Option Pricing Models

If the world of sense does not fit mathematics,
so much the worse for the world of sense.
— Bertrand Russell (1872–1970)

Black insisted that anything one could do
with a mouse could be done better
with macro redefinitions
of particular keys on the keyboard.
— Emanuel Derman,
My Life as a Quant (2004)

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.[a]
  - Known as the Black-Scholes option pricing model.

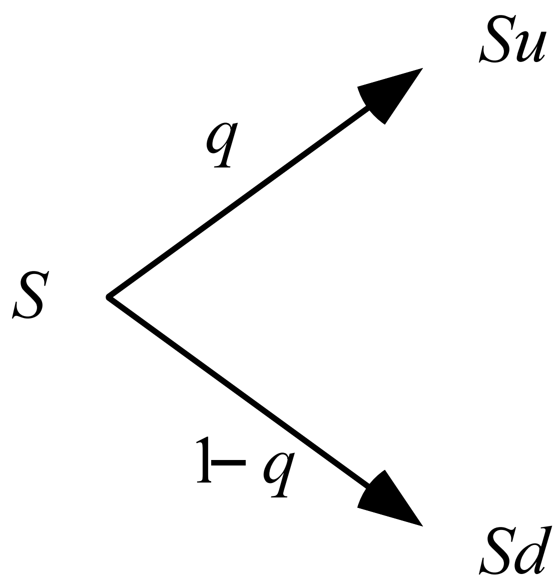
[a] The results were obtained as early as June 1969.

Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

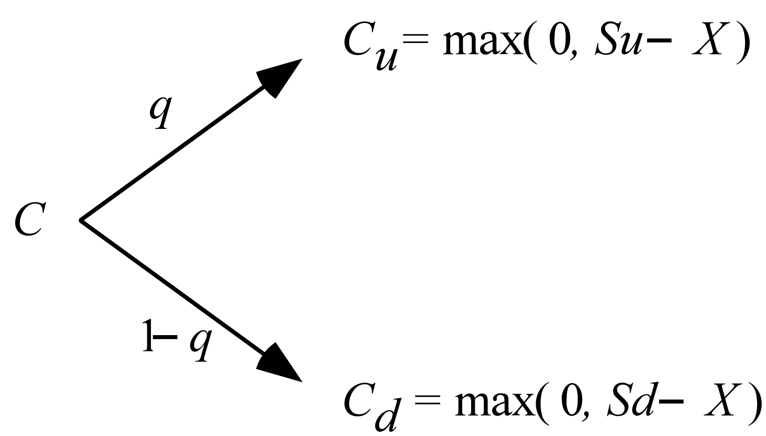
- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information will suffice to determine the option value based on arbitrage considerations: S, u, d, X, $\hat{r}$, and the number of periods to expiration.


Su
q
S
$1-q$
Sd

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time one if the stock price moves to Su.
- C_d is the call price at time one if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$


$C_u = \max(0, Su - X)$
q
C
$1 - q$
$C_d = \max(0, Sd - X)$

Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is either $hSu + RB$ or $hSd + RB$.
- Choose h and B such that the portfolio replicates the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{23}$$

$$B = \frac{uC_d - dC_u}{(u-d)\,R}. \tag{24}$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,[a]

$$C = hS + B.$$

- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

[a] Or the replicating portfolio, as it replicates the option.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 4 (p. 198).
- So $C = hS + B$.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q.[a]
- Hence it is independent of the expected gross return of the stock, $qSu + (1 - q)\, Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Note that the set of possible stock prices is the same whatever q is.

[a]More precisely, not directly dependent on q. Thanks to a lively class discussion on March 16, 2011.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}.$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as $pSu + (1 - p)\, Sd = RS$.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

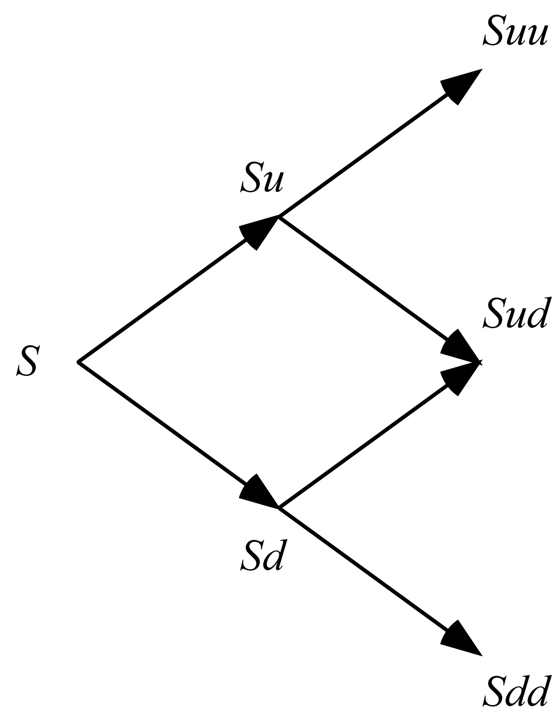
  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

- Suppose you toss a coin n times with p being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree combines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.


Suu
Su
Sud
S
Sd
Sdd

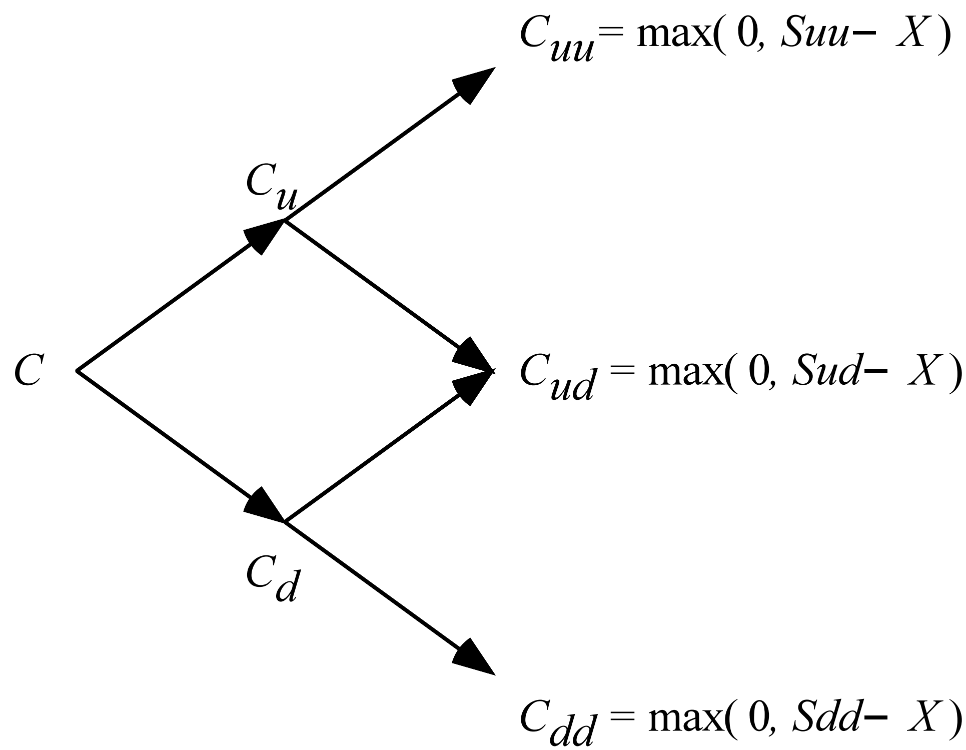
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$


$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \qquad (25)\\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (23) on p. 214.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute
$$\frac{pC_u + (1-p)\,C_d}{R}$$
as the option price.
- The values of delta h and B can be derived from Eqs. (23)–(24) on p. 214.

Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\,C_d}{R} \geq \frac{[\,pu + (1-p)\,d\,]\,S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  - The call again will not be exercised at present.[a]
- So

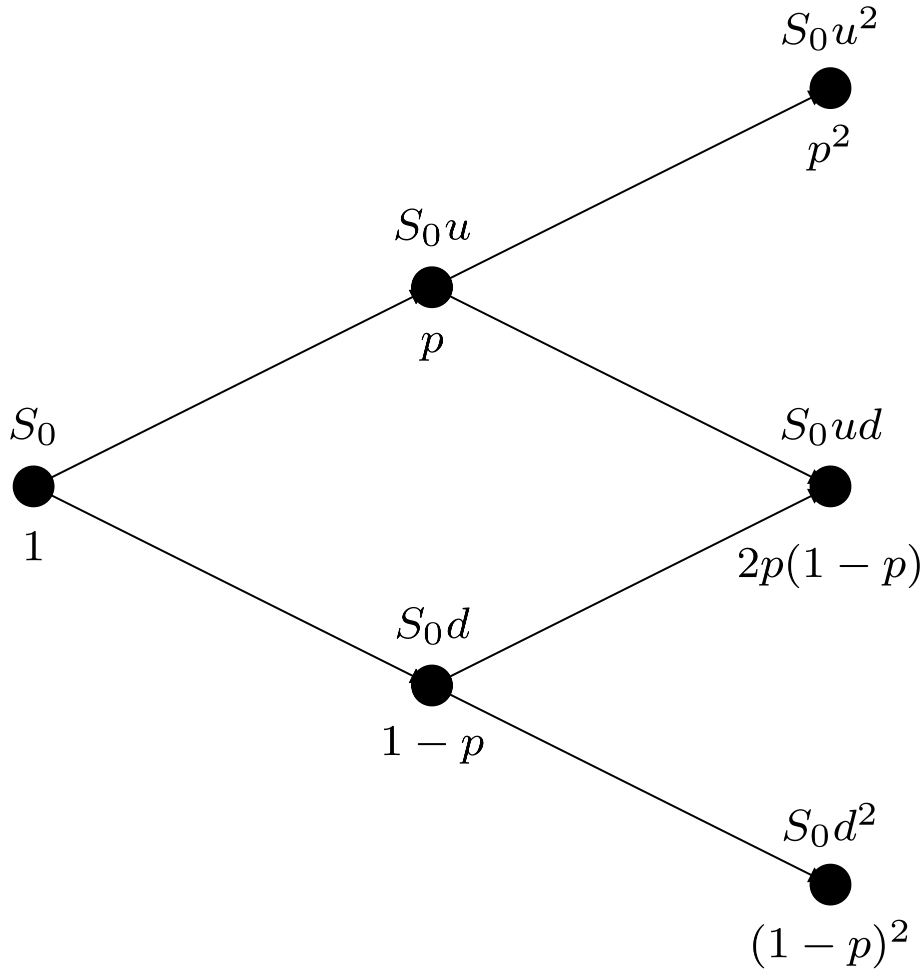
$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

[a] Consistent with Theorem 4 (p. 198).

Backward Induction of Zermelo (1871–1953)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (25) on p. 225.
- This recursive procedure is called backward induction.
- C equals

$$
\begin{aligned}
& [\, p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \; & [\, p^2 \max\left(0, Su^2 - X\right) + 2p(1-p) \max(0, Sud - X) \\
& +(1-p)^2 \max\left(0, Sd^2 - X\right)\,]/R^2.
\end{aligned}
$$


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (concluded)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  – E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

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[a]Except at the beginning, of course, when you have to put up the option value C or P before the replication starts.

Hakansson's Paradox[a]

- If options can be replicated, why are they needed at all?

[a] Hakansson (1979).

Can You Figure Out u, d without Knowing q?[a]

- Yes, you can, under BOPM.
- Let us observe the time series of past stock prices, e.g.,

$$\overbrace{S, Su,}^{u \text{ is available}} \quad Su^2, \underbrace{Su^3, Su^3 d}_{d \text{ is available}}, \ldots$$

- So with sufficiently long history, you will figure out u and d without knowing q.

[a] Contributed by Mr. Hsu, Jia-Shuo (D97945003) on March 11, 2009.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

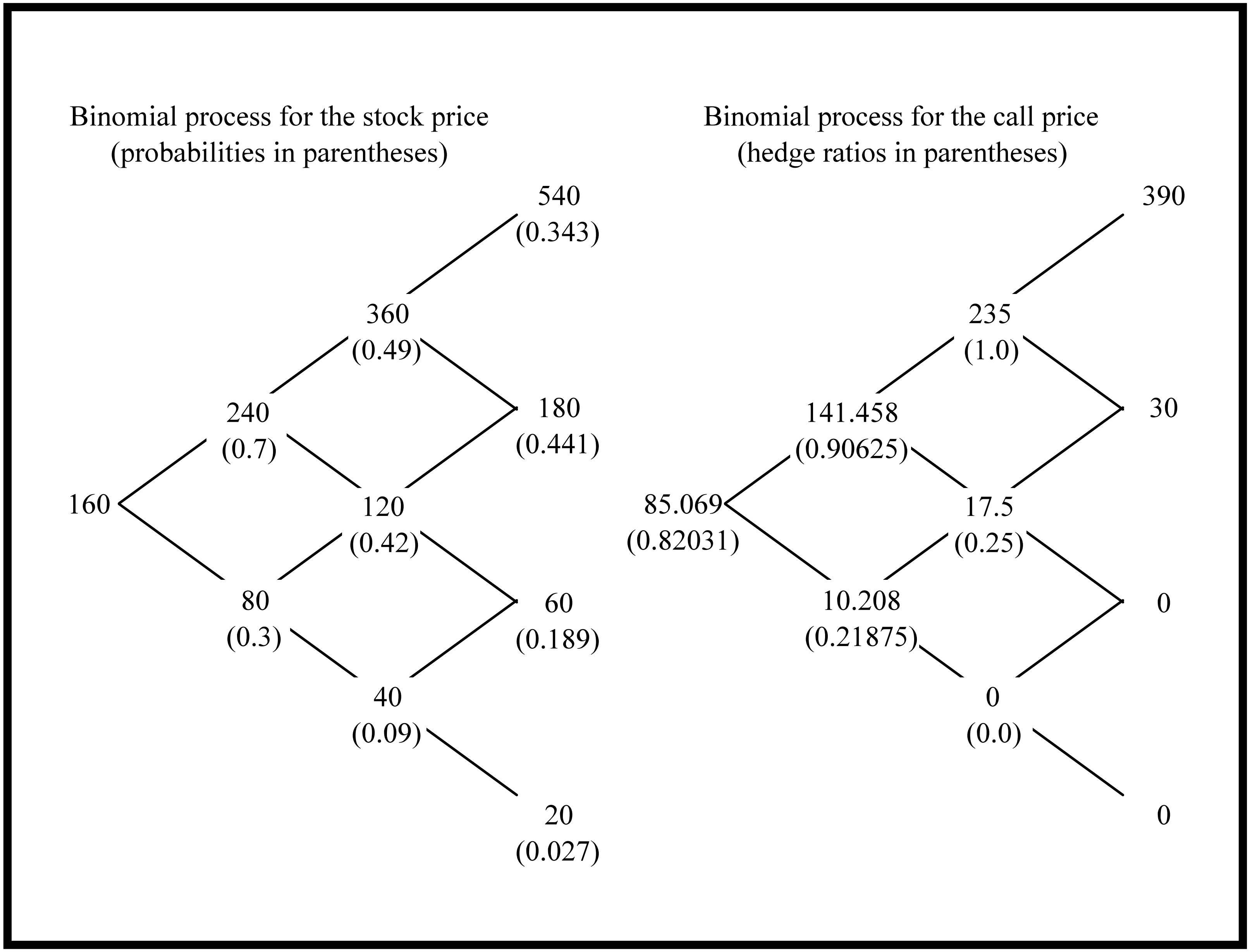
- Hence,

$$
\begin{aligned}
& C \\
= \; & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} \qquad (26) \\
= \; & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \; & S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p).
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy

$$0.90625 - 0.82031 = 0.08594$$

  more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

  dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

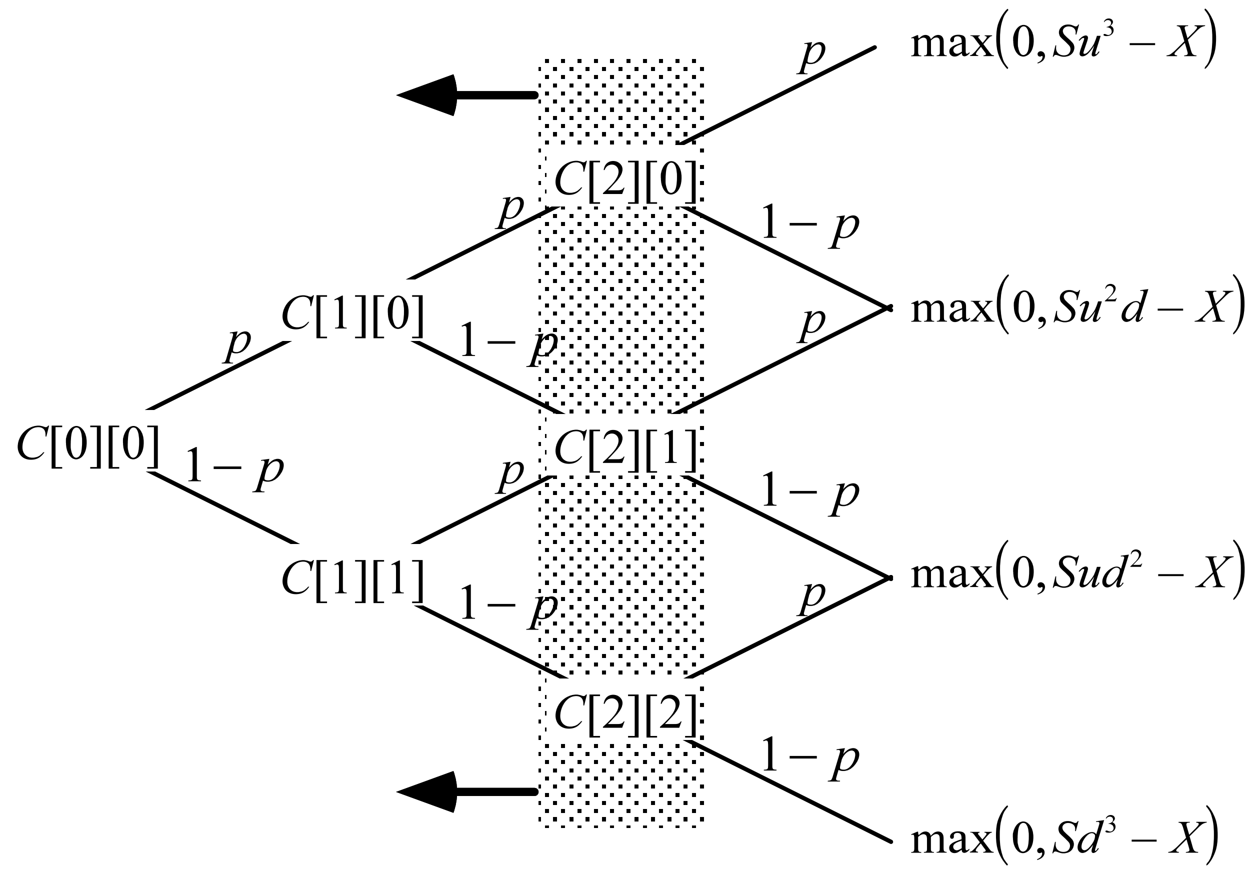
- Replicate an option using stocks and bonds.
- Hedge the options we issued (the mirror image of replication).
- $\cdots$

[a] Thanks to a lively class discussion on March 16, 2011.

Binomial Tree Algorithms for European Options

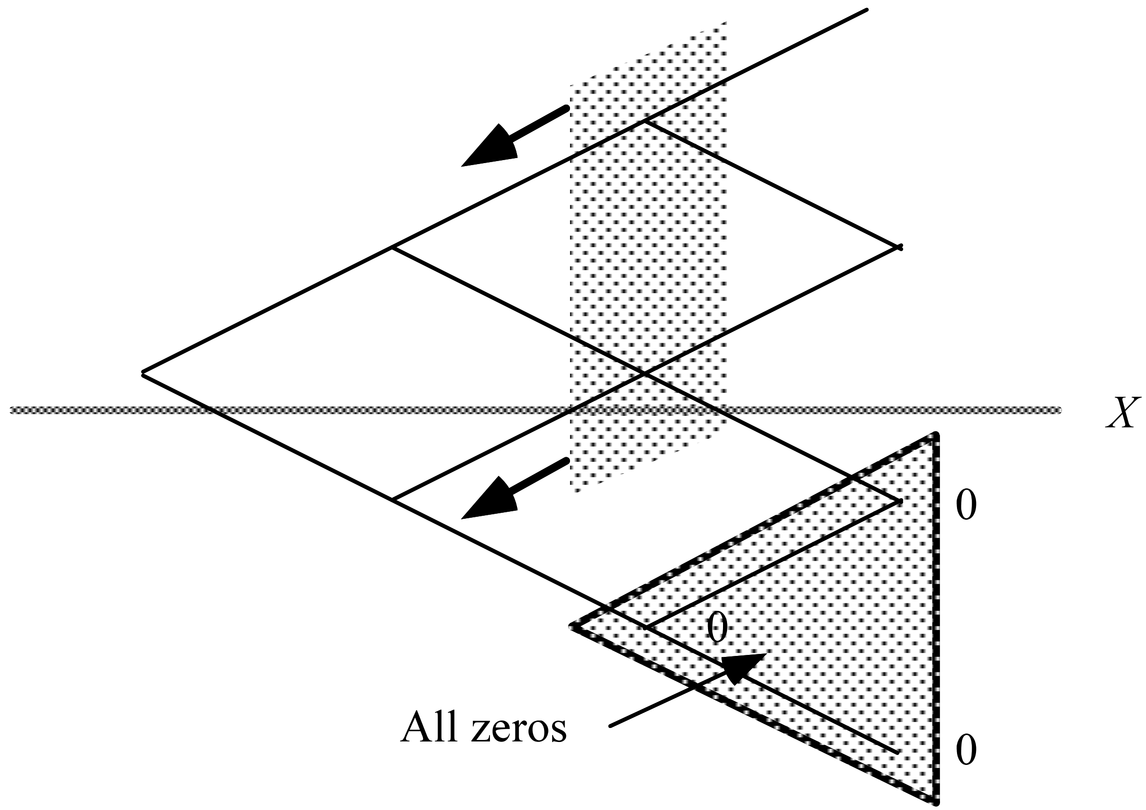
- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To price European puts, simply replace the payoff.

[a] But watch out for the proper updating of array entries.



p
$\max(0, Su^3 - X)$
$C[2][0]$
p
$1-p$
$C[1][0]$
$1-p$
p
$\max(0, Su^2d - X)$
p
$C[0][0]$
$1-p$
$C[2][1]$
p
$1-p$
$C[1][1]$
$1-p$
p
$\max(0, Sud^2 - X)$
$C[2][2]$
$1-p$
$\max(0, Sd^3 - X)$

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\, j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

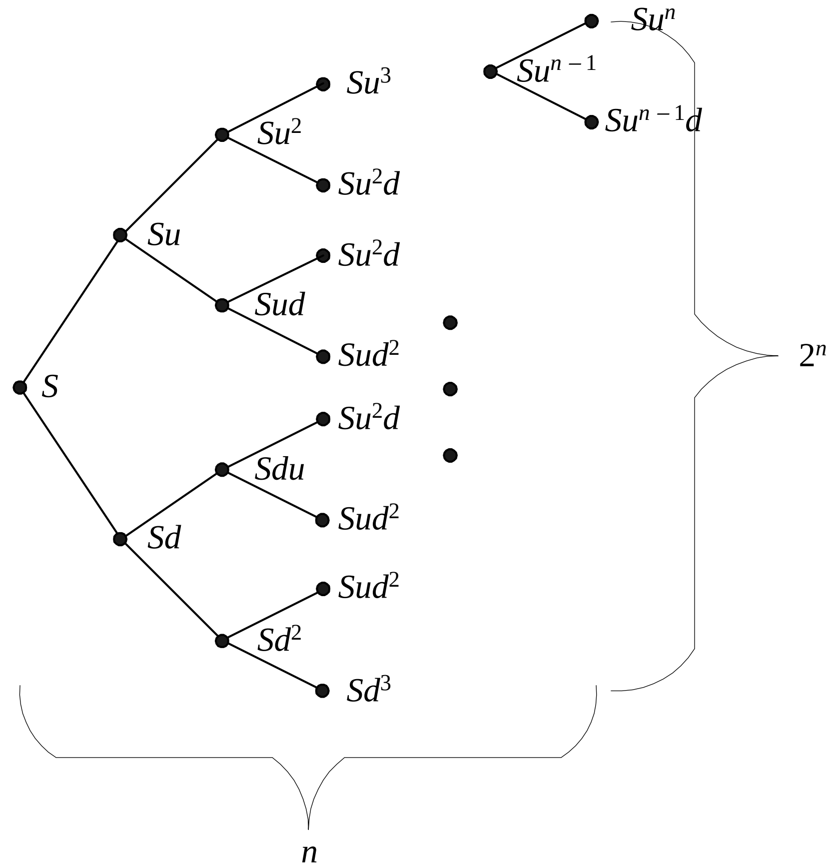
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a} p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
    $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (26) on p. 236 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Use

$$\widehat{\mu} \equiv \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right] \quad \text{and} \quad \widehat{\sigma}^2 \equiv \frac{1}{n} \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$

  to denote the expected value and variance of the continuously compounded rate of return per period.

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call σ the stock's (annualized) volatility.
- The BOPM converges to the distribution only if

$$\begin{aligned} n\widehat{\mu} &= n[\,q\ln(u/d) + \ln d\,] \to \mu\tau, \\ n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \end{aligned}$$

- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 247).
  - Other choices are possible (see text).
  - Exact solutions for u, d, q are also available.[a]

[a] Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \tag{27}$$

- With Eqs. (27), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right] \sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

- The choice (27) results in the CRR binomial model.[a]

[a] Cox, Ross, and Rubinstein (1979).

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (27) on p. 256.
  - If this happens, the risk-neutral probability may lie outside $[0, 1]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

i.e., when $n > r^2\tau/\sigma^2$ (check it).

  - So it goes away if n is large enough.
  - Other solutions will be presented later.

[a] Many papers and programs forget to check this condition!

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.
- So $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$.
- S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

Lemma 9 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \equiv (e^{r\tau/n} - d)/(u-d).$$

- Let $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$.[a]
- The stock's expected annual rate of return[b] is thus the riskless rate r.

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[a] By Lemma 9 (p. 259) and Eq. (21) on p. 154.

[b] In the sense of $(1/\tau) \ln E[\, S_\tau / S \,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S) \,]$ (geometric average rate of return).

Toward the Black-Scholes Formula (concluded)[a]

Theorem 10 (The Black-Scholes Formula)

$$
\begin{aligned}
C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\
P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x),
\end{aligned}
$$

where

$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

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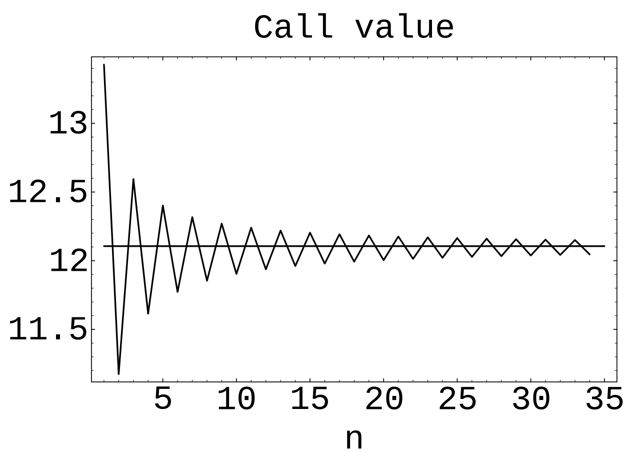
[a]On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

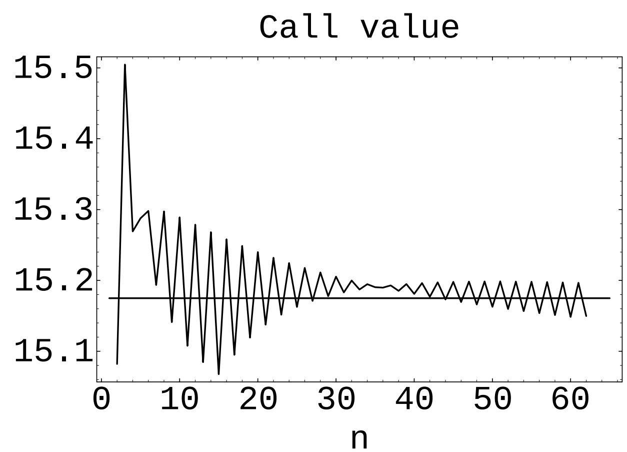
BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$u = e^{\sigma \sqrt{\tau/n}},\ d = e^{-\sigma \sqrt{\tau/n}},\ \hat{r} = r\tau/n.$$

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be dealt with by the judicious choices of u and d (see text).





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).
- The error is $O(1/n)$.[a]

[a]Chang and Palmer (2007).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility[b] in practice.

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[a] Implied volatility is hard to compute when τ is small (why?).

[b] Using the historical volatility is like driving a car with your eyes on the rearview mirror?