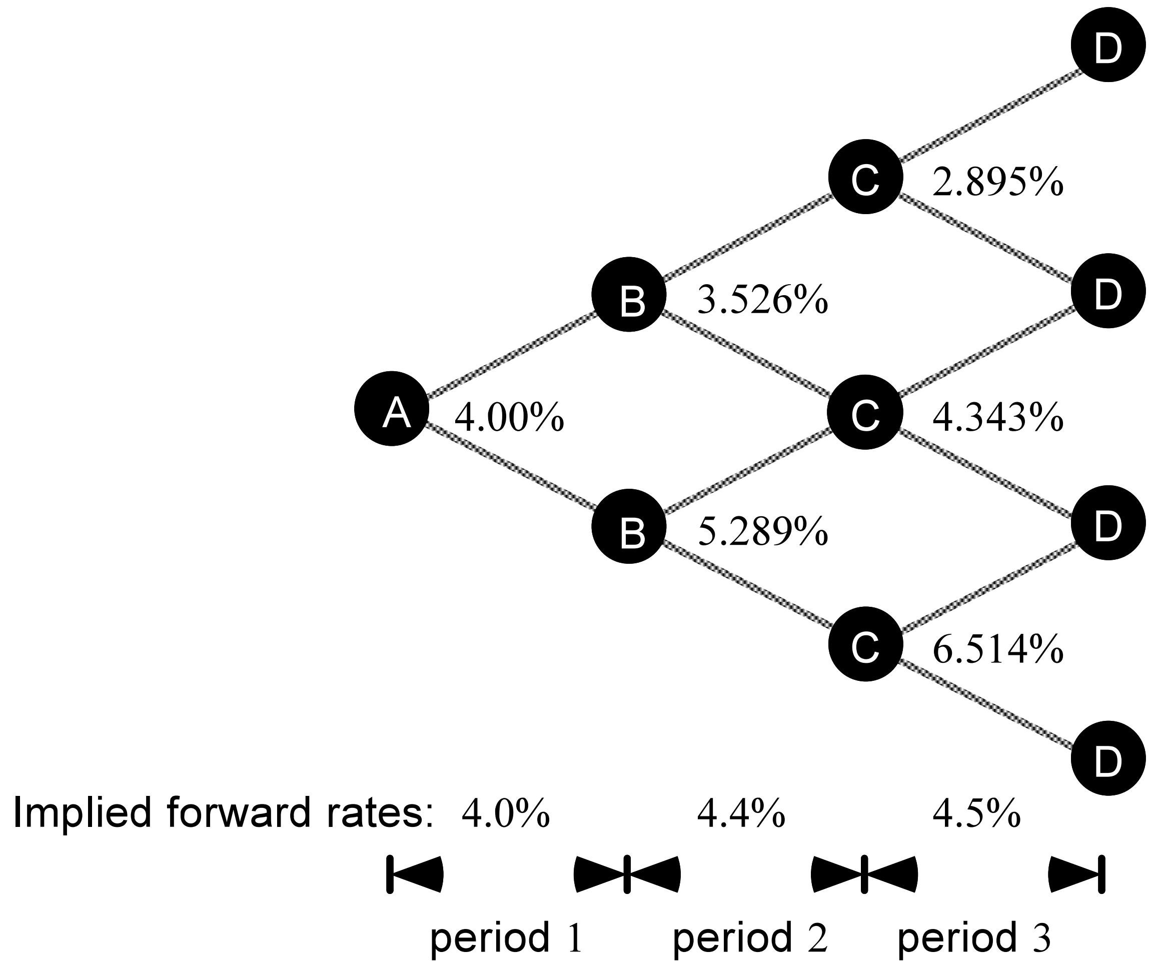
A
B
B
C
C
C
D
D
D
D
1
r
rv
P_1
P_2
$\frac{P_1}{2(1+r)}$
$\frac{P_1}{2(1+r)}+\frac{P_2}{2(1+rv)}$
$\frac{P_2}{2(1+rv)}$

(a)



4.00%
3.526%
2.895%
1
0.480769
0.480769
0.232197
0.460505
0.228308
0.112832
0.333501
0.327842
0.107173
Implied forward rates:
4.0%
4.4%
4.5%
period 1
period 2
period 3

(b)



D
C
2.895%
B
3.526%
D
A
4.00%
C
4.343%
B
5.289%
D
C
6.514%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the r in Eq. (98) on p. 858 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The total running time is $O(n^2)$, as each root-finding routine consumes $O(j)$ time.
- With a good initial guess,[a] the Newton-Raphson method converges in only a few steps.[b]

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[a]Such as the $r_j = (\frac{2}{1+v_j})^{j-1} f_j$ on p. 851.

[b]Lyuu (1999).

A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, r_2, satisfies
$$\frac{0.480769}{1+r_2} + \frac{0.480769}{1+1.5 \times r_2} = 0.92101.$$
- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in figure (b) on p. 860 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

A Numerical Example (concluded)

- The baseline rate for the third period, r_3, satisfies

$$\frac{0.232197}{1+r_3}+\frac{0.460505}{1+1.5\times r_3}+\frac{0.228308}{1+(1.5)^2\times r_3}=0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 852 using the new rates:

$$\frac{1}{4}\times\frac{1}{1.04}\times[\frac{1}{1.03526}\times(\frac{1}{1.02895}+\frac{1}{1.04343})+\frac{1}{1.05289}\times(\frac{1}{1.04343}+\frac{1}{1.06514})],$$

which equals 0.88135, an exact match.

- The tree on p. 861 prices without bias the benchmark securities.

Spread of Nonbenchmark Bonds

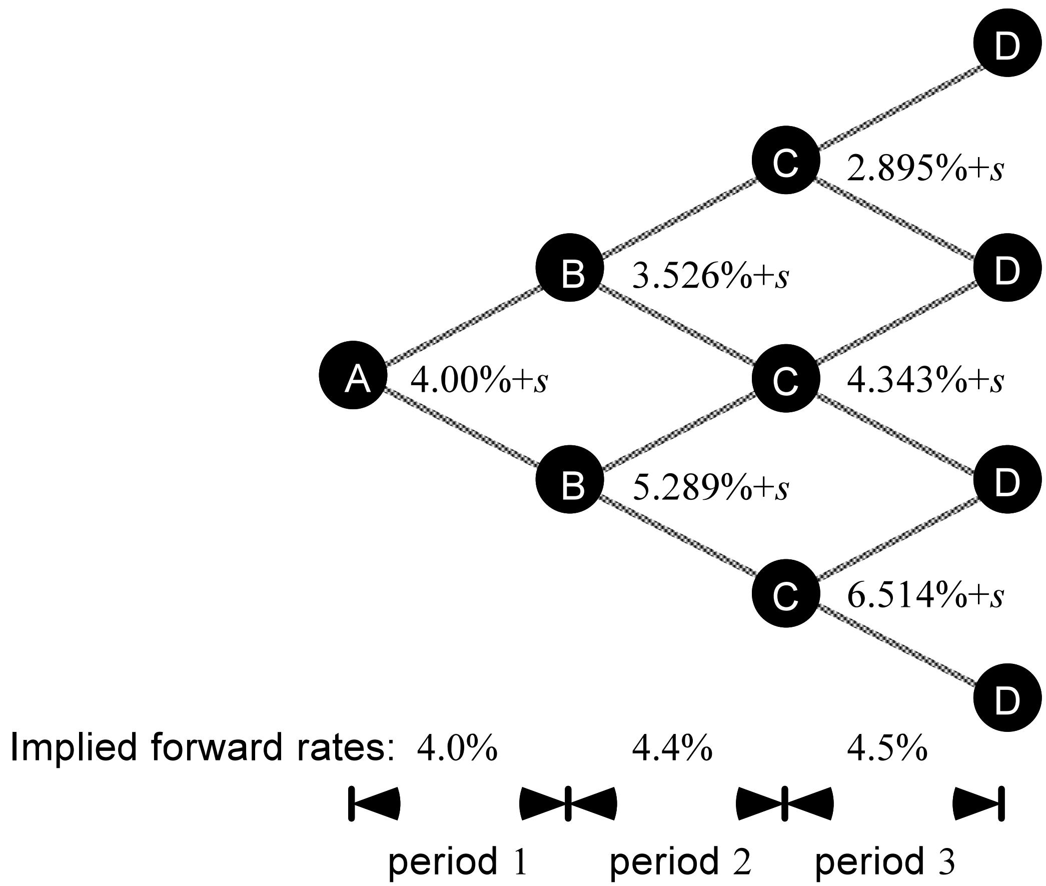
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 867.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.



D
C
2.895%+s
D
B
3.526%+s
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- We will employ the Newton-Raphson root-finding method to solve $p(s) - P = 0$ for s.
- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

Spread of Nonbenchmark Bonds (continued)

- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates
$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1+r+s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1+r+s)^2}. \qquad (99)$$
- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 871).
- This is called the differential tree method.[a]

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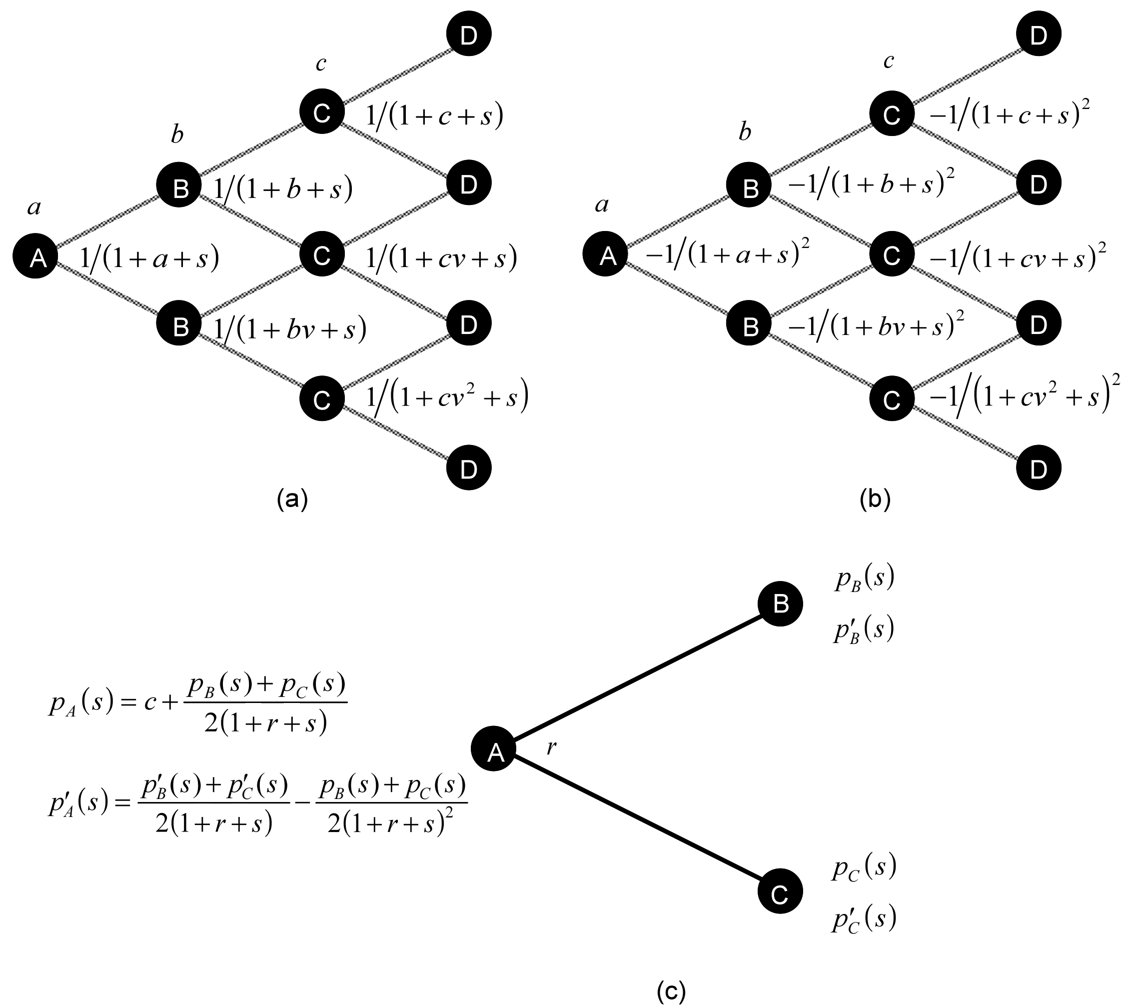
[a] Lyuu (1999).



a
b
c
A
B
C
D
1/(1+a+s)
1/(1+b+s)
1/(1+bv+s)
1/(1+c+s)
1/(1+cv+s)
1/(1+cv^2+s)
(a)
-1/(1+a+s)^2
-1/(1+b+s)^2
-1/(1+bv+s)^2
-1/(1+c+s)^2
-1/(1+cv+s)^2
-1/(1+cv^2+s)^2
(b)
r
p_B(s)
p'_B(s)
p_C(s)
p'_C(s)
p_A(s) = c + (p_B(s) + p_C(s)) / (2(1+r+s))
p'_A(s) = (p'_B(s) + p'_C(s)) / (2(1+r+s)) - (p_B(s) + p_C(s)) / (2(1+r+s)^2)
(c)

Spread of Nonbenchmark Bonds (continued)

- The total running time is $O(n^2)$.
- The memory requirement is $O(n)$.

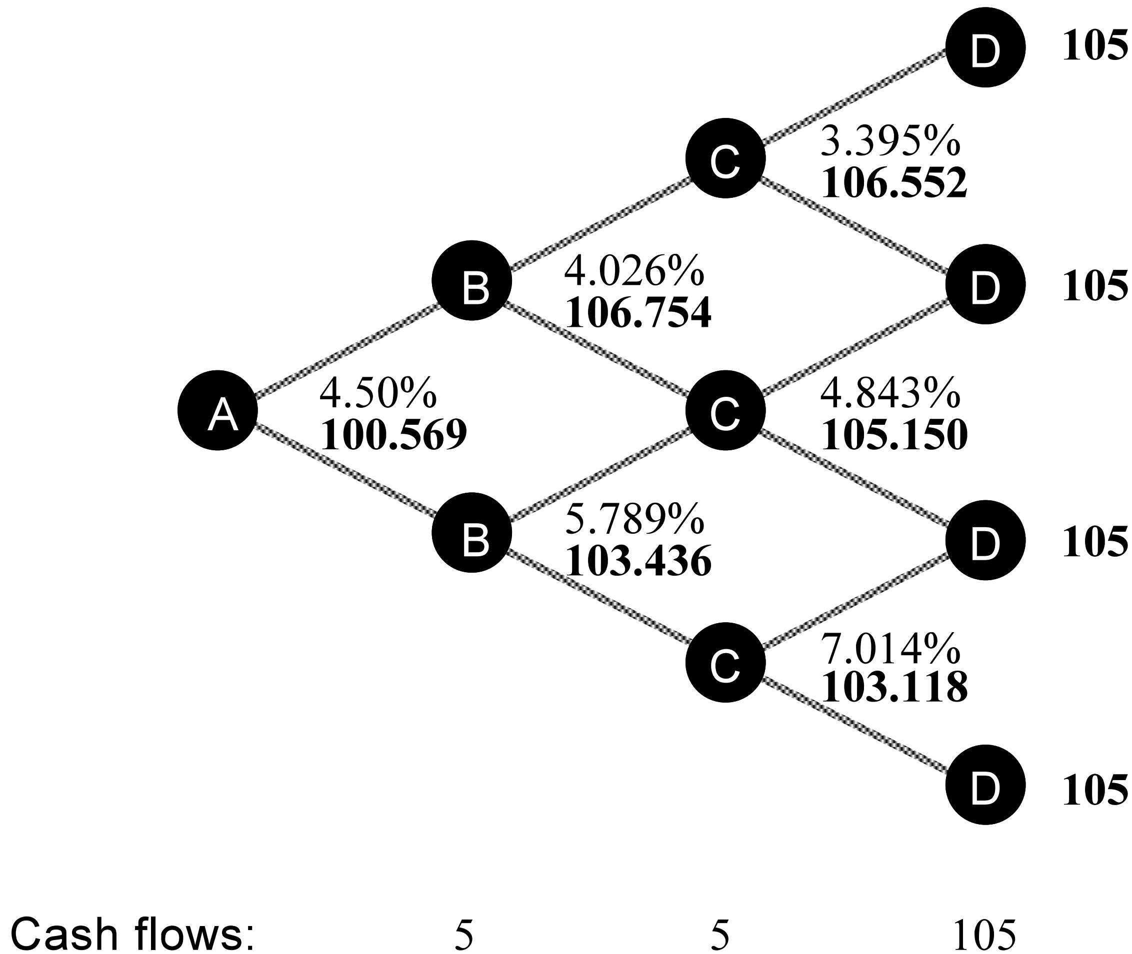
Spread of Nonbenchmark Bonds (continued)

Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 875).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread (p. 116) and static spread (p. 117) of the nonbenchmark bond over an otherwise identical benchmark bond.


A
4.50%
100.569
B
4.026%
106.754
B
5.789%
103.436
C
3.395%
106.552
C
4.843%
105.150
C
7.014%
103.118
D
105
D
105
D
105
D
105
Cash flows:
5
5
105

More Applications of the Differential Tree: Calibrating Black-Derman-Toy (in seconds)

Number of years	Running time	Number of years	Running time	Number of years	Running time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

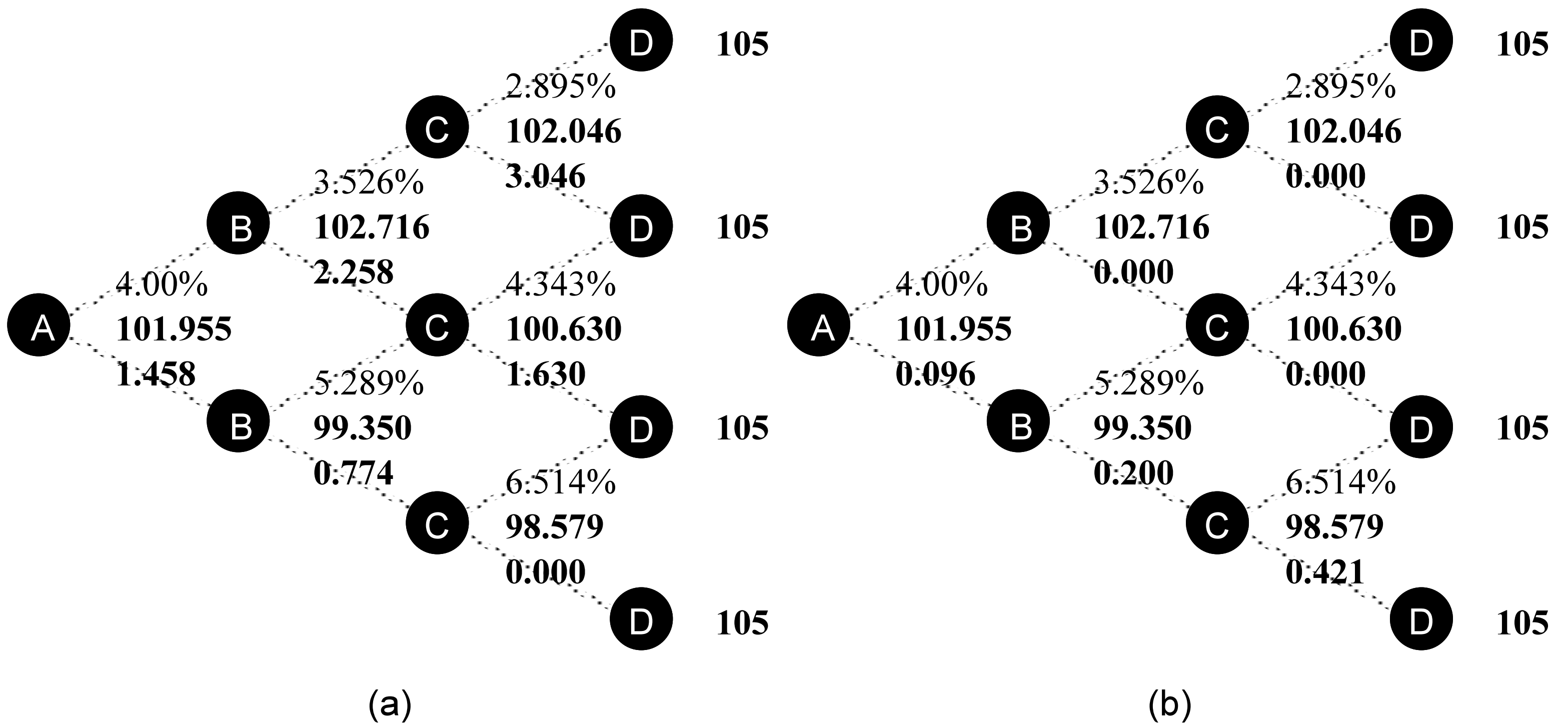
More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)

American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

Intel 166MHz Pentium, running on Microsoft Windows 95.

Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 879 the three-year Treasury's price minus the \$5 interest at year 2 could be \$102.046, \$100.630, or \$98.579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of \$3.046 and \$1.630, and out of the money in the third scenario.


A
4.00%
101.955
1.458
B
3.526%
102.716
2.258
B
5.289%
99.350
0.774
C
2.895%
102.046
3.046
C
4.343%
100.630
1.630
C
6.514%
98.579
0.000
D
105
(a)
A
4.00%
101.955
0.096
B
3.526%
102.716
0.000
B
5.289%
99.350
0.200
C
2.895%
102.046
0.000
C
4.343%
100.630
0.000
C
6.514%
98.579
0.421
D
105
(b)

Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 879(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 879(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\mathrm{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\mathrm{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \mathrm{PV}(I) - \mathrm{PV}(X).$$

Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above P_{h} and P_{ℓ} denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_{ℓ} denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 879 as examples.
- Their deltas are

$$\begin{aligned} \frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441, \\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059, \end{aligned}$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_ℓ, respectively) at the end of the initial period if the short rate rises (declines, respectively).
- The yield volatility for our model is defined as $(1/2) \ln(y_{\mathrm{h}}/y_\ell)$.

Volatility Term Structures (continued)

- For example, based on the tree on p. 861, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2}\ln\left(\frac{0.05289}{0.03526}\right) = 20.273\%.$$

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the short rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the short rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

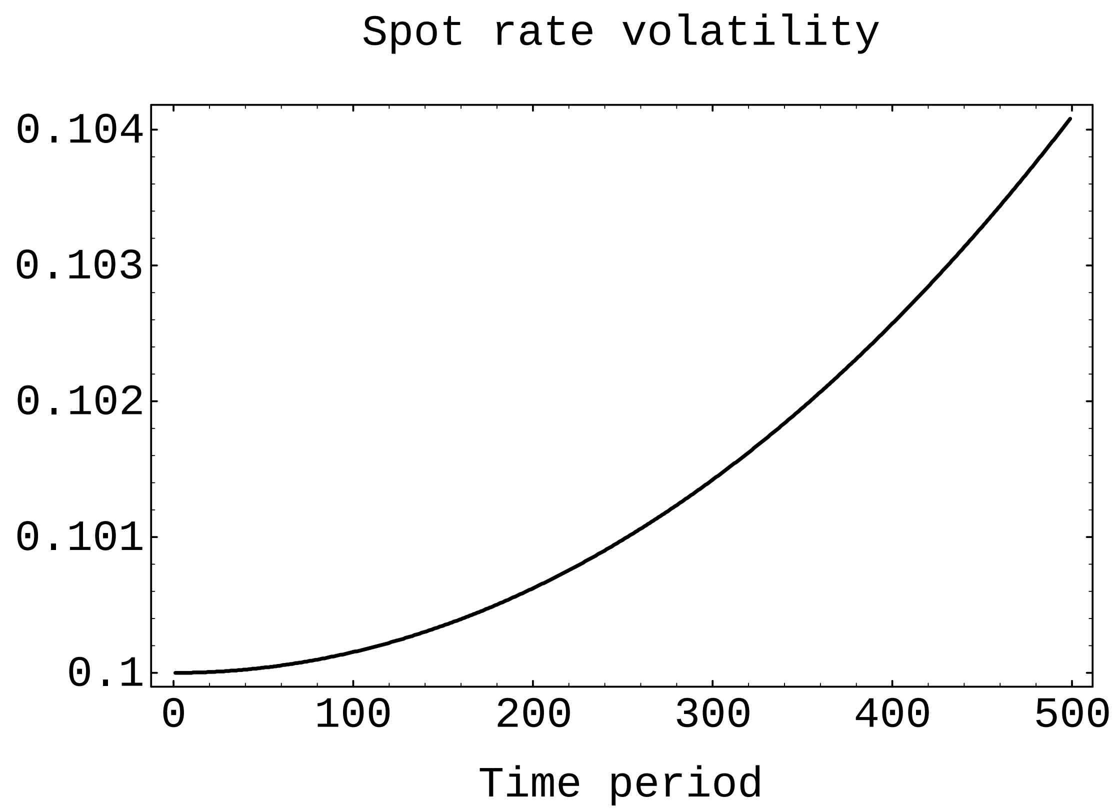
Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

$$\frac{1}{2} \ln \left(\frac{0.053531}{0.0357} \right) = 20.256\%,$$

slightly less than the one-year yield volatility.

- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.



Short rate volatility given flat %10 volatility term structure.

Volatility Term Structures (concluded)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (95) on p. 840—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.[a]

[a]Black, Derman, and Toy (1990).

Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.

— Roger Lowenstein,
When Genius Failed (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader $[\cdots]$
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
My Life as a Quant (2004)

Bond market terminology was designed less
to convey meaning than to bewilder outsiders.
— Michael Lewis, *The Big Short* (2011)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one — unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later

(the instantaneous spot rate, or short rate, at time t).

$P(t, T)$**:** the present value at time t of one dollar at time T.

Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period—in other words, the $(T - t)$-period spot rate at time t.

$F(t, T, M)$: the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{100}$$

  - The forward price equals the future value at time T of the underlying asset (see text for proof).
- Equation (100) holds whether the model is discrete-time or continuous-time.

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)} \right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)} \right)^{1/L} - 1 \tag{101}$$

  in discrete time.

  – The analog to Eq. (101) under simple compounding is

$$f(t,T,L) = \frac{1}{L} \left(\frac{P(t,T)}{P(t,T+L)} - 1 \right).$$

Fundamental Relations (continued)

- In continuous time,
$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{102}$$
by Eq. (100) on p. 898.
- Furthermore,
$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{103}$$

- Because Eq. (103) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{104}$$

the spot rate curve is

$$r(t,T) = \frac{1}{T-t} \int_t^T f(t,s)\,ds.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (104) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t+1<T$,
  $$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \qquad (105)$$
  - Relation (105) in fact follows from the risk-neutral valuation principle.[a]

[a] Theorem 16 on p. 463.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (105) as

$$\frac{E_t^{\pi}[\, P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t,T) \\
= \quad & E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \quad & E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \quad & E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right]. \qquad (106)
\end{aligned}
$$

Risk-Neutral Pricing (concluded)

- Equation (105) on p. 903 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

  – Verify that with, e.g., Eq. (100) on p. 898.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{107}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} (f_{i-1} - c)\, \Delta t \right] \\
= \;& \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= \;& \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= \;& P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{108}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Term Structure Equation

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p\, dt + \sigma_p\, dW.$$

- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\ = \quad & (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\ & + (-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW. \end{aligned}$$

- Pick

$$\alpha \equiv \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \equiv \lambda(r,t) \tag{109}$$

  for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

- The term $\lambda(r,t)$ is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 in the text,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} \right) \Big/ P, \tag{110}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r} \right) \Big/ P, \tag{110$'$}$$

subject to $P(\,\cdot\,,T,T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (109) on p. 916 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{111}$$

- This is called the term structure equation.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

- Equation (111) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.