Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

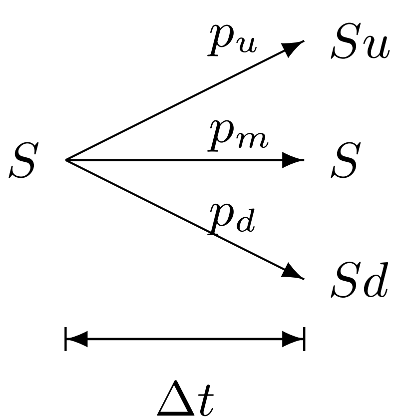
$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &= (p_u u + p_m + (p_d/u))\,S, \\ S^2V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2. \end{aligned}$$

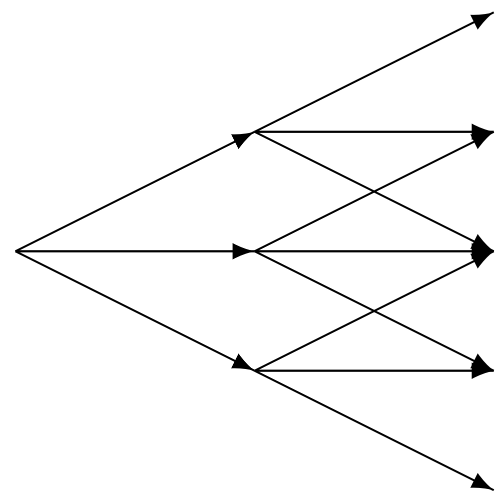
[a] Boyle (1988).

- Above,

$$
\begin{aligned}
M &\equiv e^{r\Delta t}, \\
V &\equiv M^2(e^{\sigma^2\Delta t} - 1),
\end{aligned}
$$

by Eqs. (21) on p. 154.


p_u
Su
S
p_m
S
p_d
Sd
Δt



Trinomial Tree (concluded)

- Use linear algebra to verify that

$$
\begin{aligned}
p_u &= \frac{u\left(V+M^2-M\right)-(M-1)}{(u-1)\left(u^2-1\right)},\\
p_d &= \frac{u^2\left(V+M^2-M\right)-u^3(M-1)}{(u-1)\left(u^2-1\right)}.
\end{aligned}
$$

  – In practice, we must also make sure the probabilities lie between 0 and 1.

- Countless variations.

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$
\begin{aligned}
p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\
p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

---

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.

  – This is because $Se^{-h\lambda\sigma\sqrt{\Delta t}} = H$.

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.[a]
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

  - Such a λ may not exist for very small n's.
  - This is not hard to check.

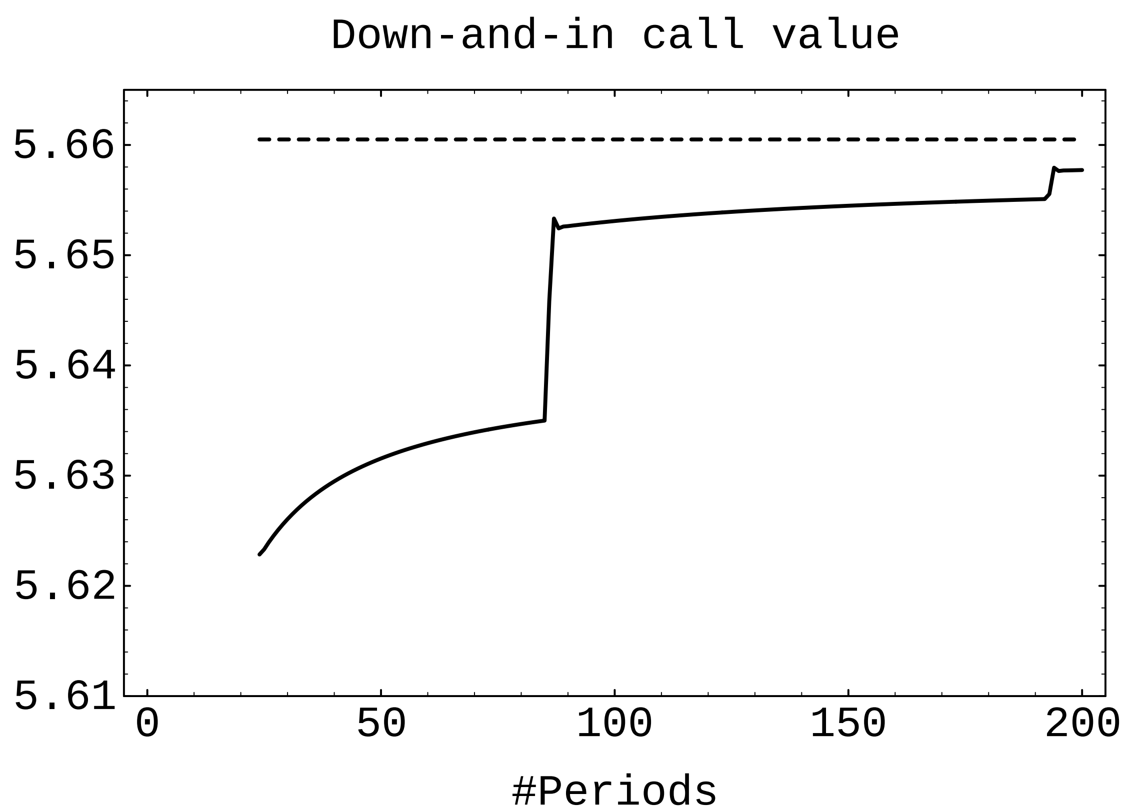
[a] Why must $\lambda \geq 1$?

Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.
- The following probabilities may be used,

$$
\begin{aligned}
p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\
p_m &= 1 - \frac{1}{\lambda^2}, \\
p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

  - $\mu' \equiv r - \sigma^2/2$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not n.[b]

[a] Lyuu (1998).
[b] Patterson and Hennessy (1994).

Algorithms Comparison (continued)

- Pages 337 and 607 seem to show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But does it make the trinomial model better then?

Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 596.
- The barrier-too-close problem is also too hard for a quadratic-time trinomial tree algorithm.[a]
- In fact, the trinomial model also has a linear-time algorithm![b]

---

[a] Lyuu (1998).
[b] Chen (`R94922003`) (2007).

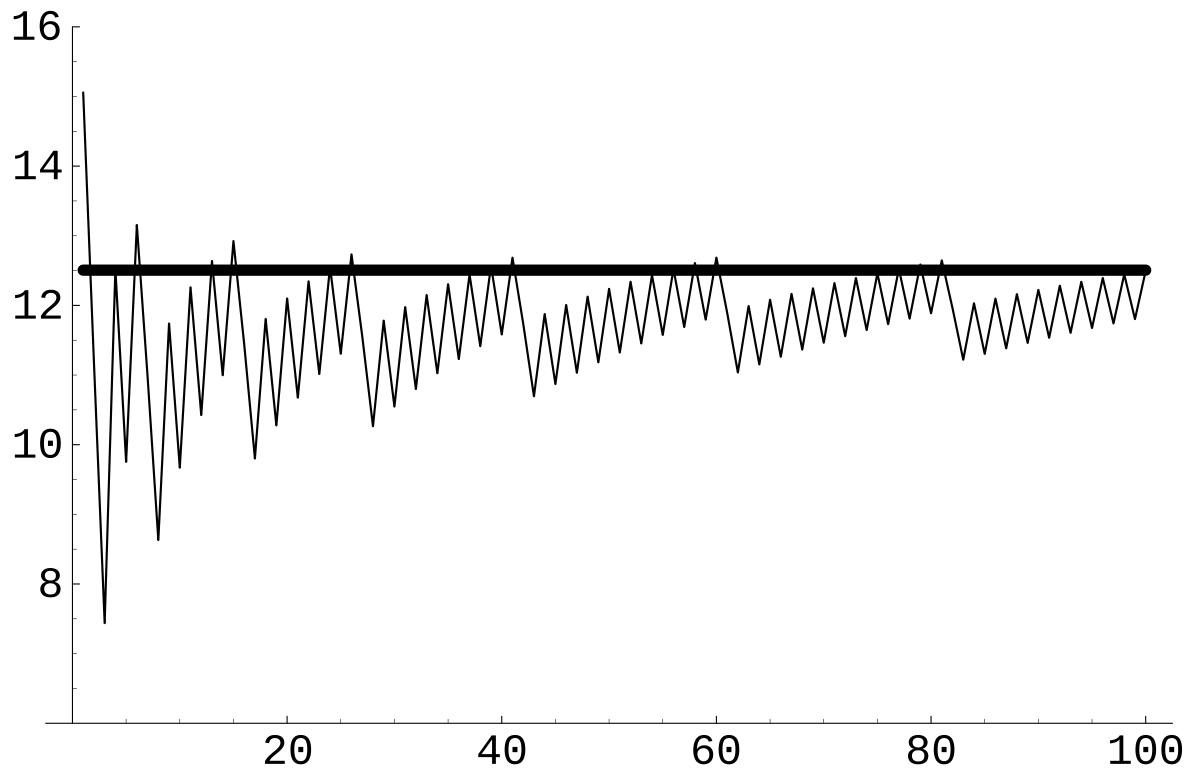
n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot on next page).[a]
- The combinatorial method gives a linear-time algorithm (see text).

[a] Chao (`R86526053`) (1999); Dai (`R86526008`, `D8852600`) and Lyuu (2005).


16
14
12
10
8
20
40
60
80
100

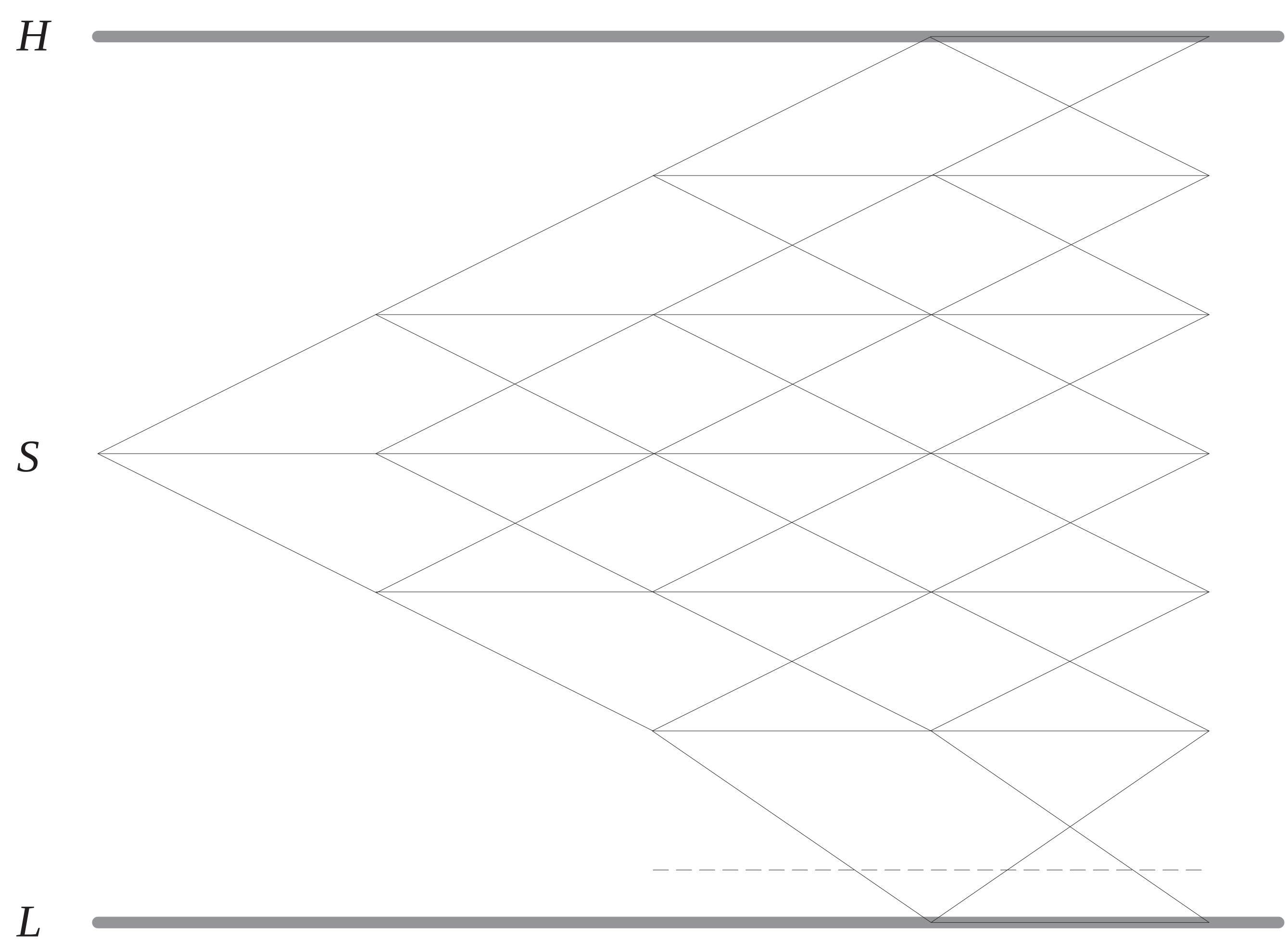
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995).

[b] Hsu (R7526001, D89922012) and Lyuu (2006). Dai (R86526008, D8852600) and Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 619ff).


H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.[a]

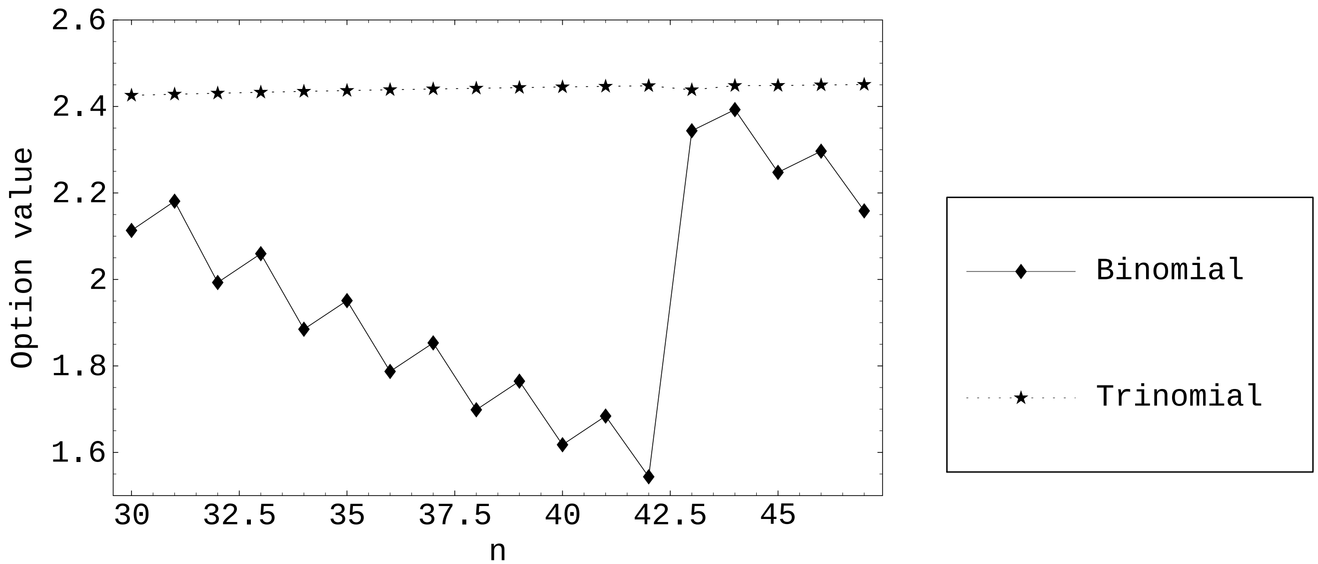
---

[a]You probably can do the same thing for binomial models. But the benefits are most likely nil (why?). Thanks to a lively discussion on April 25, 2012.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying
$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$
  - The prices between the barriers are
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$
- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are
$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$
where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



The Binomial-Trinomial Tree

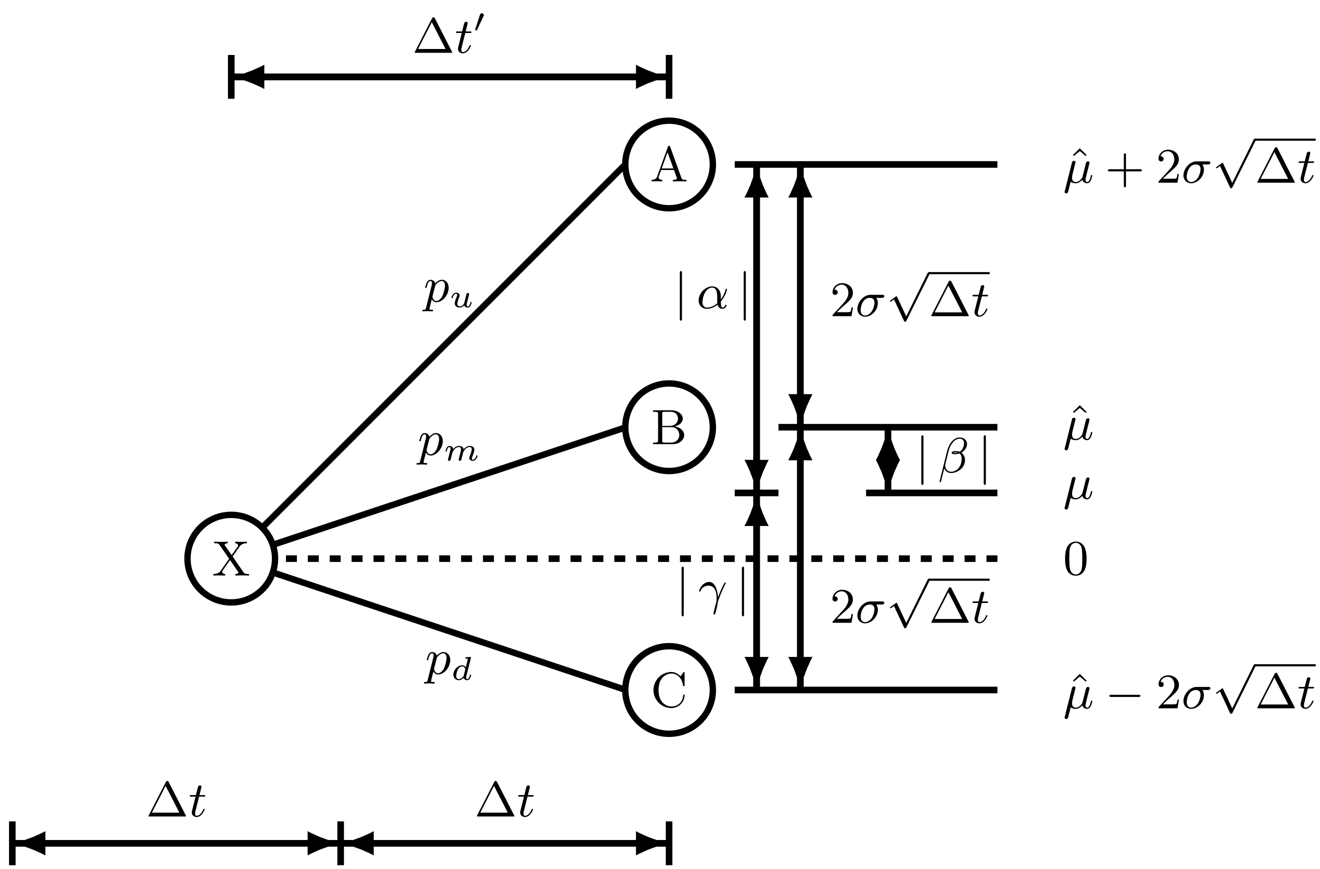
- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.[a]
- The resulting tree is called the binomial-trinomial tree.[b]
- Suppose a binomial tree will be built with Δt as the duration of one period.
- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - $\Delta t \leq \Delta t' < 2\Delta t$.

---

[a] Dai (`R86526008`, `D8852600`) and Lyuu (2006, 2008, 2010).

[b] The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)



The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.
- Let S be the stock price at node X.
- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall (p. 259, e.g.) that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t + \Delta t'$ equals

$$\mu \equiv \left(r - \sigma^2/2\right) \Delta t'. \tag{66}$$

- Its variance equals

$$\text{Var} \equiv \sigma^2 \Delta t'. \tag{67}$$

- Let node B be the node whose logarithmic return $\hat{\mu} \equiv \ln(s(\text{B})/S)$ is closest to μ among all the nodes on the binomial tree at time $t + \Delta t'$.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.
- Review the figure on p. 620 for illustration.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.
- Let $\hat{\mu} \equiv \ln(s(B)/S)$ be the logarithmic return of the middle node B.
- Also, let α, β, and γ be the differences between μ and the logarithmic returns $\ln(s(\mathrm{Z})/S)$ of nodes $\mathrm{Z} = \mathrm{A}, \mathrm{B}, \mathrm{C}$, in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}\,, \tag{68}$$

$$\beta \equiv \hat{\mu} - \mu, \tag{69}$$

$$\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}\,. \tag{70}$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \tag{71}$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \text{Var}, \tag{72}$$

$$p_u + p_m + p_d = 1. \tag{73}$$

The Binomial-Trinomial Tree (concluded)

- Equation (71) matches the mean (66) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 622.
- Equation (72) matches its variance (67) on p. 622.
- The three probabilities can be proved to lie between 0 and 1.

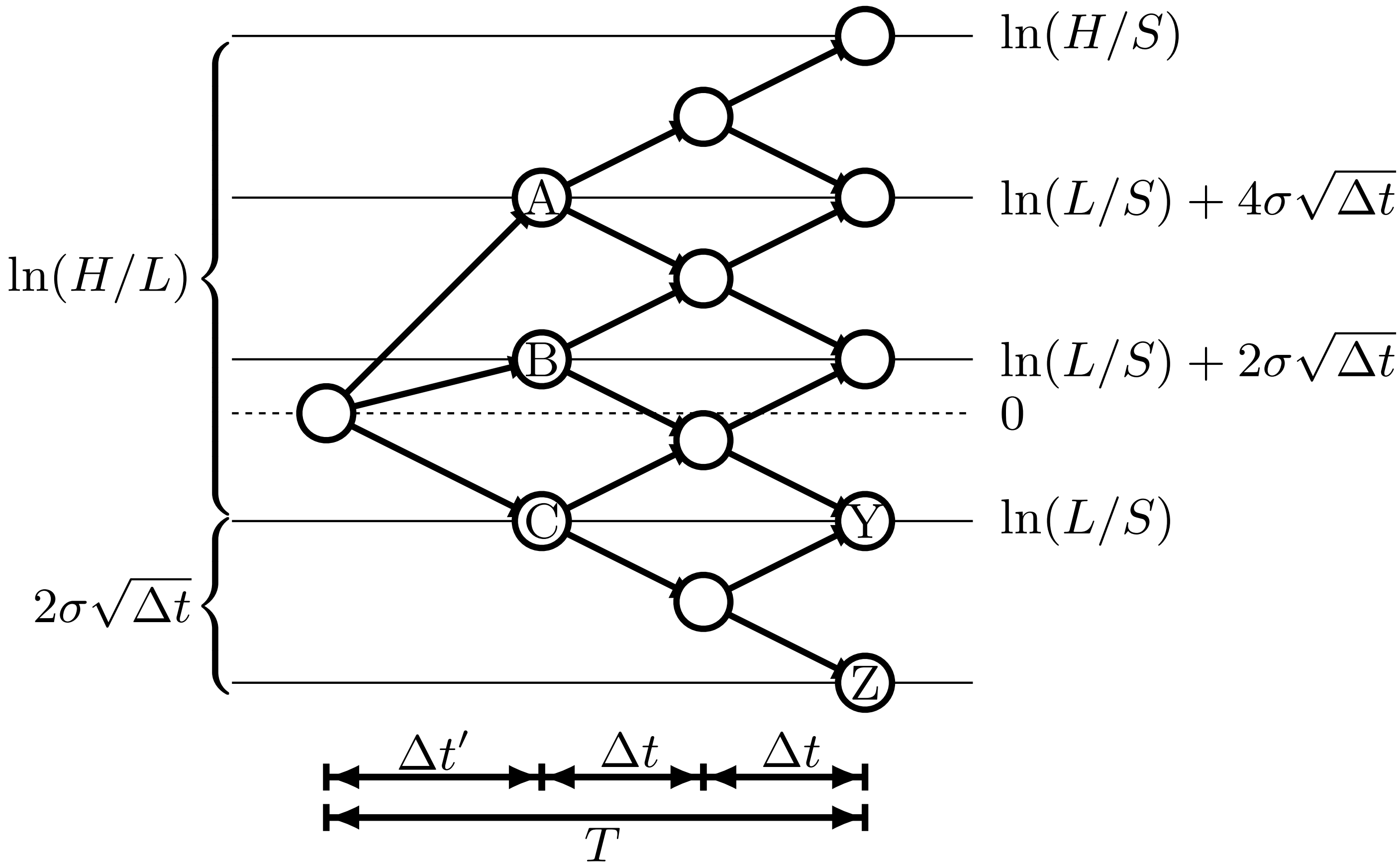
Pricing Double-Barrier Options

- Consider a double-barrier option with two barriers L and H, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- This means choosing a Δt such that

$$\kappa \equiv \frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$$

  is a positive integer.
  - The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 628 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)



Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta\tau \equiv T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.
- So we pick a Δt that is close to, but does not exceed, $\Delta\tau$ and makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ an integer.
- Specifically, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma}\right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier L upward and downward.
- Automatically, a layer coincides with the high barrier H.
- It is unlikely that Δt divides T, however.
- As a consequence, the position at time 0 and with logarithmic return $\ln(S/S) = 0$ is not occupied by a binomial node to serve as the root node.

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.
- Between time 0 and time T, the binomial tree spans $T/\Delta t$ periods.
- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.
- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.
- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.
- Calculate the three branching probabilities to them.
- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.
- See the figure on p. 628 for illustration.

Pricing Double-Barrier Options (continued)

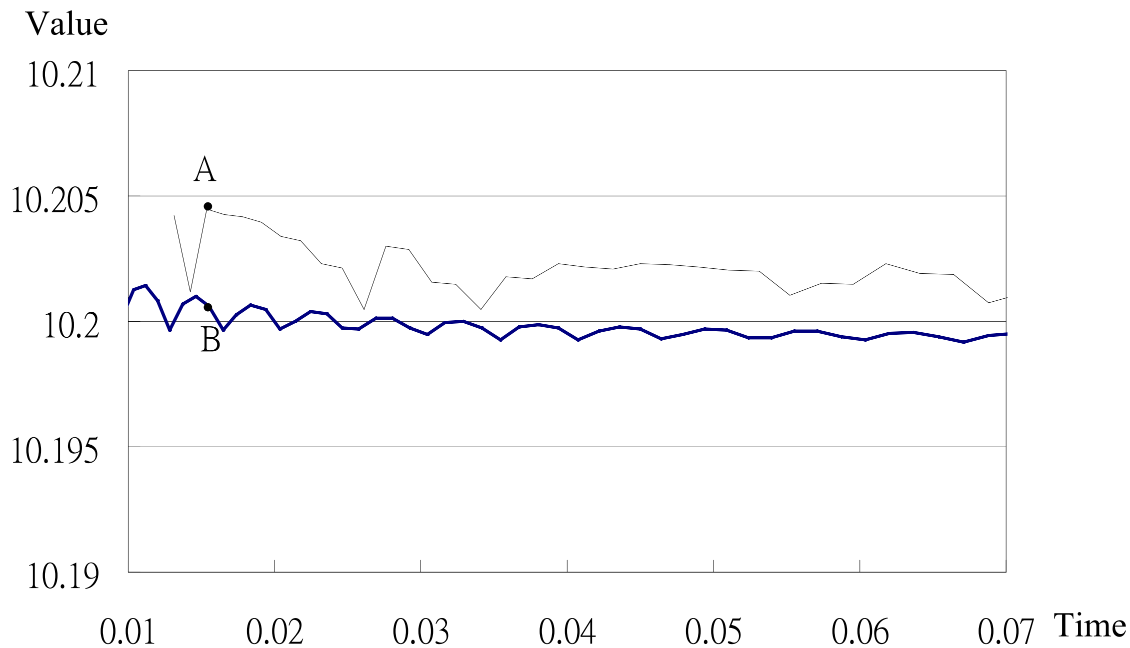
- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But we know a linear-time algorithm exists for double-barrier options on the binomial tree (see text).
- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.
  - That is, nodes A, B, and C on p. 628.
- Then calculate their expected discounted value for the root node.
- The overall running time is only linear.

Pricing Double-Barrier Options (continued)

- Binomial trees have troubles with pricing barrier options (see p. 337, p. 613, and p. 618).
- Even pit against the much better trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 635).
- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[a]

[a] Lyuu and Palmer (2010).

Pricing Double-Barrier Options (concluded)



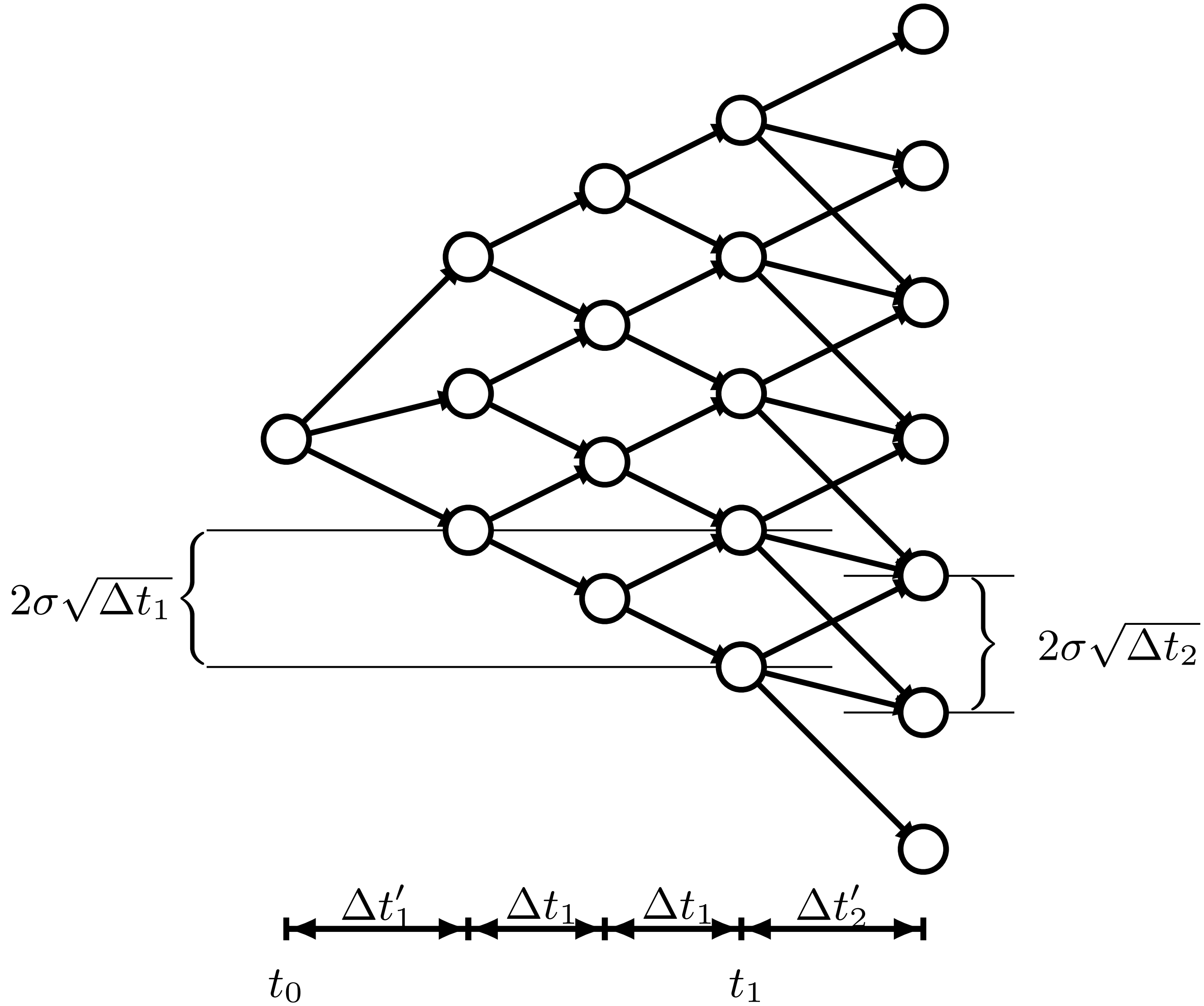
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).

Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- The main difficulty with pricing discrete barrier options lies in matching the monitored times.
- Here is why.
- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are $t_0 = 0, t_1, t_2, \ldots, t_\ell = T$.

Pricing Discrete Barrier Options (continued)

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, meaning Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.
- See p. 638.


$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t'_1$
Δt_1
Δt_1
$\Delta t'_2$
t_0
t_1

Pricing Discrete Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[\, t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]

  1. The one we saw earlier models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

---

[a] Frishling (2002).

Options on a Stock That Pays Known Dividends (continued)

- The most realistic model assumes the stock price decreases by the amount of the dividend paid at the ex-dividend date.
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But this model results in binomial trees that grow exponentially (recall p. 273).
- The binomial-trinomial tree can often avoid the exponential explosion for the known-dividends case.

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \equiv t/\Delta t$ periods between time 0 and the ex-dividend date.
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D$.
  - Equivalently,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)} \right]^2.$$

Options on a Stock That Pays Known Dividends (continued)

- Build a binomial tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \ldots ,$$

same as the binomial tree.

Options on a Stock That Pays Known Dividends (concluded)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 620.
- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

[a] Dai (R86526008, D8852600) and Lyuu (2004).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Option pricing with time-varying volatilities and time-varying barriers.[d]
- Defaultable bond pricing.[e]

---

[a] Wu (R96723058) (2009).
[b] Huang (R97922073) (2010).
[c] Huang (R97922081) (2010).
[d] Chou (R97944012) (2010) and Chen (R98922127) (2011).
[e] Dai (R86526008, D8852600), Lyuu, and Wang (F95922018) (2009, 2010).

General Properties of Trees[a]

- Consider the Ito process,

$$dX = a(X, t)\, dt + \sigma\, dW,$$

  where $a(X, t) = O(1)$ and σ is a constant.

- The mean and volatility of the next move's size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.

- Note that $\sqrt{\Delta t} \gg \Delta t$.

- The tree spacing must be in the order of $\sigma \sqrt{\Delta t}$ if the variance is to be matched.[b]

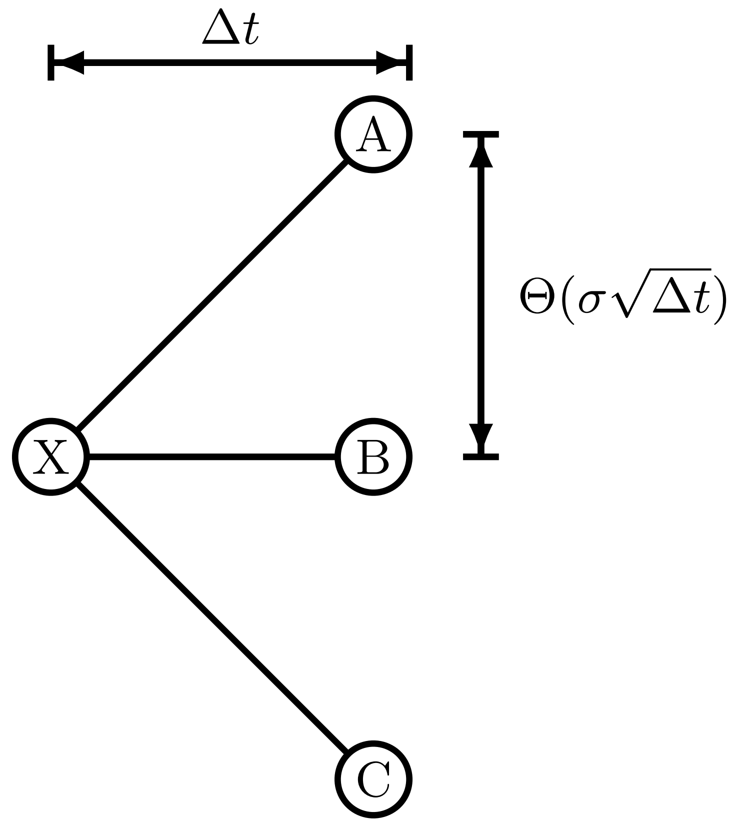
---

[a] Chiu (`R98723059`) (2012) and Wu (`R99922149`) (2012).

[b] Lyuu and Wang (`F95922018`) (2009, 2011) and Lyuu and Wen (`D94922003`) (2012).

General Properties of Trees (concluded)

- It can also be proved that either B is a tree node or both A and C are tree nodes.



Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff
$$\max\left(\sum_{i=1}^{m}\alpha_i S_i(\tau) - X, 0\right),$$
where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Multivariate Contingent Claims (concluded)

From Lyuu and Teng (R91723054) (2011):

Name	Payoff
Exchange option	$\max(S_1(\tau) - S_2(\tau), 0)$
Better-off option	$\max(S_1(\tau), \ldots, S_k(\tau), 0)$
Worst-off option	$\min(S_1(\tau), \ldots, S_k(\tau), 0)$
Binary maximum option	$I\{\max(S_1(\tau), \ldots, S_k(\tau)) > X\}$
Maximum option	$\max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Minimum option	$\max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0)$
Spread option	$\max(S_1(\tau) - S_2(\tau) - X, 0)$
Basket average option	$\max((S_1(\tau), \ldots, S_k(\tau))/k - X, 0)$
Multi-strike option	$\max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0)$
Pyramid rainbow option	$\max(\mid S_1(\tau) - X_1 \mid + \cdots + \mid S_k(\tau) - X_k \mid - X, 0)$
Madonna option	$\max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0)$

Correlated Trinomial Model[a]

- Two risky assets S_1 and S_2 follow $dS_i/S_i = r\,dt + \sigma_i\,dW_i$ in a risk-neutral economy, $i = 1, 2$.
- Let

$$
\begin{aligned}
M_i &\equiv e^{r\Delta t}, \\
V_i &\equiv M_i^2(e^{\sigma_i^2 \Delta t} - 1).
\end{aligned}
$$

  - $S_i M_i$ is the mean of S_i at time Δt.
  - $S_i^2 V_i$ the variance of S_i at time Δt.

[a] Boyle, Evnine, and Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is (as usual, we impose $u_i d_i = 1$)

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$
\begin{aligned}
1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\
S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\
S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2.
\end{aligned}
$$

Correlated Trinomial Model (concluded)

- Let $R \equiv M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$.
- Match the variances and covariance:

$$\begin{aligned}
S_1^2 V_1 &= (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5(S_1^2 - (S_1 M_1)^2) \\
&\quad + (p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2), \\
S_2^2 V_2 &= (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5(S_2^2 - (S_2 M_2)^2) \\
&\quad + (p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2), \\
S_1 S_2 R &= (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5)\, S_1 S_2.
\end{aligned}$$

- The solutions are complex (see text).

Correlated Trinomial Model Simplified[a]

- Let $\mu_i' \equiv r - \sigma_i^2/2$ and $u_i \equiv e^{\lambda\sigma_i\sqrt{\Delta t}}$ for $i = 1, 2$.
- The following simpler scheme is good enough:

$$\begin{aligned}
p_1 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_2 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_3 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} - \frac{\mu_2'}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_4 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1'}{\sigma_1} + \frac{\mu_2'}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_5 &= 1 - \frac{1}{\lambda^2}.
\end{aligned}$$

[a]Madan, Milne, and Shefrin (1989).

Correlated Trinomial Model Simplified (continued)

- All of the probabilities lie between 0 and 1 if and only if

$$-1+\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}+\frac{\mu_2'}{\sigma_2}\right| \leq \rho \leq 1-\lambda\sqrt{\Delta t}\left|\frac{\mu_1'}{\sigma_1}-\frac{\mu_2'}{\sigma_2}\right|, \tag{74}$$

$$1 \leq \lambda \tag{75}$$

- We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as

$$-1+O(\sqrt{\Delta t}) < \rho < 1-O(\sqrt{\Delta t}),$$

such as the above one.[a]

[a]Kao (R98922093) (2011) and Kao (R98922093), Lyuu, and Wen (D94922003) (2012).

Correlated Trinomial Model Simplified (concluded)

- But this model cannot price 2-asset 2-barrier options accurately.[a]
- Few multivariate trees are both optimal and able to handle multiple barriers.[b]
- An alternative is to use orthogonalization.[c]

[a]See Chang, Hsu, and Lyuu (2006) and Kao (`R98922093`), Lyuu and Wen (`D94922003`) (2012) for solutions.

[b]See Kao (`R98922093`), Lyuu, and Wen (`D94922003`) (2012) for one.

[c]Hull and White (1990) and Dai (`R86526008`, `D8852600`), Lyuu, and Wang (`F95922018`) (2012).

Extrapolation

- It is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{76}$$

- Assume c is an unknown constant independent of n.
  - Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and ignoring the smaller terms,

$$\begin{aligned} f(n_1) &= f + \frac{c}{n_1}, \\ f(n_2) &= f + \frac{c}{n_2}. \end{aligned}$$

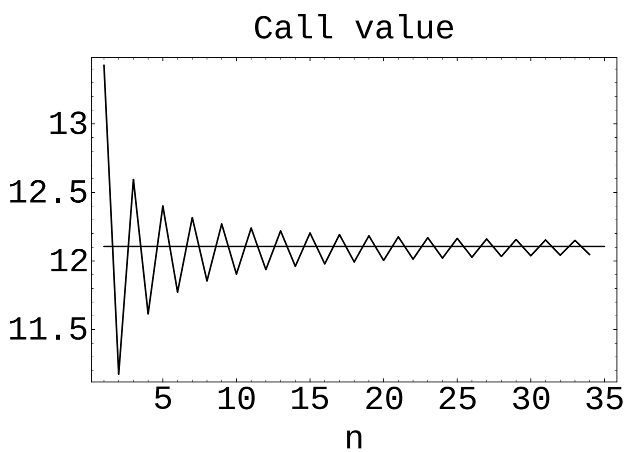
- A better approximation to the desired f is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \tag{77}$$

- This estimate should converge faster than $1/n$.
- The Richardson extrapolation uses $n_2 = 2n_1$.

Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (76) on p. 658.
- But the plots on p. 263 (redrawn on next page) demonstrate that convergence to the true option value oscillates with n.
- Extrapolation is inapplicable at this stage.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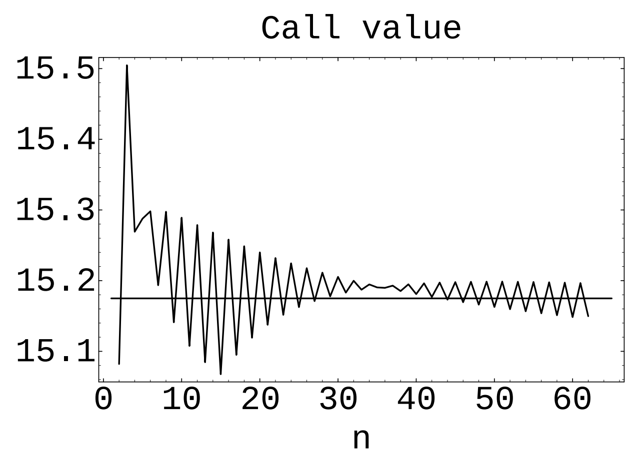
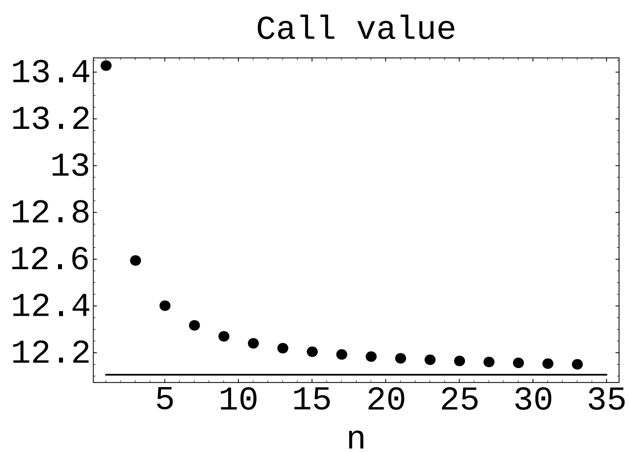
13
12.5
12
11.5
5 10 15 20 25 30 35
n


Call value
15.5
15.4
15.3
15.2
15.1
0 10 20 30 40 50 60
n

Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 661.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 663).[a]
- Apply extrapolation (77) on p. 659 with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 663.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

[a] This can be proved; see Chang and Palmer (2007).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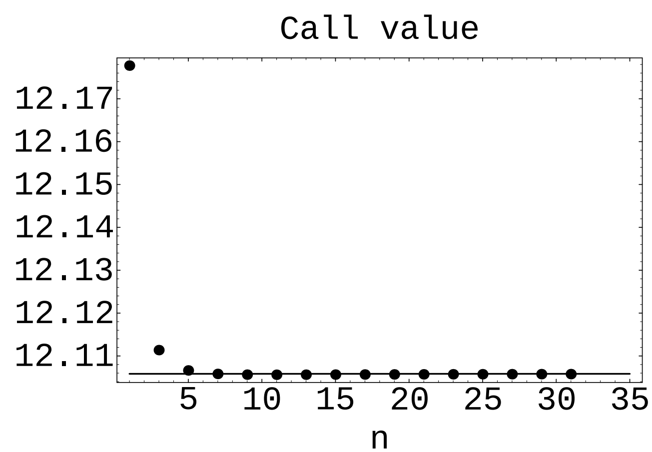
13
12.8
12.6
12.4
12.2
5 10 15 20 25 30 35
n


Call value
12.17
12.16
12.15
12.14
12.13
12.12
12.11
5 10 15 20 25 30 35
n