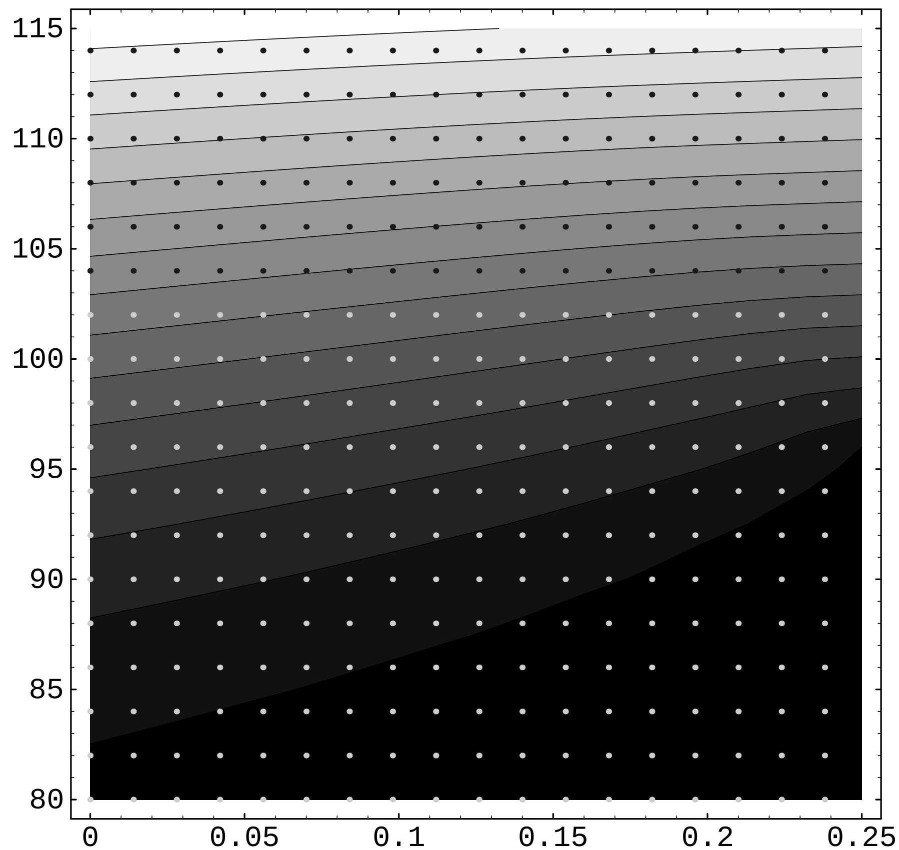
Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 667).
- Solve the equation numerically by introducing difference equations in place of derivatives.


115
110
105
100
95
90
85
80
0
0.05
0.1
0.15
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0.25

Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite difference form is

$$
\begin{aligned}
-\rho(x_i, y_j) = {} & \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} \\
& + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.
\end{aligned}
$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$-h^2 \rho(x_i, y_j) = \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) + \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0.$
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x,t_{j+1}) - \theta(x,t_j)}{\Delta t} + \cdots, \tag{78}$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots. \tag{79}$$

Explicit Methods (continued)

- Next, assemble Eqs. (78) and (79) into a single equation at (x_i, t_j).
- But we need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (79), we might as well use x_i for x in Eq. (78).
- Two choices are possible for t in Eq. (79).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \tag{80}$$

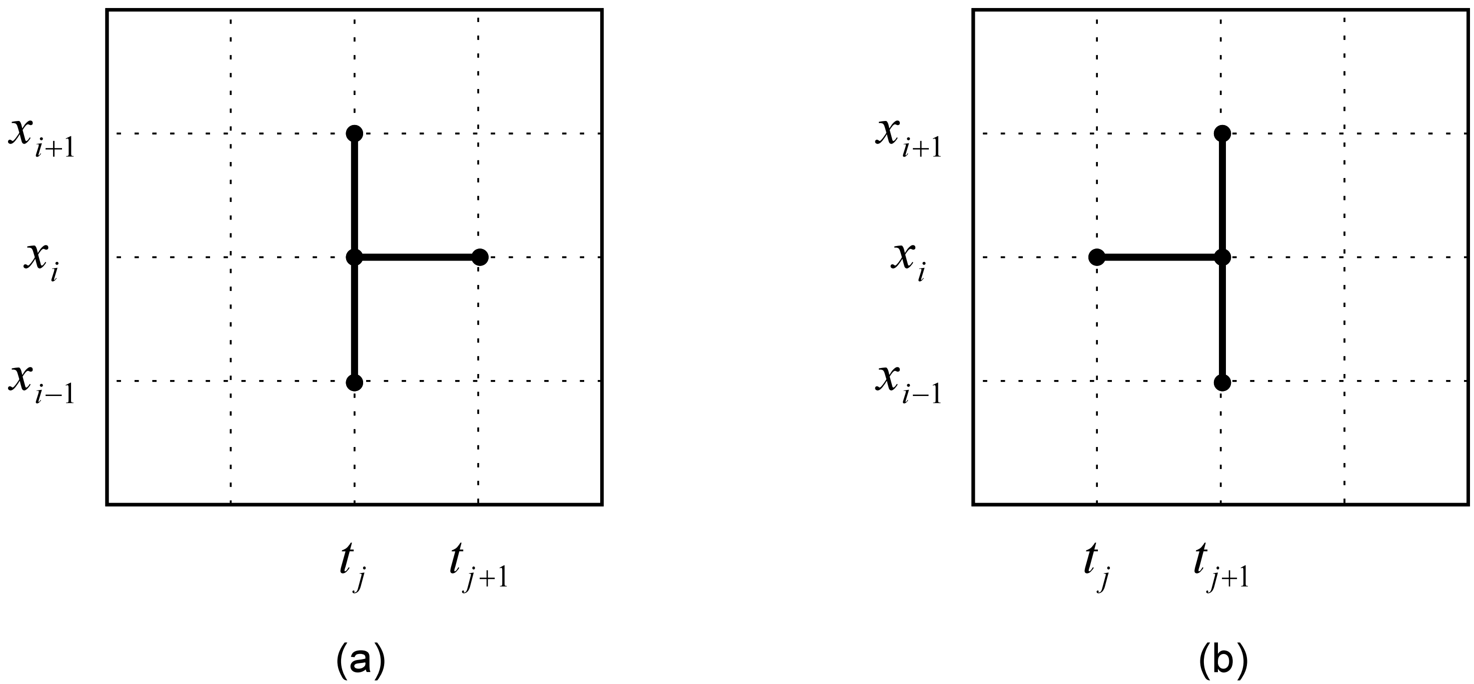
Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- Rearrange Eq. (80) on p. 671 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see exhibit (a) on next page).

Stencils



Explicit Methods (concluded)

- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

$$\theta_{i,1}, \quad i = 1, 2, \ldots .$$

- And then

$$\theta_{i,2}, \quad i = 1, 2, \ldots .$$

- And so on.

Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2/(2D).$$

  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.
- The stability condition may lead to high running times and memory requirements.
- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time 8 times as much.

Explicit Method and Trinomial Tree

- Recall that

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!
- The freedom in choosing Δx corresponds to similar freedom in the construction of trinomial trees.

Implicit Methods

- Suppose we use $t = t_{j+1}$ in Eq. (79) on p. 670 instead.
- The finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}. \tag{81}$$

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is implicit:
  - The value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known.
  - See exhibit (b) on p. 673.

Implicit Methods (continued)

- Equation (81) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma)\,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma\theta_{i,j},$$

  where $\gamma \equiv (\Delta x)^2/(D\Delta t)$.

- This equation is unconditionally stable.
- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.
- After $\theta_{i,j}$ has been calculated for $i = 1, 2, \ldots, N$, the values of $\theta_{i,j+1}$ at time t_{j+1} can be computed as the solution to the following tridiagonal linear system,

Implicit Methods (continued)

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} -\gamma\theta_{1,j} - \theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j} - \theta_{N+1,j+1} \end{bmatrix},$$

where $a \equiv -2 - \gamma$.

Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
  - Never invert a matrix to solve a tridiagonal system.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.

Crank-Nicolson Method

- Take the average of explicit method (80) on p. 671 and implicit method (81) on p. 677:

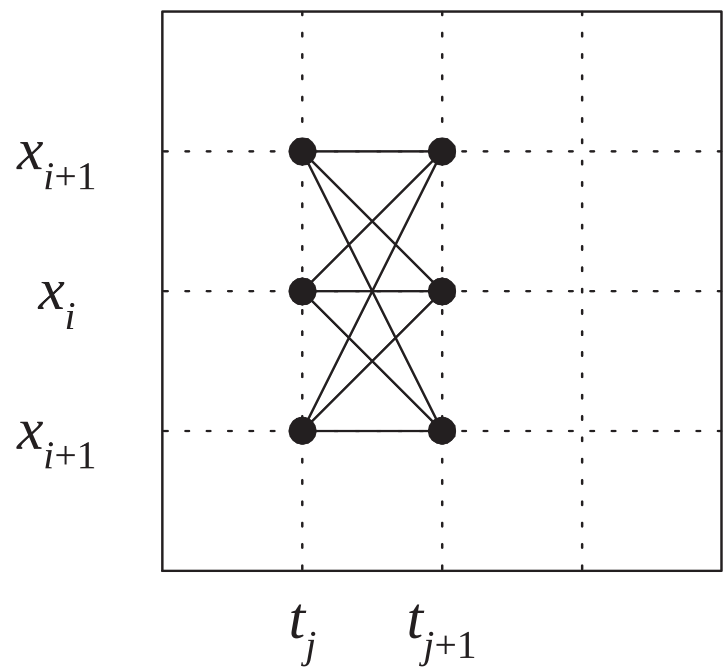
$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = \frac{1}{2}\left(D\, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2} + D\, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2} \right).$$

- After rearrangement,

$$\gamma\theta_{i,j+1} - \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{2} = \gamma\theta_{i,j} + \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{2}.$$

- This is an unconditionally stable implicit method with excellent rates of convergence.

Stencil



Numerically Solving the Black-Scholes PDE

- See text.
- Brennan and Schwartz (1978) analyze the stability of the implicit method.

Monte Carlo Simulation[a]

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.
- When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo may very well be the only strategy that succeeds consistently.

[a] A top 10 algorithm according to Dongarra and Sullivan (2000).

The Big Idea

- Assume $X_1, X_2, \ldots, X_n$ have a joint distribution.
- $\theta \equiv E[\, g(X_1, X_2, \ldots, X_n)\,]$ for some function g is desired.
- We generate

$$\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\right), \quad 1 \leq i \leq N$$

  independently with the same joint distribution as $(X_1, X_2, \ldots, X_n)$.
- Set

$$Y_i \equiv g\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\right).$$

The Big Idea (concluded)

- $Y_1, Y_2, \ldots, Y_N$ are independent and identically distributed random variables.
- Each Y_i has the same distribution as

$$Y \equiv g(X_1, X_2, \ldots, X_n).$$

- Since the average of these N random variables, $\overline{Y}$, satisfies $E[\,\overline{Y}\,] = \theta$, it can be used to estimate θ.
- The strong law of large numbers says that this procedure converges almost surely.
- The number of replications (or independent trials), N, is called the sample size.

Accuracy

- The Monte Carlo estimate and true value may differ owing to two reasons:
  1. Sampling variation.
  2. The discreteness of the sample paths.[a]
- The first can be controlled by the number of replications.
- The second can be controlled by the number of observations along the sample path.

[a]This may not be an issue if the financial derivative only requires discrete sampling along the time dimension, such as the discrete barrier option.

Accuracy and Number of Replications

- The statistical error of the sample mean $\overline{Y}$ of the random variable Y grows as $1/\sqrt{N}$.
  - Because $\text{Var}[\,\overline{Y}\,] = \text{Var}[\,Y\,]/N$.
- In fact, this convergence rate is asymptotically optimal.[a]
- So the variance of the estimator $\overline{Y}$ can be reduced by a factor of $1/N$ by doing N times as much work.
- This is amazing because the same order of convergence holds independently of the dimension n.

[a]The Berry-Esseen theorem.

Accuracy and Number of Replications (concluded)

- In contrast, classic numerical integration schemes have an error bound of $O(N^{-c/n})$ for some constant $c > 0$.
  - n is the dimension.
- The required number of evaluations thus grows exponentially in n to achieve a given level of accuracy.
  - The curse of dimensionality.
- The Monte Carlo method, for example, is more efficient than alternative procedures for multivariate derivatives.

Monte Carlo Option Pricing

- For the pricing of European options on a dividend-paying stock, we may proceed as follows.
- Assume $dS/S = \mu\, dt + \sigma\, dW$.
- Stock prices $S_1, S_2, S_3, \ldots$ at times $\Delta t, 2\Delta t, 3\Delta t, \ldots$ can be generated via

$$S_{i+1} = S_i e^{(\mu - \sigma^2/2)\, \Delta t + \sigma \sqrt{\Delta t}\, \xi}, \quad \xi \sim N(0,1). \tag{82}$$

Monte Carlo Option Pricing (continued)

- If we discretize $dS/S = \mu\, dt + \sigma\, dW$ directly, we will obtain

$$S_{i+1} = S_i + S_i \mu\, \Delta t + S_i \sigma \sqrt{\Delta t}\, \xi.$$

- But this is locally normally distributed, not lognormally, hence biased.[a]

- In practice, this is not expected to be a major problem as long as Δt is sufficiently small.

[a] Contributed by Mr. Tai, Hui-Chin (R97723028) on April 22, 2009.

Monte Carlo Option Pricing (continued)

- Non-dividend-paying stock prices in a risk-neutral economy can be generated by setting $\mu = r$ and $\Delta t = T$.

$C := 0$; {Accumulated terminal option value.}
for $i = 1, 2, 3, \ldots, m$ **do**
    $P := S \times e^{(r - \sigma^2/2)\, T + \sigma \sqrt{T}\, \xi}$, $\xi \sim N(0, 1)$;
    $C := C + \max(P - X, 0)$;
end for
`return` Ce^{-rT}/m;

Monte Carlo Option Pricing (concluded)

- Pricing Asian options is also easy.

$C := 0;$
for $i = 1, 2, 3, \ldots, m$ **do**
    $P := S; \; M := S;$
    for $j = 1, 2, 3, \ldots, n$ **do**
        $P := P \times e^{(r - \sigma^2/2)(T/n) + \sigma \sqrt{T/n}\, \xi};$
        $M := M + P;$
    end for
    $C := C + \max(M/(n+1) - X, 0);$
end for
`return` $Ce^{-rT}/m;$

How about American Options?

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise (why?).
- It is difficult to determine the early-exercise point based on one single path.
- But Monte Carlo simulation can be modified to price American options with small biases (pp. 743ff).[a]

[a] Longstaff and Schwartz (2001).

Delta and Common Random Numbers

- In estimating delta, it is natural to start with the finite-difference estimate
$$e^{-r\tau}\, \frac{E[\,P(S+\epsilon)\,] - E[\,P(S-\epsilon)\,]}{2\epsilon}.$$
  - $P(x)$ is the terminal payoff of the derivative security when the underlying asset's initial price equals x.
- Use simulation to estimate $E[\,P(S+\epsilon)\,]$ first.
- Use another simulation to estimate $E[\,P(S-\epsilon)\,]$.
- Finally, apply the formula to approximate the delta.
- This is also called the bump-and-revalue method.

Delta and Common Random Numbers (concluded)

- This method is not recommended because of its high variance.
- A much better approach is to use common random numbers to lower the variance:

$$e^{-r\tau}\, E\left[\, \frac{P(S+\epsilon)-P(S-\epsilon)}{2\epsilon} \,\right].$$

- Here, the *same* random numbers are used for $P(S+\epsilon)$ and $P(S-\epsilon)$.
- This holds for gamma and cross gammas (for multivariate derivatives).

Problems with the Bump-and-Revalue Method

- Consider the binary option with payoff

$$\begin{cases} 1, & \text{if } S(T) > X, \\ 0, & \text{otherwise.} \end{cases}$$

- Then

$$P(S+\epsilon) - P(S-\epsilon) = \begin{cases} 1, & \text{if } P(S+\epsilon) > X \text{ and} \\ & \quad P(S-\epsilon)\,] < X\ , \\ 0, & \text{otherwise.} \end{cases}$$

- So the finite-difference estimate per run for the (undicsounted) delta is 0 or $O(1/\epsilon)$.
- This means high variance.

Gamma

- The finite-difference formula for gamma is

$$e^{-r\tau}\, E\left[\,\frac{P(S+\epsilon)-2\times P(S)+P(S-\epsilon)}{\epsilon^2}\,\right].$$

- For a correlation option with multiple underlying assets, the finite-difference formula for the cross gamma $\partial^2 P(S_1, S_2, \ldots)/(\partial S_1 \partial S_2)$ is:

$$e^{-r\tau}\, E\left[\,\frac{P(S_1+\epsilon_1, S_2+\epsilon_2)-P(S_1-\epsilon_1, S_2+\epsilon_2)}{4\epsilon_1\epsilon_2}\right.$$

$$\left.\frac{-P(S_1+\epsilon_1, S_2-\epsilon_2)+P(S_1-\epsilon_1, S_2-\epsilon_2)}{}\,\right].$$

Gamma (continued)

- Choosing an ϵ of the right magnitude can be challenging.
  - If ϵ is too large, inaccurate Greeks result.
  - If ϵ is too small, unstable Greeks result.
- This phenomenon is sometimes called the curse of differentiation.

Gamma (continued)

- In general, suppose

$$\frac{\partial^i}{\partial \theta^i} e^{-r\tau} E[\, P(S)\,] = e^{-r\tau} E\left[\, \frac{\partial^i P(S)}{\partial \theta^i} \,\right]$$

  holds for all $i > 0$, where θ is a parameter of interest.

- Then formulas for the Greeks become integrals.

- As a result, we avoid ϵ, finite differences, and resimulation.

Gamma (concluded)

- This is indeed possible for a broad class of payoff functions.[a]
  - Roughly speaking, any payoff function that is equal to a sum of products of differentiable functions and indicator functions with the right kind of support.
  - For example, the payoff of a call is

$$\max(S(T) - X, 0) = (S(T) - X) I_{\{ S(T) - X \geq 0 \}}.$$

  - The results are too technical to cover here.

[a]Teng (R91723054) (2004) and Lyuu and Teng (R91723054) (2011).

Biases in Pricing Continuously Monitored Options with Monte Carlo

- We are asked to price a continuously monitored up-and-out call with barrier H.
- The Monte Carlo method samples the stock price at n discrete time points $t_1, t_2, \ldots, t_n$.
- A sample path $S(t_0), S(t_1), \ldots, S(t_n)$ is produced.
  - Here, $t_0 = 0$ is the current time, and $t_n = T$ is the expiration time of the option.

Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

- If all of the sampled prices are below the barrier, this sample path pays $\max(S(t_n) - X, 0)$.
- Repeating these steps and averaging the payoffs yield a Monte Carlo estimate.

$C := 0;$

for $i = 1, 2, 3, \ldots, m$ **do**

  $P := S;$ hit $:= 0;$

  for $j = 1, 2, 3, \ldots, n$ **do**

    $P := P \times e^{(r - \sigma^2/2)\,(T/n) + \sigma \sqrt{(T/n)}\,\xi};$

    if $P \geq H$ **then**

      hit $:= 1;$

      `break`;

    end if

  end for

  if hit $= 0$ **then**

    $C := C + \max(P - X, 0);$

  end if

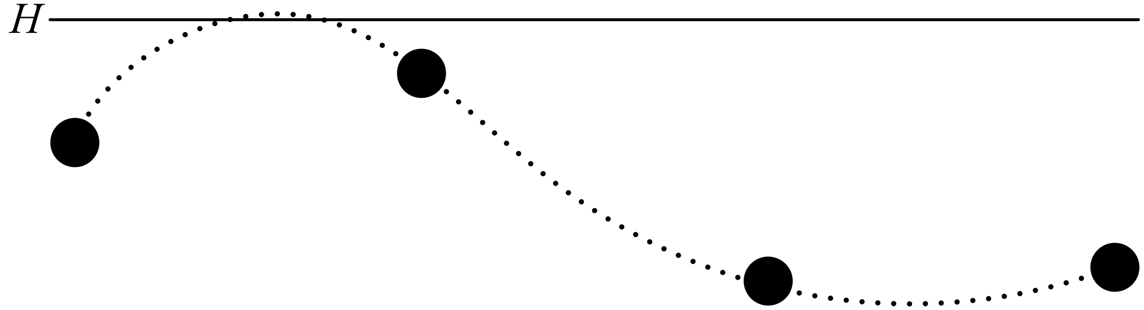
end for

`return` $C e^{-rT}/m;$

Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

- This estimate is biased.[a]
  - Suppose none of the sampled prices on a sample path equals or exceeds the barrier H.
  - It remains possible for the continuous sample path that passes through them to hit the barrier between sampled time points (see plot on next page).

[a]Shevchenko (2003).


H

Biases in Pricing Continuously Monitored Options with Monte Carlo (concluded)

- The bias can certainly be lowered by increasing the number of observations along the sample path.
- However, even daily sampling may not suffice.
- The computational cost also rises as a result.

Brownian Bridge Approach to Pricing Barrier Options

- We desire an unbiased estimate which can be calculated efficiently.
- The above-mentioned payoff should be multiplied by the probability p that a continuous sample path does *not* hit the barrier conditional on the sampled prices.
- This methodology is called the Brownian bridge approach.
- Formally, we have

$$p \equiv \text{Prob}[\, S(t) < H, 0 \leq t \leq T \mid S(t_0), S(t_1), \ldots, S(t_n)\,].$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

- As a barrier is hit over a time interval if and only if the maximum stock price over that period is at least H,

$$p = \text{Prob}\left[\max_{0 \leq t \leq T} S(t) < H \mid S(t_0), S(t_1), \ldots, S(t_n)\right].$$

- Luckily, the conditional distribution of the maximum over a time interval given the beginning and ending stock prices is known.

Brownian Bridge Approach to Pricing Barrier Options (continued)

Lemma 21 *Assume* S *follows* $dS/S = \mu\, dt + \sigma\, dW$ *and define*

$$\zeta(x) \equiv \exp\left[-\frac{2 \ln(x/S(t)) \ln(x/S(t+\Delta t))}{\sigma^2 \Delta t} \right].$$

(1) If $H > \max(S(t), S(t+\Delta t))$*, then*

$$\text{Prob}\left[\max_{t \leq u \leq t+\Delta t} S(u) < H \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(H).$$

(2) If $h < \min(S(t), S(t+\Delta t))$*, then*

$$\text{Prob}\left[\min_{t \leq u \leq t+\Delta t} S(u) > h \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(h).$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

- Lemma 21 gives the probability that the barrier is not hit in a time interval, given the starting and ending stock prices.
- For our up-and-out call, choose $n = 1$.
- As a result,

$$p = \begin{cases} 1 - \exp\left[-\frac{2 \ln(H/S(0)) \ln(H/S(T))}{\sigma^2 T} \right], & \text{if } H > \max(S(0), S(T)), \\ 0, & \text{otherwise.} \end{cases}$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

$C := 0;$

for $i = 1, 2, 3, \ldots, m$ **do**

  $P := S \times e^{(r-q-\sigma^2/2)\,T+\sigma\sqrt{T}\;\xi(\,)};$

  if $(S < H\ \text{ and }\ P < H)$ or $(S > H\ \text{ and }\ P > H)$ **then**

    $C := C + \max(P - X, 0) \times \left\{ 1 - \exp\left[-\frac{2\ln(H/S) \times \ln(H/P)}{\sigma^2 T} \right] \right\};$

  end if

end for

`return` $Ce^{-rT}/m;$

Brownian Bridge Approach to Pricing Barrier Options (concluded)

- The idea can be generalized.
- For example, we can handle more complex barrier options.
- Consider an up-and-out call with barrier H_i for the time interval $(t_i, t_{i+1}]$, $0 \leq i < n$.
- This option thus contains n barriers.
- Multiply the probabilities for the n time intervals to obtain the desired probability adjustment term.

Variance Reduction

- The statistical efficiency of Monte Carlo simulation can be measured by the variance of its output.
- If this variance can be lowered without changing the expected value, fewer replications are needed.
- Methods that improve efficiency in this manner are called variance-reduction techniques.
- Such techniques become practical when the added costs are outweighed by the reduction in sampling.

Variance Reduction: Antithetic Variates

- We are interested in estimating $E[\,g(X_1, X_2, \ldots, X_n)\,]$, where $X_1, X_2, \ldots, X_n$ are independent.
- Let Y_1 and Y_2 be random variables with the same distribution as $g(X_1, X_2, \ldots, X_n)$.
- Then

$$\mathrm{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{\mathrm{Var}[\,Y_1\,]}{2} + \frac{\mathrm{Cov}[\,Y_1, Y_2\,]}{2}.$$

  – $\mathrm{Var}[\,Y_1\,]/2$ is the variance of the Monte Carlo method with two independent replications.
- The variance $\mathrm{Var}[\,(Y_1 + Y_2)/2\,]$ is smaller than $\mathrm{Var}[\,Y_1\,]/2$ when Y_1 and Y_2 are negatively correlated.

Variance Reduction: Antithetic Variates (continued)

- For each simulated sample path X, a second one is obtained by *reusing* the random numbers on which the first path is based.
- This yields a second sample path Y.
- Two estimates are then obtained: One based on X and the other on Y.
- If N independent sample paths are generated, the antithetic-variates estimator averages over $2N$ estimates.

Variance Reduction: Antithetic Variates (continued)

- Consider process $dX = a_t\, dt + b_t \sqrt{dt}\; \xi$.
- Let g be a function of n samples $X_1, X_2, \ldots, X_n$ on the sample path.
- We are interested in $E[\, g(X_1, X_2, \ldots, X_n)\,]$.
- Suppose one simulation run has realizations $\xi_1, \xi_2, \ldots, \xi_n$ for the normally distributed fluctuation term ξ.
- This generates samples $x_1, x_2, \ldots, x_n$.
- The estimate is then $g(\boldsymbol{x})$, where $\boldsymbol{x} \equiv (x_1, x_2 \ldots, x_n)$.

Variance Reduction: Antithetic Variates (concluded)

- The antithetic-variates method does not sample n more numbers from ξ for the second estimate $g(\boldsymbol{x}')$.
- Instead, generate the sample path $\boldsymbol{x}' \equiv (x'_1, x'_2 \ldots, x'_n)$ from $-\xi_1, -\xi_2, \ldots, -\xi_n$.
- Compute $g(\boldsymbol{x}')$.
- Output $(g(\boldsymbol{x}) + g(\boldsymbol{x}'))/2$.
- Repeat the above steps for as many times as required by accuracy.

Variance Reduction: Conditioning

- We are interested in estimating $E[X]$.
- Suppose here is a random variable Z such that $E[X \mid Z = z]$ can be efficiently and precisely computed.
- $E[X] = E[E[X \mid Z]]$ by the law of iterated conditional expectations.
- Hence the random variable $E[X \mid Z]$ is also an unbiased estimator of $E[X]$.

Variance Reduction: Conditioning (concluded)

- As
$$\text{Var}[\,E[\,X \mid Z\,]\,] \leq \text{Var}[\,X\,],$$
$E[\,X \mid Z\,]$ has a smaller variance than observing X directly.
- First obtain a random observation z on Z.
- Then calculate $E[\,X \mid Z = z\,]$ as our estimate.
  - There is no need to resort to simulation in computing $E[\,X \mid Z = z\,]$.
- The procedure can be repeated a few times to reduce the variance.

Control Variates

- Use the analytic solution of a similar yet simpler problem to improve the solution.
- Suppose we want to estimate $E[\,X\,]$ and there exists a random variable Y with a known mean $\mu \equiv E[\,Y\,]$.
- Then $W \equiv X + \beta(Y - \mu)$ can serve as a "controlled" estimator of $E[\,X\,]$ for any constant β.
  - However β is chosen, W remains an unbiased estimator of $E[\,X\,]$ as

$$E[\,W\,] = E[\,X\,] + \beta E[\,Y - \mu\,] = E[\,X\,].$$

Control Variates (continued)

- Note that
$$\mathrm{Var}[\,W\,] = \mathrm{Var}[\,X\,] + \beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,], \tag{83}$$
- Hence W is less variable than X if and only if
$$\beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,] < 0. \tag{84}$$

Control Variates (concluded)

- The success of the scheme clearly depends on both β and the choice of Y.
  - For example, arithmetic average-rate options can be priced by choosing Y to be the otherwise identical geometric average-rate option's price and $\beta = -1$.
- This approach is much more effective than the antithetic-variates method.

Choice of Y

- In general, the choice of Y is ad hoc, and experiments must be performed to confirm the wisdom of the choice.
- Try to match calls with calls and puts with puts.[a]
- On many occasions, Y is a discretized version of the derivative that gives μ.
  - Discretely monitored geometric average-rate option vs. the continuously monitored geometric average-rate option given by formulas (32) on p. 355.
- For some choices, the discrepancy can be significant, such as the lookback option.[b]

[a]Contributed by Ms. Teng, Huei-Wen (R91723054) on May 25, 2004.
[b]Contributed by Mr. Tsai, Hwai (R92723049) on May 12, 2004.

Optimal Choice of β

- Equation (83) on p. 722 is minimized when

$$\beta = -\text{Cov}[\,X, Y\,]/\text{Var}[\,Y\,].$$

  - It is called beta in the book.

- For this specific β,

$$\text{Var}[\,W\,] = \text{Var}[\,X\,] - \frac{\text{Cov}[\,X, Y\,]^2}{\text{Var}[\,Y\,]} = \left(1 - \rho_{X,Y}^2\right) \text{Var}[\,X\,],$$

  where $\rho_{X,Y}$ is the correlation between X and Y.

- Note that the variance can never be increased with the optimal choice.

Optimal Choice of β (continued)

- Furthermore, the stronger X and Y are correlated, the greater the reduction in variance.
- For example, if this correlation is nearly perfect (± 1), we could control X almost exactly.
- Typically, neither $\mathrm{Var}[\, Y\,]$ nor $\mathrm{Cov}[\, X, Y\,]$ is known.
- Therefore, we cannot obtain the maximum reduction in variance.
- We can guess these values and hope that the resulting W does indeed have a smaller variance than X.

Optimal Choice of β (continued)

- A second possibility is to use the simulated data to estimate these quantities.
  - How to do it efficiently in terms of time and space?
- Observe that $-\beta$ has the same sign as the correlation between X and Y.
- Hence, if X and Y are positively correlated, $\beta < 0$, then X is adjusted downward whenever $Y > \mu$ and upward otherwise.
- The opposite is true when X and Y are negatively correlated, in which case $\beta > 0$.

Optimal Choice of β (concluded)

- Suppose a suboptimal $\beta + \epsilon$ is used instead.
- The variance increases by only $\epsilon^2 \text{Var}[\,Y\,]$.[a]

[a] Han and Lai (2010).

A Pitfall

- A potential pitfall is to sample X and Y independently.
- In this case, $\text{Cov}[\,X, Y\,] = 0$.
- Equation (83) on p. 722 becomes

$$\text{Var}[\,W\,] = \text{Var}[\,X\,] + \beta^2 \,\text{Var}[\,Y\,].$$

- So whatever Y is, the variance is *increased*!
- Lesson: X and Y must be correlated.

Problems with the Monte Carlo Method

- The error bound is only probabilistic.
- The probabilistic error bound of $\sqrt{N}$ does not benefit from regularity of the integrand function.
- The requirement that the points be independent random samples are wasteful because of clustering.
- In reality, pseudorandom numbers generated by completely deterministic means are used.
- Monte Carlo simulation exhibits a great sensitivity on the seed of the pseudorandom-number generator.

Matrix Computation

To set up a philosophy against physics is rash;
philosophers who have done so
have always ended in disaster.
— Bertrand Russell

Definitions and Basic Results

- Let $A \equiv [\, a_{ij} \,]_{1 \leq i \leq m, 1 \leq j \leq n}$, or simply $A \in \boldsymbol{R}^{m \times n}$, denote an $m \times n$ matrix.
- It can also be represented as $[\, a_1, a_2, \ldots, a_n \,]$ where $a_i \in \boldsymbol{R}^m$ are vectors.
  - Vectors are column vectors unless stated otherwise.
- A is a square matrix when $m = n$.
- The rank of a matrix is the largest number of linearly independent columns.

Definitions and Basic Results (continued)

- A square matrix A is said to be symmetric if $A^{\mathrm{T}} = A$.
- A real $n \times n$ matrix

$$A \equiv [\, a_{ij} \,]_{i,j}$$

  is diagonally dominant if $|\, a_{ii} \,| > \sum_{j \neq i} |\, a_{ij} \,|$ for $1 \leq i \leq n$.

  – Such matrices are nonsingular.
- The identity matrix is the square matrix

$$I \equiv \mathrm{diag}[\, 1, 1, \ldots, 1 \,].$$

Definitions and Basic Results (concluded)

- A matrix has full column rank if its columns are linearly independent.
- A real symmetric matrix A is positive definite if

$$x^{\mathrm{T}} A x = \sum_{i,j} a_{ij} x_i x_j > 0$$

  for any nonzero vector x.
- A matrix A is positive definite if and only if there exists a matrix W such that $A = W^{\mathrm{T}} W$ and W has full column rank.

Cholesky Decomposition

- Positive definite matrices can be factored as

$$A = LL^{\mathrm{T}},$$

called the Cholesky decomposition.

  – Above, L is a lower triangular matrix.

Generation of Multivariate Distribution

- Let $\boldsymbol{x} \equiv [\, x_1, x_2, \ldots, x_n \,]^{\mathrm{T}}$ be a vector random variable with a positive definite covariance matrix C.
- As usual, assume $E[\, \boldsymbol{x} \,] = \mathbf{0}$.
- This covariance structure can be matched by $P\boldsymbol{y}$.
  - $C = PP^{\mathrm{T}}$ is the Cholesky decomposition of C.[a]
  - $\boldsymbol{y} \equiv [\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$ is a vector random variable with a covariance matrix equal to the identity matrix.

[a]What if C is not positive definite? See Lai (`R93942114`) and Lyuu (2007).

Generation of Multivariate Normal Distribution

- Suppose we want to generate the multivariate normal distribution with a covariance matrix $C = PP^{\mathrm{T}}$.
  - First, generate independent standard normal distributions $y_1, y_2, \ldots, y_n$.
  - Then
    $$P[\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$$
    has the desired distribution.
  - These steps can then be repeated.

Multivariate Derivatives Pricing

- Generating the multivariate normal distribution is essential for the Monte Carlo pricing of multivariate derivatives (pp. 648ff).

- For example, the rainbow option on k assets has payoff

$$\max(\max(S_1, S_2, \ldots, S_k) - X, 0)$$

  at maturity.

- The closed-form formula is a multi-dimensional integral.[a]

[a] Johnson (1987); Chen (D95723006) and Lyuu (2009).

Multivariate Derivatives Pricing (concluded)

- Suppose $dS_j/S_j = r\,dt + \sigma_j\,dW_j$, $1 \leq j \leq k$, where C is the correlation matrix for $dW_1, dW_2, \ldots, dW_k$.
- Let $C = PP^{\mathrm{T}}$.
- Let ξ consist of k independent random variables from $N(0,1)$.
- Let $\xi' = P\xi$.
- Similar to Eq. (82) on p. 690,

$$S_{i+1} = S_i e^{(r-\sigma_j^2/2)\,\Delta t + \sigma_j \sqrt{\Delta t}\;\xi'_j}, \quad 1 \leq j \leq k.$$