Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_{\mathrm{A}}\%$	$Y_{\mathrm{A}}\%$
B	$D_{\mathrm{B}}\%$	$Y_{\mathrm{B}}\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\rm A}\%$ and B to borrow dollars at $D_{\rm B}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, i.e.,

$$Y_{\rm B} - Y_{\rm A} < D_{\rm B} - D_{\rm A}.$$

- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$:
  - The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.
  - The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

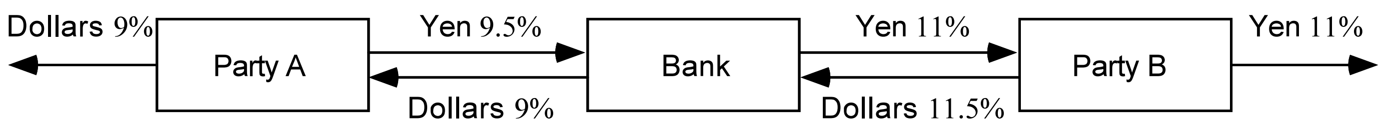
- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (continued)

- The rate differential in dollars (3%) is different from that in yen (1%).
- So a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market.
- B is relatively more competitive in the yen market.

Example (concluded)

- Next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5% (lower than 10%).
  - B borrows dollars at 11.5% (lower than 12%).
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.


Dollars 9%
Party A
Yen 9.5%
Dollars 9%
Bank
Yen 11%
Dollars 11.5%
Party B
Yen 11%

As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 432 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on:
  - The term structures of interest rates in the currencies involved.
  - The spot exchange rate.
- It has zero value when

$$SP_{\mathrm{Y}} = P_{\mathrm{D}}.$$

Example

- Take a 3-year swap on p. 432 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\begin{aligned}
& \frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09 \times 2} + 111 \times e^{-0.09 \times 3}\right) \\
- & \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08 \times 2} + 1.115 \times e^{-0.08 \times 3}\right) \quad = \quad 0.074.
\end{aligned}$$

As a Package of Forward Contracts

- From Eq. (37) on p. 402, the forward contract maturing i years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{42}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

As a Package of Forward Contracts (concluded)

- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 435.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.

Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind
has confronted and has overcome in the last
fifteen hundred years, the one which seems to me
to have been the most amazing in character and
the most stupendous in the scope of its
consequences is the one relating to
the problem of motion.
— Herbert Butterfield (1900–1979)

Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

  is a time series of random variables.

- $X(t)$ (or X_t) is a random variable for each time t and is usually called the state of the process at time t.

- A realization of X is called a sample path.

- A sample path defines an ordinary function of t.

Stochastic Processes (concluded)

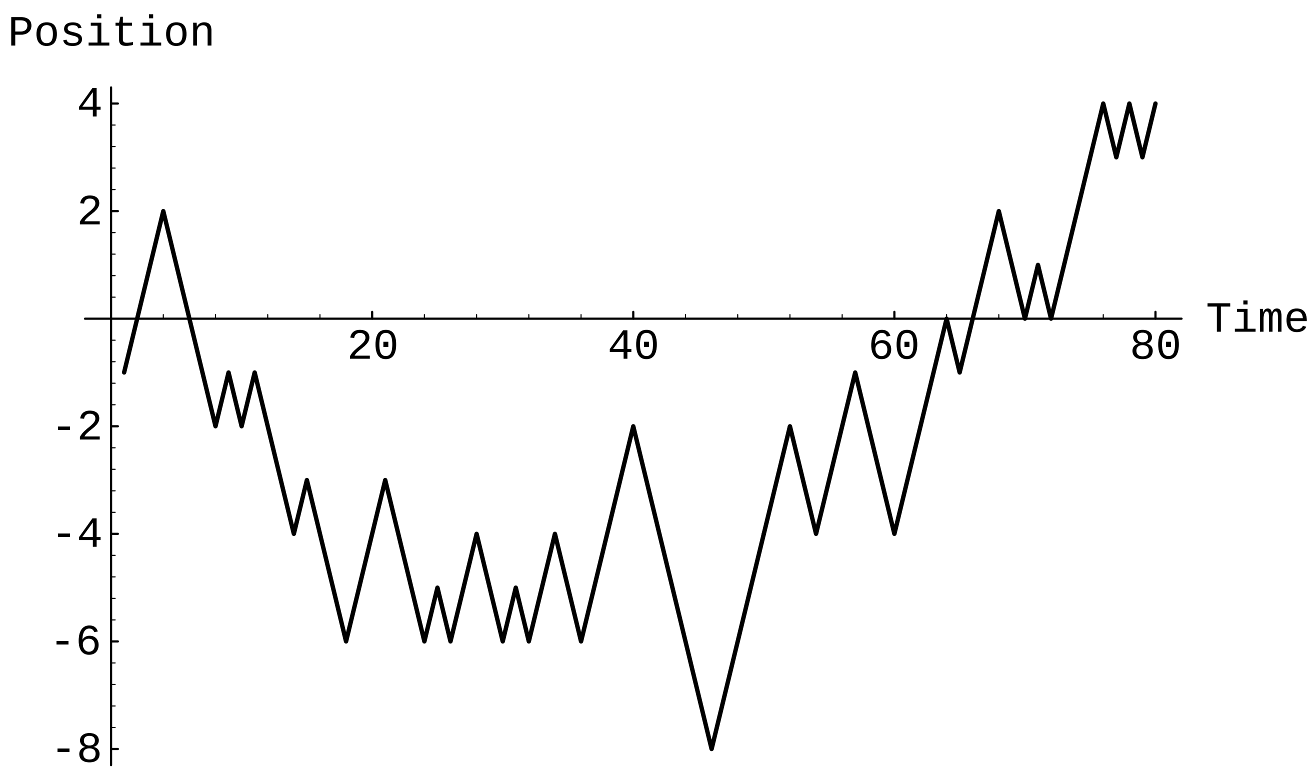
- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

- If the times form a continuum, X is called a continuous-time stochastic process.

Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability p or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the total number of up moves minus that of down moves up to that time.


Position
4
2
-2
-4
-6
-8
20
40
60
80
Time

Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- ξ_n are independent and identically distributed with zero mean.
- Drift μ is the expected change per period.
- Note that this process is continuous in space.

Martingales[a]

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and

$$E[\,X(t)\,|\,X(u), 0 \leq u \leq s\,] = X(s), \quad s \leq t. \tag{43}$$

- In the discrete-time setting, a martingale means

$$E[\,X_{n+1}\,|\,X_1, X_2, \ldots, X_n\,] = X_n. \tag{44}$$

- X_n can be interpreted as a gambler's fortune after the nth gamble.
- Identity (44) then says the expected fortune after the $(n+1)$th gamble equals the fortune after the nth gamble regardless of what may have occurred before.

[a] The origin of the name is somewhat obscure.

Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (44) on p. 446 to yield

$$E[\,X_n\,] = E[\,X_1\,] \tag{45}$$

  for all n.
- Similarly, $E[\,X(t)\,] = E[\,X(0)\,]$ in the continuous-time case.

Still a Martingale?

- Suppose we replace Eq. (44) on p. 446 with

$$E[\, X_{n+1} \,|\, X_n \,] = X_n.$$

- It also says past history cannot affect the future.
- But is it equivalent to the original definition (44) on p. 446?[a]

---

[a]Contributed by Mr. Hsieh, Chicheng (`M9007304`) on April 13, 2005.

Still a Martingale? (continued)

- Well, no.[a]
- Consider this random walk with drift:

$$X_i = \begin{cases} X_{i-1} + \xi_i, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- Above, ξ_n are random variables with zero mean.

[a] Contributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

Still a Martingale? (concluded)

- It is not hard to see that

$$E[\,X_i \mid X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

  – It is a martingale by the "new" definition.

- But

$$E[\,X_i \mid \ldots, X_{i-2}, X_{i-1}\,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

  – It is not a martingale by the original definition.

Example

- Consider the stochastic process

$$\{\, Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \,\},$$

where X_i are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned} & E[\, Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & E[\, Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & E[\, Z_n \,|\, Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & Z_n + E[\, X_{n+1} \,] = Z_n. \end{aligned}$$

Probability Measure

- A probability measure assigns probabilities to states of the world.
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- A martingale is also defined with respect to an information set.
  - In the characterizations (43)–(44) on p. 446, the information set contains the current and past values of X by default.
  - But it need not be so.

Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\,|\,X(t)\,|\,] < \infty$ and

$$E[\, X(u) \,|\, I_t \,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\, X_{n+1} \,|\, I_n \,] = X_n,$$

given the information sets $\{ I_n \}$.

Probability Measure (concluded)

- The above implies

$$E[\, X_{n+m} \,|\, I_n \,] = X_n$$

for any $m > 0$ by Eq. (19) on p. 146.

  – A typical I_n is the price information up to time n.

  – Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \equiv \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean μ.
- Now,

$$
\begin{aligned}
& E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\
= \;& E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \;& E[\, Z_n + X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \;& Z_n + \mu - (n+1)\,\mu \\
= \;& Z_n - n\mu.
\end{aligned}
$$

Example (concluded)

- Define
$$I_n \equiv \{ X_1, X_2, \ldots, X_n \}.$$
- Then
$$\{ Z_n - n\mu, n \geq 1 \}$$
is a martingale with respect to $\{ I_n \}$.

Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\, pC_u + (1-p)\, C_d \,]/R.$$

  - p is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i.
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \ldots, n \right\}.$$

- Then,

$$E\left[\left. \frac{C(i+1)}{R^{i+1}} \right| C(i) = C \right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$

Martingale Pricing (continued)

- It is easy to show that

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\,C(i)=C\right]=\frac{C}{R^i},\quad i\leq k. \tag{46}$$

- This formulation assumes:[a]
  1. The model is Markovian: The distribution of the future is determined by the present (time i) and not the past.
  2. The payoff depends only on the terminal price of the underlying asset (Asian options do not qualify).

---

[a]Contributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (`B90902081`) on May 3, 2006.

Martingale Pricing (continued)

- In general, the discount process is a martingale in that[a]

$$E_i^{\pi}\left[\frac{C(k)}{R^k}\right] = \frac{C(i)}{R^i}, \quad i \leq k. \tag{47}$$

  - E_i^{π} is taken under the risk-neutral probability conditional on the price information *up to time* i.

- This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

[a] In this general formulation, Asian options do qualify.

Martingale Pricing (continued)

- Equation (47) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \leq k. \tag{48}$$

  - $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
  - It is called the bank account process.
- It says the discount process is a martingale under π.

Martingale Pricing (continued)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.
- Identity (48) on p. 461 is the general formulation of risk-neutral valuation.

Martingale Pricing (concluded)

Theorem 16 *A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale.*[a]

[a]This probability measure is called the risk-neutral probability measure.

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is
  $$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F \left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F$$
  (p. 421).
- Can be generalized to
  $$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k,$$
  where F_i is the futures price at time i.
- This equation holds under stochastic interest rates, too.

Martingale Pricing and Numeraire[a]

- The martingale pricing formula (48) on p. 461 uses the money market account as numeraire.[b]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is positive at all times.

[a] John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value."

[b] Leon Walras (1834–1910).

Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of S at time j.
- So the discount process remains a martingale.

Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2.
- In a period, asset two's price can go from P to P_1 or P_2.
- Both assets must move up or down at the same time.
- Assume
$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}$$
to rule out arbitrage opportunities.

Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1.
- Let C_2 be its price at time one if asset one's price moves to S_2.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using α units of asset one and β units of asset two.

Example (continued)

- This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (concluded)

- It is easy to verify that

$$\frac{C}{P} = p\,\frac{C_1}{P_1} + (1-p)\,\frac{C_2}{P_2}.$$

  – Above,

$$p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- The derivative's price using asset two as numeraire (i.e., C/P) is a martingale under the risk-neutral probability p.

- The expected returns of the two assets are irrelevant.

Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.

[a]Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{49}$$

- Note that $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$.

Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_n \right).$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here
    $$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$
  - X_i are independent with
    $\text{Prob}[\, X_i = 1 \,] = p = 1 - \text{Prob}[\, X_i = -1 \,]$.
- Recall $E[\, X_i \,] = 2p - 1$ and $\text{Var}[\, X_i \,] = 1 - (2p - 1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$

$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2 \left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} & \mathrm{Var}[\, Y(t+\Delta t) - Y(t) \,] \\ = & \mathrm{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \mathrm{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (27) on p. 256 and $\Delta x = \ln u$.

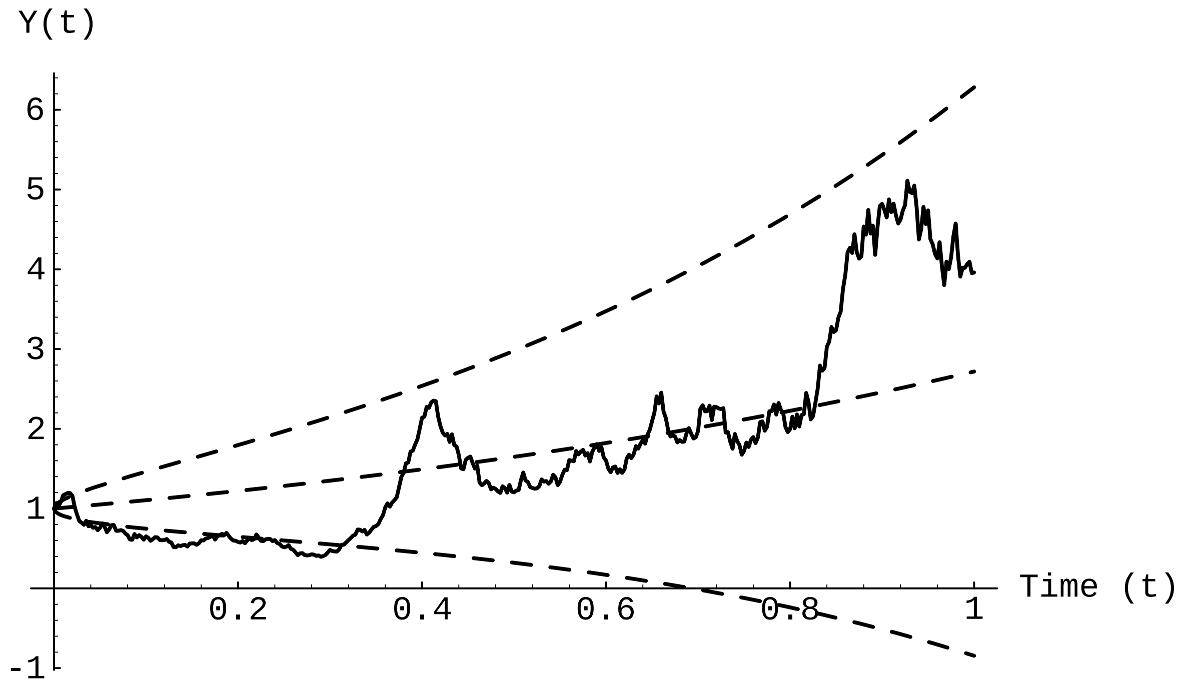
Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process
$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$
is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function
$$E\left[e^{sX(t)} \right] = E\left[Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$
from Eq. (20) on p 148.

Geometric Brownian Motion (continued)

- In particular,

$$\begin{aligned} E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)}, \\ \mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t}-1\right). \end{aligned}$$


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns

$$X_i \equiv \frac{Y_i}{Y_{i-1}}$$

are independent and identically distributed.

Geometric Brownian Motion (concluded)

- Then
$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$
is a sum of independent, identically distributed random variables.
- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.
  - And $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

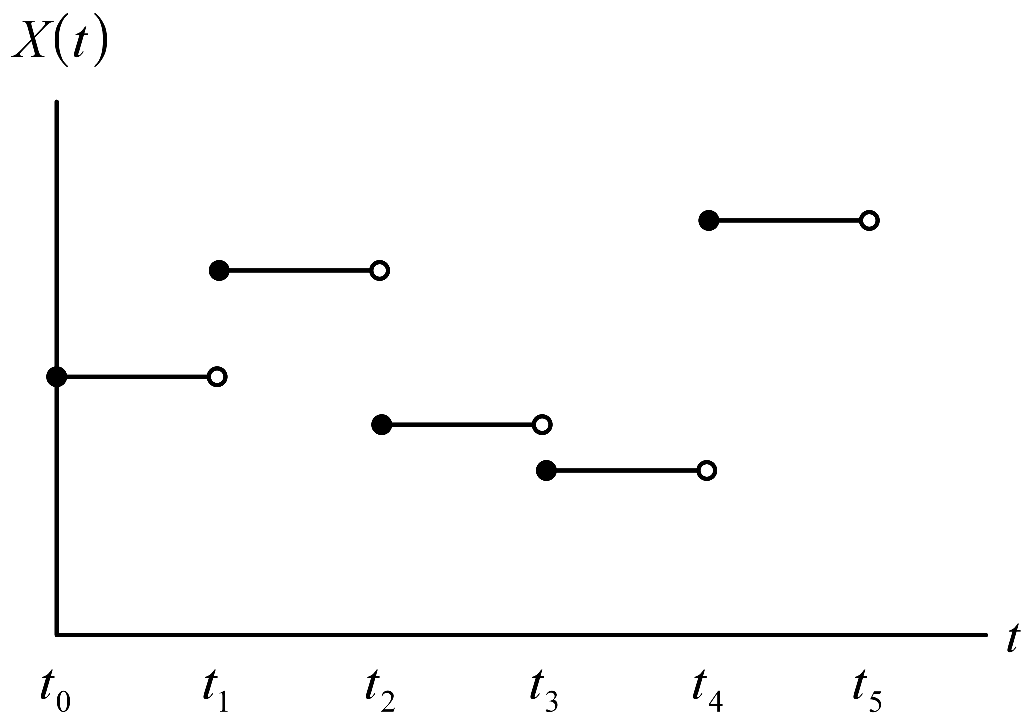
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{ X(s), 0 \leq s \leq t \}$ is independent of $\{ W(t+u) - W(t), u > 0 \}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \quad \text{for} \quad t \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots$$

for any realization (see figure on next page).



$X(t)$
t
t_0
t_1
t_2
t_3
t_4
t_5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\, W(t_{k+1}) - W(t_k)\,], \tag{50}$$

  where $t_n = t$.

  – The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\,dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
  - A corollary is the mean value formula

$$E\left[\int_a^b X\,dW\right] = 0.$$

Theorem 17 *The Ito integral* $\int X\,dW$ *is a martingale.*

Discrete Approximation

- Recall Eq. (50) on p. 489.
- The following simple stochastic process $\{ \widehat{X}(t) \}$ can be used in place of X to approximate the stochastic integral $\int_0^t X \, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{ \widehat{X}(t), W(t), 0 \leq t \leq s \},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

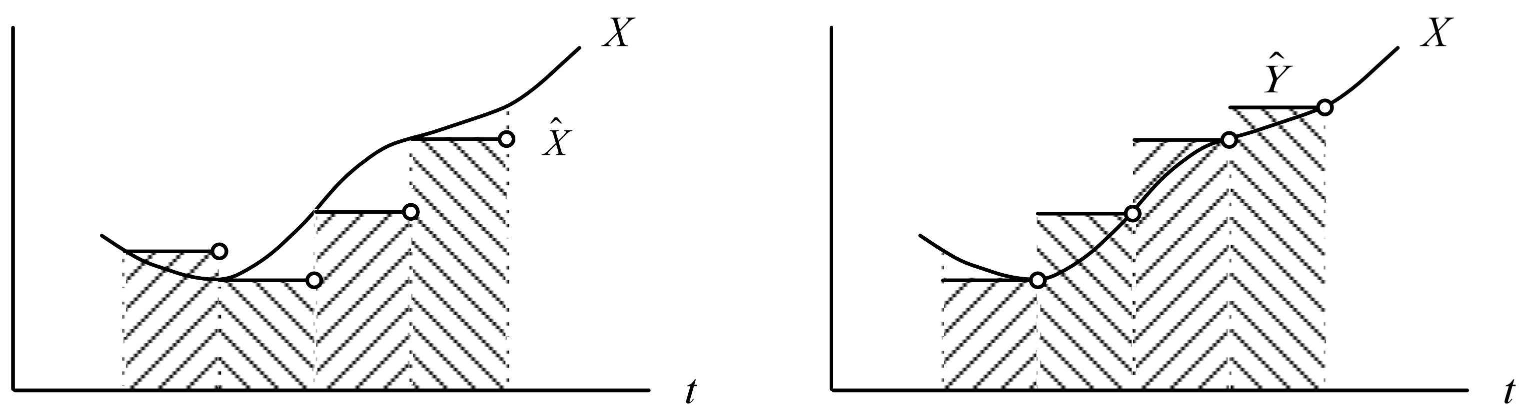
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \;\; \text{for} \;\; s \in [\,t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.



(a) (b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

- $a(X_t, t)$: the drift.
- $b(X_t, t)$: the diffusion.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{51}$$

  - Or simply
$$dX_t = a_t\, dt + b_t\, dW_t.$$
  - This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.

- X is a martingale if $a_t = 0$ (Theorem 17 on p. 491).

---

[a]Paul Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form of Eq. (51) is

$$dX_t = a_t \, dt + b_t \sqrt{dt} \, \xi, \tag{52}$$

where $\xi \sim N(0, 1)$.

Euler Approximation

- The following approximation follows from Eq. (52),

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\,\Delta W(t_n),\end{aligned} \tag{53}$$

  where $t_n \equiv n\Delta t$.

- It is called the Euler or Euler-Maruyama method.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$, not $W(t_n) - W(t_{n-1})$.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.

More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (53) on p. 498 can be replaced by

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$

  – $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned} &\widehat{X}(t_{n+1}) \\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,\xi. \end{aligned}$$

  - $\text{Prob}[\,\xi = 1\,] = \text{Prob}[\,\xi = -1\,] = 1/2$.
  - Note that $E[\,\xi\,] = 0$ and $\text{Var}[\,\xi\,] = 1$.

- This is a binomial model.

- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t (\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 18 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{54}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.

- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.
- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$.
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 19 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

where $f_i \equiv \partial f / \partial X_i$ *and* $f_{ik} \equiv \partial^2 f / \partial X_i \partial X_k$*.*

Ito's Lemma (continued)

- The multiplication table for Theorem 19 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is time t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- As an example, let

$$dX_t = a_t\, dt + b_t\, dW_t.$$

- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$
\begin{aligned}
df &= \frac{\partial f}{\partial X_t}\,dX_t + \frac{\partial f}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(dX_t)^2 \\
&= \frac{\partial f}{\partial X_t}\,(a_t\,dt + b_t\,dW_t) + \frac{\partial f}{\partial t}\,dt \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(a_t\,dt + b_t\,dW_t)^2 \\
&= \left(\frac{\partial f}{\partial X_t}\,a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,b_t^2\right) dt \\
&\quad + \frac{\partial f}{\partial X_t}\,b_t\,dW_t. \qquad (55)
\end{aligned}
$$

Ito's Lemma (continued)

Theorem 20 (Alternative Ito's Lemma) *Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\, dt + b_i\, dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 20 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Above, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider geometric Brownian motion $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
  - Hence $dX = \mu\, dt + \sigma\, dW$ by Eq. (49) on p. 473.
- As $\partial Y / \partial X = Y$ and $\partial^2 Y / \partial X^2 = Y$, Ito's formula (54) on p. 504 implies

$$\begin{aligned} dY &= Y\, dX + (1/2)\, Y\, (dX)^2 \\ &= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y\, (\mu\, dt + \sigma\, dW)^2 \\ &= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y \sigma^2\, dt. \end{aligned}$$

Geometric Brownian Motion (concluded)

- Hence
$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW. \tag{56}$$
- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not μ.